Supplementary Materials for "Optimizing (s, S) Policies for Multi-period Inventory Models with Demand Distribution Uncertainty: Robust Dynamic Programming Approaches"

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A.1 Proof of Theorem 1

The following two lemmas using the definition of a $\mathcal{K}$-convex function are used to prove Theorem 1.

Lemma 1. *For any $t = 1, 2, ..., T$, the function $\psi^t(x)$ (7) is continuous with respect to x and $\lim_{|x|\to\infty} \psi^t(x) = \infty$. Specifically, $\psi^t(x)$ is a $\mathcal{K}$-convex function of x.*

Lemma 2. *Under Lemma 1, let S_t be a minimum point of $\psi^t(x)$ and s_t be any element of the set*

$$\left\{x \,\middle|\, x \le S_t, \psi^t(x) = \psi^t(S_t) + \mathcal{K}\right\} \tag{37}$$

The following results hold

(i) $\psi^t(S_t) \le \psi^t(x)$, for all $x \in R$.

(ii) $\psi^t(S_t) + \mathcal{K} = \psi^t(s_t) \le \psi^t(x)$, for all $x \le s_t$.

(iii) $\psi^t(x) \le \psi^t(y) + \mathcal{K}$, for all x and y with $s_t \le x \le y$.

The proof of Lemma 1 follows a similar logic in Scarf (1960). A different $\mathcal{K}$-convex function from that in Scarf (1960) and mathematical induction are used to complete the proof. First, consider the last period, i.e., $t = T$, $\psi^T(x) = c_T x + G^T(x) = c_T x + \max_{\boldsymbol{p}_T} \{C_T(x)'\boldsymbol{p}_T\}$. Obviously, $\psi^T(x)$ is continuous and $\lim_{|x|\to\infty} \psi^T(x) = \infty$. Specifically, $\psi^T(x)$ is convex and, hence, a $\mathcal{K}$-convex function.

Next, assume $\psi^t(x)$ is continuous, $\mathcal{K}$-convex and $\lim_{|x|\to\infty} \psi^t(x) = \infty$, then there exist two parameters s_t and S_t with $s_t < S_t$ such that S_t minimizes $\psi^t(x)$ and $\psi^t(s_t) = \psi^t(S_t) + \mathcal{K}$. From the definitions of $G^t(x)$ and $z^t(x)$, $z^t(x) = \psi^t(S_t) + \mathcal{K} - c_t x$ if $x \le s_t$ and $z^t(x) = \psi^t(x) - c_t x$ otherwise. Since $\psi^t(s_t) = \psi^t(S_t) + \mathcal{K}$, $z^t(x)$ is continuous and $\mathcal{K}$-convex.

Finally, consider period $t - 1$

$$\begin{aligned}\psi^{t-1}(x) &= c_{t-1}x + G^{t-1}(x) \\ &= c_{t-1}x + \max_{\boldsymbol{p}_{t-1}} \left\{C_{t-1}(x)'\boldsymbol{p}_{t-1} + \gamma z^t(x - D_{t-1})\right\}\end{aligned}$$

Therefore, $\psi^{t-1}(x)$ is continuous. Since $z^t(x)$ is continuous and -convex, according to the properties of the $\mathcal{K}$-convex functions (Zipkin, 2000, P398), $z^t(x - D_{t-1})$ is -convex and thus $\psi^{t-1}(x)$ is also -convex. Furthermore, $\lim_{|x|\to\infty} \psi^{t-1}(x) = \infty$.

Based on Lemma 1 and the properties of a -convex function, Lemma 2 is straightforward. The proof is omitted and readers are referred to Simchi et al. (2014). Lemmas 1 and 2 then lead to Theorem 1. □

A.2 Proof of Theorem 2

Let $(x^*, \delta_T^*, \boldsymbol{\tau}_T^*, \boldsymbol{\nu}_T^*)$ be an optimal solution to Problem (14) with an optimal objective value θ^*. It can be found by comparing the constraints of Problems (13) and (14) that $(\delta_T^*, \boldsymbol{\tau}_T^*, \boldsymbol{\nu}_T^*)$ is also feasible to Problem (13). Given x^*, $\Upsilon^*(x^*) = \underline{\boldsymbol{\xi}}_T\boldsymbol{\tau}_T^* + \overline{\boldsymbol{\xi}}_T\boldsymbol{\nu}_T^*$ by the strong duality of linear programming. If x^* is not optimal to Problem (11), there exists another solution $\tilde{x}^*$ to Problem (11) such that

$$\begin{aligned}c_T\tilde{x}^* + C_T(\tilde{x}^*)'\bar{\boldsymbol{p}}_T + \Upsilon^*(\tilde{x}^*) &\le c_T x^* + C_T(x^*)'\bar{\boldsymbol{p}}_T + \Upsilon^*(x^*) \\ &= c_T x^* + C_T(x^*)'\bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T\boldsymbol{\tau}_T^* + \overline{\boldsymbol{\xi}}_T\boldsymbol{\nu}_T^* \\ &= \theta^*.\end{aligned}$$

Given $\tilde{x}^*$, let $(\tilde{\delta}_T^*, \tilde{\boldsymbol{\tau}}_T^*, \tilde{\boldsymbol{\nu}}_T^*)$ be an optimal solution to Problem (13). By comparing the constraints of Problems (13) and (14), it can be found that $(\tilde{x}^*, \tilde{\delta}_T^*, \tilde{\boldsymbol{\tau}}_T^*, \tilde{\boldsymbol{\nu}}_T^*)$ is also feasible to Problem (14). Similarly, $\Upsilon^*(\tilde{x}^*) = \underline{\boldsymbol{\xi}}_T\tilde{\boldsymbol{\tau}}_T^* + \overline{\boldsymbol{\xi}}_T\tilde{\boldsymbol{\nu}}_T^*$ by the strong duality of linear programming. Therefore, the objective value of Problem (14) at $(\tilde{x}^*, \tilde{\delta}_T^*, \tilde{\boldsymbol{\tau}}_T^*, \tilde{\boldsymbol{\nu}}_T^*)$, denoted by $\tilde{\theta}^*$, is not larger than θ^*, i.e., $\tilde{\theta}^* \le \theta^*$. This contradicts the assumption that $(x^*, \delta_T^*, \boldsymbol{\tau}_T^*, \boldsymbol{\nu}_T^*)$ is an optimal solution to Problem (14). Hence, x^* is an optimal solution to Problem (11).

Conversely, if $\hat{x}^*$ solves Problem (11), $(\hat{\delta}_T^*, \hat{\boldsymbol{\tau}}_T^*, \hat{\boldsymbol{\nu}}_T^*)$ is an optimal solution to Problem (13) with $x = \hat{x}^*$. If $(\hat{x}^*, \hat{\delta}_T^*, \hat{\boldsymbol{\tau}}_T^*, \hat{\boldsymbol{\nu}}_T^*)$ is not an optimal solution to Problem (14), there exists another solution $(\tilde{\hat{x}}^*, \tilde{\hat{\delta}}_T^*, \tilde{\hat{\boldsymbol{\tau}}}_T^*, \tilde{\hat{\boldsymbol{\nu}}}_T^*)$ that solves Problem (14). According to the discussion above, $\tilde{\hat{x}}^*$ is an optimal solution to Problem (11), contradicting the assumption that $\hat{x}^*$ solves Problem (11). Therefore, solving Problem (14) is equivalent to solving Problem (11). $\square$

A.3 Proof of Theorem 3

The reorder point s_T can be found by solving

$$\begin{aligned} \max_{y \le S_T} \;& G^T(y) \\ s.t. \;& G^T(y) \le \mathcal{K} + c_T(S_T - y) + G^T(S_T), \end{aligned} \tag{38}$$

where the optimal value of the variable y is the reorder point s_T, i.e., $s_T = y^*$, and $G^T(y)$ is the optimal objective value of the following problem with variables $(\delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T) \in R \times R^{K_T} \times R^{K_T}$

$$\begin{aligned} \min_{\delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T} \;& C_T(y)'\bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T \boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T \boldsymbol{\nu}_T \\ s.t. \;& \boldsymbol{e}'\delta_T + \boldsymbol{\tau}_T + \boldsymbol{\nu}_T = C_T(y) \\ & \boldsymbol{\tau}_T \le \mathbf{0}, \boldsymbol{\nu}_T \ge \mathbf{0}. \end{aligned} \tag{39}$$

The constraint in Problem (38) is satisfied at equality at s_T. Therefore, Problem (38) can be rewritten as

$$\begin{aligned} \max_{y \le S_T} \min_{\delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T} \;& \mathcal{K} + c_T(S_T - y) + G^T(S_T) \\ s.t. \;& \boldsymbol{e}'\delta_T + \boldsymbol{\tau}_T + \boldsymbol{\nu}_T = C_T(y) \\ & \boldsymbol{\tau}_T \le \mathbf{0}, \boldsymbol{\nu}_T \ge \mathbf{0} \\ & C_T(y)'\bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T \boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T \boldsymbol{\nu}_T = \mathcal{K} + c_T(S_T - y) + G^T(S_T). \end{aligned}$$

In this optimization problem, the objective function is composed of a constant $\mathcal{K} + c_T S_T + G^T(S_T)$ and a linear term $-c_T y$ that are both independent of the variables $(\delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T)$. Therefore, s_T can be found by solving Problem (16). $\square$

A.4 Proof of Theorem 4

Let $(x^*, \boldsymbol{\lambda}_T^*, \rho_T^*, \gamma_T^*)$ and θ^* be an optimal solution and the optimal objective value to Problem (29), respectively. Then x^* is also feasible to Problem (25). Denote by θ_0^* the objective value of Problem (25) at $x = x^*$. If x^* is not optimal to Problem (25), there exists another solution $\tilde{x}^*$ that solves Problem (25) such that $\tilde{\theta}_0^* < \theta_0^*$. Given $x = \tilde{x}^*$, $(\tilde{\boldsymbol{\lambda}}_T^*, \tilde{\rho}_T^*, \tilde{\gamma}_T^*)$ is obtained by solving Problem (28). By the strong duality of the Lagrangian, $\Gamma^*(x^*) = -(\boldsymbol{\lambda}_T^*)'\bar{\boldsymbol{p}}_T - \rho_T^*$ and $\Gamma^*(\tilde{x}^*) = -(\tilde{\boldsymbol{\lambda}}_T^*)'\bar{\boldsymbol{p}}_T - \tilde{\rho}_T^*$. Therefore, $\tilde{\theta} = \tilde{\theta}_0 < \theta_0^* = \theta^*$. Because Problems (28) and (29) have the same set of constraints, $(\tilde{x}^*, \tilde{\boldsymbol{\lambda}}_T^*, \tilde{\rho}_T^*, \tilde{\gamma}_T^*)$ is also feasible to Problem (29). This contradicts the assumption that $(x^*, \boldsymbol{\lambda}_T^*, \rho_T^*, \gamma_T^*)$ is an optimal solution to Problem (29) because $\tilde{\theta}^* < \theta^*$. Thus, x^* is an optimal solution to Problem (25).

Conversely, if $\hat{x}^*$ is an optimal solution to Problem (25), $(\hat{\boldsymbol{\lambda}}_T^*, \hat{\rho}_T^*, \hat{\gamma}_T^*)$ can be obtained by solving Problem (28). Since Problems (28) and (29) have the same set of constraints, $(\hat{x}^*, \hat{\boldsymbol{\lambda}}_T^*, \hat{\rho}_T^*, \hat{\gamma}_T^*)$ is also feasible to Problem (29). If $(\hat{x}^*, \hat{\boldsymbol{\lambda}}_T^*, \hat{\rho}_T^*, \hat{\gamma}_T^*)$ is not an optimal solution to Problem (29), there exists another solution $(\tilde{\hat{x}}^*, \tilde{\hat{\boldsymbol{\lambda}}}_T^*, \tilde{\hat{\rho}}_T^*, \tilde{\hat{\gamma}}_T^*)$ that solves Problem (29). From the discussion above, $\tilde{\hat{x}}^*$ is an optimal solution to Problem (25), contradicting the assumption that $\hat{x}^*$ is an optimal solution to Problem (25). Therefore, $(\hat{x}^*, \hat{\boldsymbol{\lambda}}_T^*, \hat{\rho}_T^*, \hat{\gamma}_T^*)$ is an optimal solution to Problem (28). □

References

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