L S Penrose's limit theorem: Tests by simulation

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Draft 6
May 2005

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ABSTRACT

L S Penrose's Limit Theorem – which is implicit in Penrose [7, p. 72] and for which he gave no rigorous proof – says that, in simple weighted voting games, if the number of voters increases indefinitely and the relative quota is pegged, then – under certain conditions – the ratio between the voting powers of any two voters converges to the ratio between their weights. Lindner and Machover [4] prove some special cases of Penrose's Limit Theorem. They give a simple counter-example showing that the theorem does not hold in general even under the conditions assumed by Penrose; but they conjecture, in effect, that under rather general conditions it holds 'almost always' – that is with probability 1 – for large classes of weighted voting games, for various values of the quota, and with respect to several measures of voting power. We use simulation to test this conjecture. It is corroborated with respect to the Penrose–Banzhaf index for a quota of 50% but not for other values; with respect to the Shapley–Shubik index the conjecture is corroborated for all values of the quota (short of 100%).

Keywords: limit theorems, majority games, simulation, weighted voting games

JEL classifications: C71, D71

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1 Introduction

Throughout this paper, we shall be concerned with *weighted voting games*. Let us recall briefly their definition. A weighted voting game $\mathcal{W}$ consists of a finite set N together with an assignment of a non-negative real *weight* w_x to each $x \in N$; and a real $q \in (0, 1)$. We refer to N as the *assembly* of $\mathcal{W}$, to the members of N as *voters*, and to q as the *quota*.[1] Any subset $A \subseteq N$ is often referred to as a 'coalition'.

For our purposes it will be convenient, and will entail no loss of generality, to assume that all weights are positive. The *relative weight* of voter a in $\mathcal{W}$ is given by

$$\overline{w}_a := \frac{w_a}{\sum_{x \in N} w_x}. \tag{1}$$

A coalition A is said to be *winning* if

$$\sum_{x \in A} \overline{w}_x \geq q. \tag{2}$$

L S Penrose's Limit Theorem is an assertion about the asymptotic behaviour of the voting power of voters in weighted voting games with a large number of voters. Here we shall consider the two major indices of voting power: the so-called *Banzhaf* index β (which is obtained by normalization from the absolute measure of voting power first proposed by Penrose [6]); and the *Shapley–Shubik* index ϕ proposed by these two authors in [8] (which is a special case of the Shapley value for co-operative games). For the definitions of these indices see, for example, Felsenthal and Machover [3].

Penrose [7, p. 72] gives an approximation formula for the voting power (as defined by him) of a voter in a weighted voting game $\mathcal{W}$ with quota $\frac{1}{2}$, according to which voters' powers are approximately proportional to their respective weights. He claims that this approximation is valid provided the number of voters in $\mathcal{W}$ is large, and the relative weights of the voters in question are small. He offers no rigorous proof of his claim, merely an outline of an argument, obviously based on some version of the central limit theorem of probability theory.

[1] In the voting-power literature, q is often referred to as the *relative* quota, as distinct from the *absolute* quota, which equals $q \cdot \sum_{x \in N} w_x$.

However, Lindner and Machover [4] show by means of a simple counter-example that these conditions are insufficient for Penrose's approximation formula and the version of Penrose's Limit Theorem implied by it. On the other hand, they prove the approximation formula (in a somewhat improved form) as well as Penrose's Limit Theorem under more stringent conditions: both with respect to β for $q = \frac{1}{2}$ (see [4, Theorem 3.6]); and with respect to the Shapley–Shubik index ϕ for arbitrary $q \in (0, 1)$ (see [4, Theorem 2.3]).

Furthermore, they conjecture that Penrose's Limit Theorem holds 'almost always' with respect to both β and ϕ for all $q \in (0, 1)$.

For a rigorous treatment of their special cases, as well as for a precise statement of their conjecture, they use the concept of a *q-chain* of weighted voting games. This is an infinite sequence $\mathcal{W}^{(k)}(k = 0, 1, \ldots)$ of such games in which the assembly of each $\mathcal{W}^{(k)}$ is a proper subset of the assembly of its successor, $\mathcal{W}^{(k+1)}$; the voters of $\mathcal{W}^{(k)}$ keep their old weights in $\mathcal{W}^{(k+1)}$; and the quota q is held fixed.

Their conjecture is that under a 'reasonable' probability measure on the space of all q-chains, Penrose's Limit Theorem holds with probability 1 with respect to both β and ϕ.

In this paper we report the results of simulation designed to test a version of this conjecture. Here is an outline of how we go about it. (A more detailed account will be given in Section 2.)

Obviously, we cannot use simulation to test the conjecture directly in the form stated above, because we cannot select at random an entire q-chain, which is an infinite object. Instead, we set up a finite framework that will allow us to state and test a hypothesis that is a suitably modified version of the conjecture.

For any given n, consider the $(n-1)$-dimensional simplex $\Delta^{(n-1)}$ of all real n-vectors $\mathbf{x}$ with non-negative components x_i that add up to 1:

$$\Delta^{(n-1)} := \left\{ \mathbf{x} \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \ldots, n;\ \sum_{i=1}^{n} x_i = 1 \right\}. \tag{3}$$

We endow this set with a reasonable probability measure, thus making it into a probability space. On this space, we define an n-dimensional random variable $\mathbf{w} = (\overline{w}_1, \ldots, \overline{w}_n)$. For any fixed $q \in (0, 1)$, this gives us a random weighted voting game $\mathcal{W}$, with assembly $N = \{1, \ldots, n\}$, with $\mathbf{w}$ as its vector of relative weights and q as quota.

This random weighted voting game determines the corresponding vector of values of the Banzhaf index $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ and that of the Shapley–Shubik index $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$. These n-dimensional vectors are also random variables on $\Delta^{(n-1)}$. In what follows, '$\boldsymbol{\xi}$' stands for either $\boldsymbol{\beta}$ or $\boldsymbol{\phi}$.

The *discrepancy* (measured by a suitable metric) between the vector $\boldsymbol{\xi}$ and the relative weight vector $\mathbf{w}$ is then a scalar non-negative random variable. Our hypothesis concerns the asymptotic behaviour of the distribution of this random variable: it says that as n increases this distribution tends to become increasingly concentrated near 0. More precisely, both the expected value and standard deviation of this random variable approach 0 as n increases.[2]

We test this hypothesis by simulation, as follows. First, we fix some 'large' values of n, which will be the number of voters. (We allow n to get as large as feasibility of computation allows.) We also fix various values of the quota q, spaced at fairly close intervals.

Next, for each of our n, we select at random positive weights $w_1, \dots, w_n$. Replacing these by the corresponding relative weights, we get a vector $\mathbf{w} = (\overline{w}_1, \dots, \overline{w}_n)$ in $\Delta^{(n-1)}$.

This random selection is repeated a large number of times, so that for each of our n we obtain a large random sample of vectors $\mathbf{w} \in \Delta^{(n-1)}$.

For each randomly selected $\mathbf{w}$ and fixed q we compute the corresponding vector of values of the Banzhaf index $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ and that of the Shapley–Shubik index $\boldsymbol{\phi} = (\phi_1, \dots, \phi_n)$. These vectors $\boldsymbol{\xi}$ also belong to $\Delta^{(n-1)}$.

Next, we compute the discrepancy between the vector $\boldsymbol{\xi}$ and the relative weight vector $\mathbf{w}$.

Finally, for each of our n and q, we compute the mean and standard error of this discrepancy, over our large sample of $\mathbf{w} \in \Delta^{(n-1)}$.

If, for a given value of q, the mean and standard error of the discrepancy between $\boldsymbol{\xi}$ and $\mathbf{w}$ approach 0 as n increases, then this corroborates our hypothesis with respect to $\boldsymbol{\xi}$ for this value of q. If the mean discrepancy shows no tendency to approach 0 as n increases, this provides evidence against that hypothesis.

In Section 2 we fill in the details of the method outlined above. In particular, we specify our choice of 'reasonable' probability measure on $\Delta^{(n-1)}$ (see Subsection 2.1), and 'reasonable' metric for measuring the discrepancy between $\boldsymbol{\xi}$ and $\mathbf{w}$ (see Subsection 2.4).

In Section 3 we present and discuss the results of our simulation. We shall see that our hypothesis is corroborated with respect to β for $q = \frac{1}{2}$ (but not for other values of q); and with respect to ϕ for all $q \in (0, 1)$.

[2]This hypothesis is in fact equivalent to the conjecture made in [4]. The equivalence can be proved using the well-known theorem (due to Andersen and Jessen) about the probability measure on the product of an infinite sequence of probability spaces (see, for example, Halmos [5, p. 157]). However, we do not wish to press this point, as in our opinion the present hypothesis is of obvious interest per se.

We shall also point out some additional interesting features of the statistical behaviour of β and ϕ which our simulation reveals as a sort of by-product. In particular, in the case of β our simulation provides independent corroboration of a phenomenon observed by Życzkowski and Słomczyński [10].

2 Description of the method

2.1. Random selection of weights In fact, we use two different methods of random selection, corresponding to two probability measures on $\Delta^{(n-1)}$. The first method selects n positive integer weights w_i independently of one another, with a Poisson probability distribution, shifted so as to avoid 0 weights; thus, for each $i = 1, 2, \ldots, n$ we have

$$\text{Prob}\{w_i = k\} = \frac{e^{-1}}{(k-1)!}, \ k = 1, 2, \ldots. \tag{4}$$

Our second method selects the random vector $\mathbf{w}$ from an $(n-1)$-dimensional uniform distribution on $\Delta^{(n-1)}$. There are of course various ways for achieving this. We use the following method, which is very efficient computationally.[3] We select positive real weights w_i independently of one another, each with an exponential probability density

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The exponential distribution is a special case of the gamma distribution; and the fact that this probability density for the w_i yields the required uniform distribution for the normalized vector $\mathbf{w}$ follows from a property of the so-called Dirichlet composition. (For details, see [1, pp. 59f].)

Thus we have two separate series of samples. We shall refer to them as the *Poisson* and *uniform* samples, respectively.

In each of these two series, the size of our random sample of vectors $\mathbf{w} \in \Delta^{(n-1)}$ is 1,000 for every chosen value of n.

2.2. Choice of n As lowest value for n we took $n = 15$, because experience suggests that in cases where the asymptotic behaviour asserted by Penrose's Limit Theorem occurs, it begins to manifest itself at about this value.

The choice of highest value for n was dictated by computational feasibility. The time needed to compute the vectors of values $\boldsymbol{\beta}$ and $\boldsymbol{\phi}$ for given $\mathbf{w}$ and

[3] We are grateful to Friedrich Pukelsheim for suggesting this method to us.

q increases very steeply with n. We were able to go up to $n = 57$; for larger n the computation (using the means at our disposal) became impractically slow.

Fortunately, this range of values of n is sufficient for providing results from which quite firm conclusions can be drawn.

2.3. Choice of q Although the hypothesis we are testing concerns values of q in the open interval $(0, 1)$, we need only consider values of q in the half-open interval $[0.5, 1)$.

To see this, note that both β and ϕ are *self-dual* (see [3, p. 180]). Thus, let $\mathcal{W}$ be a weighted voting game with weights w_x $(x \in N)$ and quota q; and let $\mathcal{W}^*$ be the simple voting game with the same assembly, N, whose winning coalitions are those $A \subseteq N$ such that

$$\sum_{x \in A} \overline{w}_x > 1 - q. \tag{6}$$

($\mathcal{W}^*$ is called the *dual* of $\mathcal{W}$. It is easy to see that it is a weighted voting game with the same weights as $\mathcal{W}$ and quota $1 - q + \varepsilon$, for any sufficiently small positive ε.) The self-duality of β and ϕ implies that

$$\beta_x[\mathcal{W}^*] = \beta_x[\mathcal{W}] \text{ and } \phi_x[\mathcal{W}^*] = \phi_x[\mathcal{W}] \text{ for all } x \in N. \tag{7}$$

Although in our definition (2) of a weighted voting game with quota q we had a 'blunt' inequality ($\geq$), whereas here in (6) we have a sharp inequality ($>$), it is not difficult to see that this makes no difference to the asymptotic behaviour. More precisely: the asymptotic behaviour, with respect to both β and ϕ, of weighted voting games with randomly chosen weights is the same for quota $1 - q$ as for q.[4]

In our initial simulation we fixed q at the following values:

$$q = 0.50,\ 0.51,\ 0.55,\ 0.60,\ 0.65,\ 0.70,\ 0.75,\ 0.80,\ 0.85,\ 0.90,\ 0.95.$$

However, as explained in Subsection 3.1, the results of this initial simulation revealed an interesting phenomenon concerning the asymptotic behaviour of $\boldsymbol{\beta}$ at lower values of q, near $q = 0.50$. Also, as explained in Subsection 3.2, the simulation revealed a noteworthy phenomenon concerning the asymptotic behaviour of $\boldsymbol{\phi}$ at higher values of q, approaching $q = 1.00$.

In order to get a better view of these phenomena, we repeated the simulation with a finer subdivision, with intervals of 0.01:

$$q = 0.50,\ 0.51,\ 0.52,\ 0.53, \ldots, 0.98,\ 0.99. \tag{8}$$

[4] In this connection note that the results of Lindner and Machover [4] hold also – with virtually the same proofs – if in the definition (2) of weighted voting game $\geq$ is replaced by $>$.

2.4. Measuring the discrepancy For each $\mathbf{w}$ in our samples and each chosen value of q, we compute the vectors $\boldsymbol{\beta}$ and $\boldsymbol{\phi}$ of the values of the Banzhaf and the Shapley–Shubik indices, respectively.

Penrose's Limit Theorem with respect to $\boldsymbol{\xi}$ claims that asymptotically $\boldsymbol{\xi}$ approaches the vector of relative weights $\mathbf{w}$. We measure the 'discrepancy' of $\boldsymbol{\xi}$ compared to $\mathbf{w}$ in two ways.

First, we measure the *overall* discrepancy between $\boldsymbol{\xi}$ and $\mathbf{w}$ by the well-known index of distortion D, commonly attributed to Loosemore and Hanby:[5]

$$D[\boldsymbol{\xi}, \mathbf{w}] := \frac{1}{2} \sum_{i=1}^{n} |\xi_i - \overline{w}_i|. \tag{9}$$

Second, we measure the *local* (or *componentwise*) discrepancy between $\boldsymbol{\xi}$ and $\mathbf{w}$ by

$$d[\boldsymbol{\xi}, \mathbf{w}] := \max_{1 \le i \le n} \left| 1 - \frac{\xi_i}{\overline{w}_i} \right|. \tag{10}$$

Note that $\boldsymbol{\xi}$ is completely determined by $\mathbf{w}$ and q. Therefore, if we fix n and q, and regard $\mathbf{w}$ as a random variable vector, then $D[\boldsymbol{\xi}, \mathbf{w}]$ and $d[\boldsymbol{\xi}, \mathbf{w}]$ are random variable scalars, whose distributions depend on that of $\mathbf{w}$. We are interested in the expected value and standard deviation of these random variable scalars, as functions of n and q.

2.5. Output of computation In our simulation, we estimate the expected value and standard deviation of $D[\boldsymbol{\xi}, \mathbf{w}]$ and $d[\boldsymbol{\xi}, \mathbf{w}]$ by computing their mean and standard error for each of our samples. This yields the following outputs for all the chosen values of n and q :

$$\mu D(n, q), \quad \sigma D(n, q), \quad \mu d(n, q), \quad \sigma d(n, q).$$

Here 'μ' and 'σ' stand for mean and standard error, respectively.

More specifically, we have a set of four such outputs for each of the two indices and each of our sample series. Thus we have altogether:

$$\begin{array}{llll} \mu_P D(\beta; n, q), & \sigma_P D(\beta; n, q), & \mu_P d(\beta; n, q), & \sigma_P d(\beta; n, q), \\ \mu_U D(\beta; n, q), & \sigma_U D(\beta; n, q), & \mu_U d(\beta; n, q), & \sigma_U d(\beta; n, q), \\ \mu_P D(\phi; n, q), & \sigma_P D(\phi; n, q), & \mu_P d(\phi; n, q), & \sigma_P d(\phi; n, q), \\ \mu_U D(\phi; n, q), & \sigma_U D(\phi; n, q), & \mu_U d(\phi; n, q), & \sigma_U d(\phi; n, q). \end{array}$$

Here 'β' and 'ϕ' and the subscripts 'P' and 'U' are labels that refer to the Banzhaf and Shapley–Shubik indices, and the Poisson and uniform samples, respectively.

[5] See however discussion by Taagepera and Grofman [9] of the authorship of this index.

In [2] we present detailed tables as well as 3-D graphs of each of these sixteen statistics. These occupy too much space to be included in the present paper; here we only include, as representative illustration, two of the tables (Tables 1 and 2) and four 3-D graphs (Figures 1, 1a, 2 and 2a).[6] In each diagram, the values of the statistic in question are plotted along the vertical axis; n is plotted along the $\searrow$ axis and q along the $\nearrow$ axis.

In addition, we present in [2] four graphs concerned with the 'dip' discussed in Subsection 3.1; and four graphs concerned with the 'regime transition' discussed in Subsection 3.2. As representative illustration we include here Figures 3 and 4 for the dip and transition, respectively.

3 Results and conclusions

The output of our simulation shows a conspicuous difference between the behaviours of the two indices. We consider these indices in turn.

3.1. The Banzhaf index The data for the statistics labelled 'β' – see Table 1 and Figures 1, 1a (and cf. fuller data in [2]) – do not corroborate the hypothesis with respect to the Banzhaf index except for $q = 0.5$ and perhaps for values of q very close to 0.5. Note that Penrose's original claim concerned only $q = 0.5$. This claim, as we know, does not hold in all cases even for $q = 0.5$, but it does now appear to hold in *almost* all such cases.

Our negative findings with respect to β for $q > 0.5$ of course do not exclude the possibility that Penrose's Limit Theorem holds for large classes of weighted voting games with $q > 0.5$. Finding 'natural' and sufficiently interesting such classes is an open problem.

One feature of these data – which we had not anticipated – ought to be pointed out. For fixed values of n near the bottom of our range, the mean discrepancy between $\boldsymbol{\beta}$ and $\mathbf{w}$ has a dip – a minimum, indicating the closest mean fit between $\boldsymbol{\beta}$ and $\mathbf{w}$ – at a value of q considerably greater than 0.5. But as n increases the dip edges towards $q = 0.5$. This is shown in Figure 3.

The same general pattern applies to both the Poisson and the uniform samples, and to both measures of discrepancy. The differences are in minor details; for these, see [2].

Extrapolating from these data, it is reasonable to expect that for still greater values on n, beyond our range, the dip of all these four quantities

[6] The tables show the values of each statistic for all values of n from 15 to 57, but only for 20 selected values of q out of the 50 listed in (8) for which we performed the simulation. The selection is different for the two indices and is designed to focus on values of q which are of special interest.

– $\mu_P D(\beta; n, q)$, $\mu_P d(\beta; n, q)$, $\mu_U D(\beta; n, q)$ and $\mu_U d(\beta; n, q)$ – should occur at $q = 0.5$.

These observations provide an independent corroboration of results announced in an unpublished report by Życzkowski and Słomczyński [10, Section 10], which came to our attention after completing an earlier draft of this paper. These two authors (using much smaller samples and considerably fewer values of n) made essentially the same observations and drew the same conclusion regarding the behaviour of the dip (which they call 'critical point') in the mean discrepancy between $\boldsymbol{\beta}$ and $\mathbf{w}$.[7]

3.2. The Shapley–Shubik index The data for the statistics labelled 'ϕ' – see Table 2 and Figures 2, 2a (and cf. fuller data in [2]) – corroborate the hypothesis with respect to the Shapley–Shubik index for all $q \in (0, 1)$. For every chosen value of q, the mean discrepancy between $\boldsymbol{\phi}$ and $\mathbf{w}$ – whether measured by $\mu D(\boldsymbol{\phi}, \mathbf{w})$ or $\mu d(\boldsymbol{\phi}, \mathbf{w})$ – seems to approach 0 as n increases. It appears that the Penrose's Limit Theorem with respect to ϕ does hold almost always.

Let us look at the behaviour of the mean discrepancy between $\boldsymbol{\phi}$ and $\mathbf{w}$ as a function of q and n.

Clearly, for any fixed n, as q gets very close to 1, we would expect a weighted voting game with n voters to behave somewhat like a unanimity game, in which all voters have the same voting power, irrespective of their weights. Therefore it is reasonable to expect that as q approaches 1, the mean discrepancy between $\boldsymbol{\phi}$ and $\mathbf{w}$ should increase. Also, it is reasonable to expect that as q gets closer to 1, it would take greater values of n to overcome this 'unanimity effect'. In other words, the closer q is to 1, the slower the convergence to 0 of the mean discrepancy as n increases.

Our data show that this is indeed the case. However, we wish to point out an additional interesting phenomenon concerning the dependence on n of the the mean discrepancy between $\boldsymbol{\phi}$ and $\mathbf{w}$. For every fixed value of n in our range, this mean discrepancy tends to increase with q, albeit with some slight fluctuations; but the rate of increase is by no means uniform. For each n, as q increases from 0.5 towards 1, we can discern two regimes: at first the increase in the mean discrepancy is very gentle, barely noticeable, and may fluctuate slightly; then, rather abruptly, the rate of increase becomes quite steep. In other words, the transition from the 'Penrose's Limit Theorem effect' to the

[7] The focus of [10] is rather different from that of the present paper. Życzkowski and Słomczyński are not primarily interested in the asymptotic behaviour of $\boldsymbol{\beta}$ (they do not consider $\boldsymbol{\phi}$ at all) but in getting the closest fit between $\boldsymbol{\beta}$ and $\mathbf{w}$ for a decision rule designed for the Council of Ministers of the EU. They consider a close fit desirable in the interest of transparency.

'unanimity effect' is rather sharp. As may be expected, it seems that the greater the value of n, the higher is the value of q at which this transition takes place. This is shown in Figure 4.[8]

This general pattern applies to both the Poisson and the uniform samples, and to both measures of discrepancy. Here too, the differences are in minor details; for which see [2].

3.3. Caveats We would like to conclude by voicing two warnings against misinterpretation and misuse of our results.

First, it must be emphasized that these results are *purely statistical*: they concern the probability with which $\boldsymbol{\beta}$ or $\boldsymbol{\phi}$ tends to be close to $\mathbf{w}$ in a *randomly chosen* weighted voting game with many voters. Even where this probability is high, there are atypical counter-examples, in which the discrepancy between $\boldsymbol{\beta}$ or $\boldsymbol{\phi}$ and $\mathbf{w}$ is quite large. This must be borne in mind especially when dealing with a weighted voting game taken from real life rather than drawn out of a hat: there may be reasons making such a game quite atypical.

Second, our results provide additional evidence that – contrary to a still common misapprehension – the Banzhaf and Shapley–Shubik indices behave quite differently, and therefore have quite different meanings. It is therefore far from being a matter of indifference which one should be used: this must depend on what exactly one wishes to measure.[9] Moreover, this holds also for weighted voting games with quota of $\frac{1}{2}$, for which the two indices display similar statistical behaviour. In particular, it must be emphasized that – unlike the Shapley–Shubik index – the Banzhaf index is merely an arithmetical artefact, obtained by normalization from the absolute measure of voting power defined by Penrose [6, 7]. And it is the latter measure, not the Banzhaf index, which must be used when comparing the powers of a voter in two different voting games.

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[8] In this graph we pinpointed the transitional values of q using two criteria. First, the magnitude of the change at the selected transitional point must be at least 0.0010. Second, the mean discrepancy increases monotonically after the selected transitional point.

[9] For a detailed discussion, see [3, Comment 7.10.2 and *passim*].

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