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Shurojit Chatterji, Arunava Sen and Huaxia Zeng

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Shurojit Chatterji [*], Arunava Sen [†] and Huaxia Zeng [‡]

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Abstract

A domain of preference orderings is a random dictatorship domain if every strategy-proof random social choice function satisfying unanimity defined on the domain, is a random dictatorship. Gibbard (1977) showed that the universal domain is a random dictatorship domain. We investigate the relationship between dictatorial and random dictatorship domains. We show that there exist dictatorial domains that are not random dictatorship domains. We provide stronger versions of the linked domain condition (introduced in Aswal et al. (2003)) that guarantee that a domain is a random dictatorship domain. A key step in these arguments that is of independent interest, is a ramification result that shows that under certain assumptions, a domain that is a random dictatorship domain for two voters is also a random dictatorship domain for an arbitrary number of voters.

1 Introduction

It has long been understood that allowing for *randomization* significantly enlarges the set of incentive-compatible social choice functions. The reason for this is clear. Outcomes in a random social choice functions are *lotteries* and it is typically assumed that player preferences over lotteries satisfy *domain restrictions*. If they satisfy the von-Neumann-Morgenstern expected utility hypothesis, then convex combinations of deterministic incentive-compatible social choice functions are also incentive-compatible in the random environment. The converse question is more interesting and much harder: for what preference domains is it the case that every incentive-compatible randomized social choice function is a convex combination of incentive-compatible deterministic social choice functions? We address a specific version of this question in voting environments.

A voting environment is one where monetary compensation for players is not feasible. According to the classic Gibbard-Satterthwaite Theorem (Gibbard (1973), Satterthwaite (1975)) in this environment, the only dominant-strategy incentive-compatible

[*]Singapore Management University, Singapore.
[†]Indian Statistical Institute, New Delhi, India.
[‡]Singapore Management University, Singapore.

(or strategy-proof) social choice functions that satisfy the mild requirement of unanimity are *dictatorial*, provided that voters' preferences belong to the universal domain.[1] In a subsequent paper, Gibbard (1977) characterized the class of strategy-proof random social choice functions over the universal domain. An immediate consequence of this powerful result is that strategy-proof random social choice functions that satisfy unanimity are *random dictatorships*. A random dictatorship is a fixed probability distribution over dictatorial social choice functions. In other words, if social choice functions are assumed to satisfy unanimity, then the universal domain has the property that we alluded to in the earlier paragraph: every strategy-proof random social choice function is a convex combination of deterministic strategy-proof social choice functions. Related results have been proved for the domain of two alternatives (see Picot and Sen (2011)) and in some auctions settings (see Mehta and Vazirani (2004), Manelli and Vincent (2007)).[2]

In our terminology, the universal domain is both a dictatorial domain as well as a random dictatorship domain. In this paper, we investigate the relationship between these kinds of domains. Does the property of the universal domain generalize? i.e., is every dictatorial domain a random dictatorship domain? We show this is false by means of an example with seven alternatives and twenty-two preference orderings. We also identify general conditions under which dictatorial domains are not random dictatorship domains. We also specify two sets of conditions that ensure that a domain is a random dictatorship domain. These conditions are significantly stronger than those that force a domain to be dictatorial. The strength of these conditions suggest that randomization permits a much richer class of strategy-proof random social choice functions than the convex hull of deterministic strategy-proof social choice functions. We note here that this occurs despite the fact that the notion of random strategy-proofness that we use (following Gibbard (1977)) is very strong. In particular, it requires the expected utility from the truth-telling lottery to be greater than the expected utility from misrepresentation for *every* utility representation of true preferences. Equivalently, the lottery from truth-telling must first-order stochastically dominate every lottery arising from a misrepresentation.

Our results rely heavily on the approach in Aswal et al. (2003). They introduced the notion of a *linked domain* and showed that every linked domain is dictatorial. We construct an example of a linked domain that admits a non-dictatorial strategy-proof random social choice function that satisfies unanimity. We then show that random dictatorship is restored when the connectivity graph induced by a linked domain, is strengthened suitably. In particular, we assume the existence of a *hub* alternative that is connected with every other alternative. We also provide an alternative sufficient condition in a different way. We strengthen the connectedness requirement underlying the definition of a linked domain to obtain the notion of a strongly linked domain; we then impose an additional condition that is however weaker than the counterpart of

[1] There is also an additional requirement that the number of alternatives is at least three .

[2] Auction settings refer to models where monetary compensation is permissible and player utility functions are quasi-linear.

the hub condition.

The proofs of our sufficiency result rely on a ramification result that states that a random dictatorship domain when there are two voters is in fact, a random dictatorship domain when there is an arbitrary number of voters. This approach was initiated in Kalai and Muller (1977) in the context of domains that permit non-dictatorial Arrovian aggregation. Corresponding results for dictatorial domains appear in Kim and Roush (1989), Sen (2001) and Aswal et al. (2003). The result for random dictatorship is however, significantly more difficult than its dictatorial domains counterpart. In fact, we are able to prove it only under an additional hypothesis which is fortunately weak and satisfied by the sufficiency conditions. We believe that this result is of independent interest.

We now proceed to details.

2 Preliminaries

The model in the paper is completely standard (see Gibbard (1973), Aswal et al. (2003)) - we therefore introduce the required notation and definitions without comment.

We let $A = \{a_1, a_2, \ldots, a_m\}$ be a finite set of alternatives where $|A| = m$ and $m \geq 3$. The set $I = \{1, 2, \ldots, N\}$ is the set of voters with $|I| = N$ and $N \geq 2$. Each voter i has a (preference) ordering $P_i \in \mathbb{D}$ over the elements of A. The set $\mathbb{D}$ is referred to as the *preference domain*. It is assumed that $\mathbb{D} \subset \mathbb{P}$ where $\mathbb{P}$ is the set of all antisymmetric orderings over the elements of A. For any $a, b \in A$, aP_ib is interpreted as "a is strictly preferred to b according to P_i." We let $r_k(P_i)$ denote the k_{th} ranked alternative in P_i, $k = 1, \ldots, m$, i.e., $[r_k(P_i) = a] \Rightarrow [|\{b \in A : bP_ia\}| = k - 1]$. Let the map $b : A \to \{1, 2, \ldots, m\}$ denote a function such that $b(a_k) = k$, $k = 1, 2, \ldots, m$, and let $e_k \in \mathbb{R}^m$ denote a unit row vector, where the k_{th} element is 1 and the remaining elements are zeros.

A preference profile $P \in \mathbb{D}^N$ is an N-tuple $(P_1, \ldots, P_N)$. Finally, let $\mathcal{L}(A)$ denote the set of lotteries over the elements of the set A.

Definition 1 *A Random Social Choice Function (RSCF) is a map* $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$.

For every profile P, $\varphi(P) \equiv [\, \varphi_{a_1}(P) \;\; \varphi_{a_2}(P) \;\; \ldots \;\; \varphi_{a_m}(P) \,]$ is a probability vector.

We follow the notion of incentive-compatibility introduced in Gibbard (1977). A RSCF is *strategy-proof* if no voter can obtain a strictly higher expected utility by misreporting her preferences for any utility representation of her true preference and any beliefs regarding the reports of other voters.

Definition 2 *A utility function* $u : A \to \mathbb{R}$ *represents the ordering* P_i *over* A*, if for all* $a, b \in A$, $[\, a\, P_i\, b \,] \Leftrightarrow [\, u(a) > u(b) \,]$.

Let $\mathbb{U}(P_i)$ denote the set of utility functions that represent P_i, while $U_i(P) = \sum_{x \in A} u(x)\varphi_x(P)$ given any $u \in \mathbb{U}(P_i)$, denotes the von-Neumann-Morgenstern expected utility function of voter i under the profile P.

Definition 3 *A RSCF* $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ *is strategy-proof, if for all* $i \in I$*;* $P_i, P'_i \in \mathbb{D}$*,* $P_{-i} \in \mathbb{D}^{N-1}$ *and* $u \in \mathbb{U}(P_i)$*, we have* $U_i(P_i, P_{-i}) \geq U_i(P'_i, P_{-i})$*.*

The notion of strategy-proofness can be equivalently formulated in terms of stochastic dominance.

Definition 4 *A RSCF* $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ *is strategy-proof, if for all* $i \in I$*;* $P_i, P'_i \in \mathbb{D}$ *and* $P_{-i} \in \mathbb{D}^{N-1}$*, we have* $\sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i, P_{-i}) \geq \sum_{k=1}^{t} \varphi_{r_k(P_i)}(P'_i, P_{-i})$ *for all* $t = 1, 2, \ldots, m$*.*

If there exists a profile P, a voter i, a preference P'_i and $t \in \{1, \ldots, m-1\}$ such that $\sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i, P_{-i}) < \sum_{k=1}^{t} \varphi_{r_k(P_i)}(P'_i, P_{-i})$, we shall say that i manipulates φ at P via P'_i.

Throughout the paper, we will assume that RSCF's under consideration satisfy *unanimity.* This property requires an alternative to be selected with certainty if it is ranked first by all voters.

Definition 5 *A RSCF* $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ *satisfies unanimity if for all* $P \in \mathbb{D}^N$ *and* $a_j \in A$*,* $[a_j = r_1(P_i)$ *for all* $i \in I] \Rightarrow [\varphi_{a_j}(P) = 1]$*.*

A RSCF of particular importance is *random dictatorship.*

Definition 6 *The RSCF* $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ *is a random dictatorship if there exists* $\beta_i \in [0, 1]$*,* $i \in I$ *with* $\sum_{i=1}^{N} \beta_i = 1$ *such that for all* $P \in \mathbb{D}^N$*,* $\varphi(P) = \sum_{i=1}^{N} \beta_i \, e_{b(r_1(P_i))}$*.*

The focus of our paper are random dictatorship domains which have the property that every strategy-proof and unanimous RSCF defined on them, is a random dictatorship.

Definition 7 *A domain* $\mathbb{D}$ *is a random dictatorship domain, if every unanimous and strategy-proof RSCF* $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ *is a random dictatorship.*

A fundamental result in random mechanism design theory proved in Gibbard (1977) is that the domain $\mathbb{P}$ is a a random dictatorship domain (see also Duggan (1996), Sen (2011)).

Theorem 1 *The domain* $\mathbb{P}$ *is a random dictatorship domain.*

The notions of a RSCF, strategy-proofness and random dictatorship have familiar deterministic counterparts. Thus a deterministic RSCF or simply a social choice function (SCF) is a RSCF whose output at every profile is a degenerate probability distribution. Similarly, dictatorship is a special case of random dictatorship where exactly one of the coefficients β_i in Definition 6 is one and all others are zero. A dictatorial domain is similarly a domain where every strategy-proof and unanimous social choice function is dictatorial. According to the Gibbard-Satterthwaite Theorem (Gibbard (1973), Satterthwaite (1975)), the domain $\mathbb{P}$ is a dictatorial domain.

A central concern of this paper is the relationship between dictatorial and random dictatorship domains. It is easily verified that a random dictatorship domain is dictatorial. The question of interest is clearly the converse question. As we have remarked, the universal domain is both a dictatorial and a random dictatorship domain. Does this relationship hold true generally? In order to investigate this question, we recall the main result of Aswal et al. (2003) on dictatorial domains.

Some additional notation will be helpful here. For every non-empty $B \subset A$, we let $\mathbb{D}^B = \{P_i \in \mathbb{D} : r_1(P_i) \in B\}$. For a mutually disjoint non-empty pair $B, C \subset A$, $\mathbb{D}^{B,C} = \{P_i \in \mathbb{D} : r_1(P_i) \in B \text{ and } r_2(P_i) \in C\}$. If B and C are singletons with $B = \{b\}$ and $C = \{c\}$, we write $\mathbb{D}^B$ as $\mathbb{D}^b$ and $\mathbb{D}^{B,C}$ as $\mathbb{D}^{b,c}$.

Definition 8 *Let $\mathbb{D}$ be a domain. A pair of alternatives a, b is connected (denoted by $a \sim b$) if $\mathbb{D}^{a,b} \neq \emptyset$ and $\mathbb{D}^{b,a} \neq \emptyset$.*

A domain induces a *connectivity graph* as follows: the set of nodes of the graph is the set of alternatives and two nodes a and b have an edge connecting them iff $a \sim b$.

Definition 9 *Let $\mathbb{D}$ be a domain. Let $B \subset A$ and $a \notin B$. Then, a is linked to B if there exist $b, c \in B$ such that $a \sim b$ and $a \sim c$.*

Definition 10 *The domain $\mathbb{D}$ is linked, if there exists an one to one function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that*

(i) $a_{\sigma(1)} \sim a_{\sigma(2)}$.

(ii) $a_{\sigma(j)}$ *is linked to* $\{a_{\sigma(1)}, \ldots a_{\sigma(j-1)}\}$, $j = 3, \ldots, m$.

The notion of a linked domain is formulated entirely in terms of alternatives that can be ranked first and second according to preferences in the domain. A domain is linked if its associated connectivity graph is rich enough. The reader is referred to Aswal et al. (2003) for details and numerous examples. The following result is their main result.

Theorem 2 *A linked domain is dictatorial.*

A natural question is whether a linked domain is a random dictatorship domain. This is addressed in the next section.

3 Examples

3.1 A Specific Case

In this section, we provide an example of a dictatorial domain that is not a random dictatorship domain.

Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$. The domain $\mathbb{D}_L$ of preferences over these seven alternatives is described in Appendix A. The following features of the domain are critical and can be readily verified.

1. Domain $\mathbb{D}_L$ is linked. Its connectivity graph is shown in Figure 1. Note that $|\mathbb{D}_L| = 22$. Since Aswal et al. (2003) have demonstrated that the minimal cardinality of a linked domain is $4m-6$, domain $\mathbb{D}_L$ is in fact a linked domain of minimal size.
2. $[P_k \in \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}_L^{a_3,a_1}] \Rightarrow [a_2 = r_m(P_k)]$. Whenever a_1 and a_3 are ranked first and second or vice-versa, a_2 is ranked last.
3. $[P_k \notin \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}_L^{a_3,a_1}] \Rightarrow [a_2 \in \{r_1(P_k), r_2(P_k), r_3(P_k)\}]$. If Condition 2 above does not apply, a_2 is ranked either first, second or third.

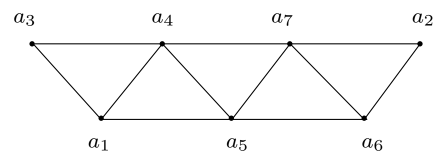


Figure 1: Connectivity Graph of Domain $\mathbb{D}_L$

According to Theorem 2, $\mathbb{D}_L$ is a dictatorial domain. However, as Proposition 1 below shows, it is not a random dictatorship domain.

Proposition 1 *The domain $\mathbb{D}_L$ is not a random dictatorship domain.*

Proof: It suffices to construct a unanimous, strategy-proof and non-dictatorial RSCF $\varphi : \mathbb{D}_L^2 \to \mathcal{L}(A)$.[3] Let $I = \{i, j\}$ and consider the RSCF φ below:

$$\varphi(P_i, P_j) = \begin{cases} \varepsilon\, e_{b(r_1(P_i))} + \alpha\, e_{b(r_2(P_j))} + (1-\varepsilon-\alpha)e_2 & \text{if } P_i \in \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}_L^{a_3,a_1} \text{ and } P_j \in \mathbb{D}_L^{a_2} \\ \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))} & \text{otherwise.} \end{cases}$$

where $0 < \varepsilon < 1$ and $0 < \alpha \leq \min(\varepsilon, 1-\varepsilon)$.

Thus, φ is a random dictatorship with weights ε and $1-\varepsilon$ on the best alternatives of voters i and j at all profiles *except* when i's first and second ranked alternatives are a_1 and a_3 or vice-versa and j's best alternative is a_2. In this case, probability weight α is transferred from a_2 to j's second ranked alternative. It is evident that φ is unanimous and not a random dictatorship. We only need to check that φ is strategy-proof. In order to do this, it suffices to consider only the seven cases below.

Case 1: The profile is (P_i, P_j) where $P_j \in \mathbb{D}_L^{a_2}$ and $P_i \notin \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}_L^{a_3,a_1}$. Voter i considers a manipulation via $P_i' \in \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}_L^{a_3,a_1}$.

Let $u \in \mathbb{U}(P_i)$. The loss from misrepresentation is $U_i(P_i, P_j) - U_i(P_i', P_j) = \varepsilon\, u(r_1(P_i)) + \alpha u(a_2) - \varepsilon\, u(r_1(P_i')) - \alpha u(r_2(P_j))$. If $a_2\, P_i\, r_2(P_j)$, then $U_i(P_i, P_j) - U_i(P_i', P_j) \geq 0$. Suppose $r_2(P_j)\, P_i\, a_2$. Note that $r_2(P_j) \in \{a_6, a_7\}$ (refer to P_4 and P_5 in Appendix A). Whenever $a_6 P_i a_2$ or $a_7 P_i a_2$, we have $a_2 P_i a_1$ and $a_2 P_i a_3$, i.e., $a_2 P_i r_1(P_i')$. Hence, $U_i(P_i, P_j) - U_i(P_i', P_j) = \alpha\Big[u(r_1(P_i)) - u(r_2(P_j))\Big] + (\varepsilon - \alpha)\Big[u(r_1(P_i)) - u(r_1(P_i'))\Big] + \alpha\Big[u(a_2) - u(r_1(P_i'))\Big] \geq 0$.

[3] In case there are more than two voters, the additional voters can be made dummies whose preferences have no bearing on the outcome.

Case 2: The profile is (P_i, P_j) where $P_j \in \mathbb{D}_L^{a_2}$ and $P_i \in \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}_L^{a_3,a_1}$. Voter i considers a manipulation via $P'_i \notin \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}_L^{a_3,a_1}$.

Let $u \in \mathbb{U}(P_i)$. Since $P_i \in \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}_L^{a_3,a_1}$, $r_2(P_j) P_i a_2$. Hence the loss from misrepresentation is $U_i(P_i, P_j) - U_i(P'_i, P_j) = \varepsilon\Big[u(r_1(P_i)) - u(r_1(P'_i))\Big] + \alpha\Big[u(r_2(P_j)) - u(a_2)\Big] \geq 0$

Case 3: The profile is (P_i, P_j) where $P_j \in \mathbb{D}_L^{a_2}$ and $P_i \in \mathbb{D}_L^{a_1,a_3}$. Voter i considers a manipulation via P'_i where $P'_i \in \mathbb{D}_L^{a_3,a_1}$.

Let $u \in \mathbb{U}(P_i)$. The loss from misrepresentation is $U_i(P_i, P_j) - U_i(P'_i, P_j) = \varepsilon\Big[u(a_1) - u(a_3)\Big] \geq 0$.

Case 4: The profile is (P_i, P_j), where $P_j \in \mathbb{D}_L^{a_2}$ and $P_i \in \mathbb{D}_L^{a_3,a_1}$. Voter i considers a manipulation via P'_i where $P'_i \in \mathbb{D}_L^{a_1,a_3}$.

Let $u \in \mathbb{U}(P_i)$. The loss from misrepresentation is $U_i(P_i, P_j) - U_i(P'_i, P_j) = \varepsilon\Big[u(a_3) - u(a_1)\Big] \geq 0$.

Case 5: The profile is (P_i, P_j) where $P_i \in \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}_L^{a_3,a_1}$ and $P_j \notin \mathbb{D}_L^{a_2}$. Voter j considers a manipulation via P'_j where $P'_j \in \mathbb{D}_L^{a_2}$.

Let $u \in \mathbb{U}(P_j)$. The loss from misrepresentation is $U_j(P_i, P_j) - U_j(P_i, P'_j) = (1 - \varepsilon - \alpha)\Big[u(r_1(P_j)) - u(a_2)\Big] + \alpha\Big[u(r_1(P_j)) - u(r_2(P'_j))\Big] \geq 0$.

Case 6: The profile is (P_i, P_j) where $P_i \in \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}^{a_3,a_1}$ and $P_j \in \mathbb{D}_L^{a_2}$. Voter j considers a manipulation via P'_j where $P'_j \notin \mathbb{D}_L^{a_2}$.

Let $u \in \mathbb{U}(P_j)$. The loss from misrepresentation is $U_j(P_i, P_j) - U_j(P_i, P'_j) = (1 - \varepsilon - \alpha)\Big[u(a_2) - u(r_1(P'_j))\Big] + \alpha\Big[u(r_2(P_j)) - u(r_1(P'_j))\Big] \geq 0$.

Case 7: The profile is (P_i, P_j) where $P_i \in \mathbb{D}_L^{a_1,a_3} \cup \mathbb{D}^{a_3,a_1}$ and $P_j \in \mathbb{D}_L^{a_2}$. Voter j considers a manipulation via P'_j where $P'_j \in \mathbb{D}_L^{a_2}$ and $P'_j \neq P_j$.

Let $u \in \mathbb{U}(P_j)$. The loss from misrepresentation is $U_j(P_i, P_j) - U_j(P_i, P'_j) = \alpha\Big[u(r_2(P_j)) - u(r_2(P'_j))\Big] \geq 0$.

We conclude that φ is strategy-proof. ■

3.2 A General Case

The example we provided in Section 3.1 is not the simplest case of a linked domain that admits strategy-proof, unanimous and non-dictatorial RSCF's, since our objective was to use one example to cover the two domains that follow in Sections 5.1 and 5.2. Here, we provide simpler but more general restrictions that generate linked domains that are not randomly dictatorial. We do not restrict the cardinality of either the set of alternatives or the domains.

Given $a \in A$, let $\mathcal{S}(a) = \{x \in A : \text{ there exists } P_k \in \mathbb{D}^a, \text{ such that } x = r_2(P_k)\}$. It is evident that $a \notin \mathcal{S}(a)$. We consider a minimally rich domain $\mathbb{D}_{NRD}$ defined below.

Definition 11 *A minimally rich domain satisfies the Condition NRD, denoted* $\mathbb{D}_{NRD}$*, if there exist* $x, y \in A$ *such that*

(i) $y \notin \mathcal{S}(x)$*,* $x \notin \mathcal{S}(y)$ *and* $\mathcal{S}(x) \cap \mathcal{S}(y) = \emptyset$*.*

(ii) *For all* $P_k \in \mathbb{D}^x_{NRD}$ *and* $z \in \mathcal{S}(y)$*,* $zP_k y$*.*

(iii) *For all* $P_k \in \mathbb{D}^{A\backslash\{x,y\}}_{NRD}$*,* [*there exists* $z \in \mathcal{S}(y)$ *such that* $zP_k y$] $\Rightarrow$ $[yP_k x]$*.*

Remark 1 The domain $\mathbb{D}_{NRD}$ must have at least 4 alternatives.

Proposition 2 *The domain* $\mathbb{D}_{NRD}$ *is not a random dictatorship domain.*

The proof of the Proposition is similar to the proof of Proposition 1 and is relegated to Appendix B.

Next, we investigate the compatibility of the restrictions needed in Condition NRD with linked domains. We first restate the definition of linked domains in terms of a graph. Let G be a graph, the set of whose nodes is A. A path in G is a sequence $\{a_{\gamma(k)}\}_{k=1}^T$ such that every pair $(a_{\gamma(k-1)}, a_{\gamma(k)})$, $k = 1, \ldots, T$, is an edge. Graph G is connected if for all $a_j, a_k \in A$, there exists a path connecting a_j and a_k.

Definition 12 *The connected graph G is linked, if there exists an one to one function* $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ *such that*

(i) $(a_{\sigma(1)}, a_{\sigma(2)})$ *is an edge.*

(ii) *For all* $j = 3, \ldots, m$*, there exist* $a, b \in \{a_{\sigma(1)}, \ldots, a_{\sigma(j-1)}\}$ *such that* $(a, a_{\sigma(j)})$ *and* $(b, a_{\sigma(j)})$ *are both edges.*

Fix a domain $\mathbb{D}$. Let $G(\mathbb{D})$ denote the connectivity graph of domain $\mathbb{D}$. A domain $\mathbb{D}$ is linked iff $G(\mathbb{D})$ is linked. The following proposition provides conditions on a linked graph G such that there exists a linked domain $\mathbb{D}_L$ with $G(\mathbb{D}_L) = G$ and furthermore $\mathbb{D}_L$ satisfies the Condition NRD.

Proposition 3 *Given a linked graph* G*, if there exist* $x, y \in A$ *such that*

(i) (x, y) *is not an edge,*

(ii) *for all* $z \in A$ *where* (x, z) *is an edge,* (y, z) *is not an edge; and for all* $z' \in A$ *where* (y, z') *is an edge,* (x, z') *is not an edge,*

then there exists a linked domain $\mathbb{D}_L$ *with* $G(\mathbb{D}_L) = G$ *such that* $\mathbb{D}_L$ *satisfies the Condition NRD.*

Proof: We construct $\mathbb{D}_L$ by following three steps:

Step 1: For all $a, b \in A$, if (a, b) is an edge, generate two preference orderings P_k, P'_k such that (i) $r_1(P_k) = a = r_2(P'_k)$, (ii) $r_2(P_k) = b = r_1(P'_k)$ and (iii) all relative rankings among $A\backslash\{a, b\}$ in both P_k and P'_k are unrestricted.

Step 2: For all *incomplete* preference ordering P_k generated in Step 1, such that $r_1(P_k) = x$, let $y = r_m(P_k)$. Next, let the relative rankings between $r_2(P_k)$ and $r_m(P_k)$ be arbitrary.

Step 3: For all *incomplete* preference ordering P_k generated in Step 1, such that $r_1(P_k) \in A\backslash\{x\}$, if $y \in \{r_1(P_k), r_2(P_k)\}$, let the relative rankings beyond $r_2(P_k)$ be arbitrary; if $y \notin \{r_1(P_k), r_2(P_k)\}$, let $y = r_3(P_k)$ and let the relative rankings beyond $r_3(P_k)$ be arbitrary .

Step 1 specifies every preference ordering's top two alternatives which indicates that for all $a, b \in A$, if (a, b) is an edge, then $a \sim b$ in domain $\mathbb{D}_L$; and if (a, b) is not an edge, then $\mathbb{D}_L^{a,b} = \emptyset$ and $\mathbb{D}_L^{b,a} = \emptyset$. Hence, by Definition 12, domain $\mathbb{D}_L$ is linked. It is evident that the restrictions on graph G imply that $\mathbb{D}_L$ satisfies the first restriction of the Condition NRD. Next, since (x, y) is not an edge, Step 2 does not contradict Step 1. Since y is ranked last in every preference ordering P_k with $r_1(P_k) = x$, domain $\mathbb{D}_L$ meets the second restriction of the Condition NRD.

It is evident that Step 3 does not contradict Steps 1 and 2. We next check whether domain $\mathbb{D}_L$ satisfies the third restriction of the Condition NRD. According to Step 1, we know that for all $a \in A$, whenever (a, y) is not an edge, $\mathbb{D}_L^{y,a} = \emptyset$ and hence $a \notin \mathcal{S}(y)$. Thus, to show that the third restriction of the Condition NRD is not violated by domain $\mathbb{D}_L$, it suffices to show that for every preference ordering $P_k \in \mathbb{D}_L^{A\backslash\{x,y\}}$ with xP_ky, there exists no alternative $a \in A$ such that (a, y) is an edge and aP_ky. Since $r_1(P_k) \neq x$ and $y \in \{r_1(P_k), r_2(P_k), r_3(P_k)\}$ by Step 3, xP_ky implies that $x = r_2(P_k)$. Then, according to Step 1, we know that $(r_1(P_k), x)$ is an edge. Furthermore, by the second restriction on G, $(r_1(P_k), y)$ is never an edge. In conclusion, domain $\mathbb{D}_L$ satisfies the third restriction of the Condition NRD. ■

Remark 2 To construct a linked domain satisfying the Condition NRD, we need at least 6 alternatives.[4] Furthermore, linked domains satisfying the Condition NRD need not be the minimal size linked domains.

In the remainder of the paper, we will provide sufficient conditions for a domain to be a random dictatorship domain.

4 Ramification from two to an arbitrary number of voters

Our first step is to show that a random dictatorship domain when there are exactly two voters, is also a random dictatorship domain when there are more than two voters, provided an additional condition is satisfied. A result of this kind was first established in Kalai and Muller (1977) which showed that a domain where all Arrovian social welfare functions are dictatorial when there are two voters also admits only dictatorial Arrovian

[4]It is possible to construct a linked domain with 5 alternatives which satisfies the connectivity graph Figure 2 (e) and is not a random dictatorship domain.

social welfare functions when there are more than two voters. A similar property has been shown for deterministic strategy-proof SCFs (Kim and Roush (1989), see also Sen (2001), Aswal et al. (2003), Chatterji and Sen (2011)). In particular, a domain where all strategy-proof SCFs satisfying unanimity are dictatorial when there are two voters, is also a domain where all strategy-proof SCFs satisfying unanimity are dictatorial for an arbitrary number of voters. A property of this kind is, of course, interesting in its own right. In addition, it is very helpful analytically; in order to determine whether a domain is dictatorial, it suffices to verify that every two-voter strategy-proof SCF satisfying unanimity defined on it is dictatorial.

In this section we provide a result relating the two-voter with the many-voter case for RSCF's. Unfortunately, this appears to be a significantly more difficult question to resolve than the corresponding one in the deterministic case. We are able to prove it only by making an additional assumption.

Definition 13 *The domain $\mathbb{D}$ satisfies the triple-property if there exist $a, b, c \in A$ such that $a \sim b$, $b \sim c$ and $c \sim a$.*

A domain satisfies the triple property if its connectivity graph has a triangle subgraph. We also need an assumption that is very mild and is standard in the literature.

Definition 14 *The domain $\mathbb{D}$ is minimally rich if $\mathbb{D}^a \neq \emptyset$ for all $a \in A$.*

A minimally rich domain has the property that every alternative is ranked first by some preference ordering in the domain. This assumption excludes some trivial cases from consideration.

Theorem 3 *Let $\mathbb{D}$ be a minimally rich domain, satisfying the triple property. The following two statements are equivalent:*

(*a*) $\varphi : \mathbb{D}^2 \to \mathcal{L}(A)$ *is unanimous and strategy-proof* $\Rightarrow$ φ *is a random dictatorship.*

(*b*) $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$, $N \geq 2$, *is unanimous and strategy-proof* $\Rightarrow$ φ *is a random dictatorship.*

The proof of the result is in Appendix C.

Remark 3 In Appendix C, we prove a stronger version of Theorem 3 by using a weaker version of the triple condition. The triple condition is a fairly weak condition satisfied by all linked domains. We can therefore use Theorem 3 to prove our random dictatorship results. The circular domain (Sato (2010)) is a dictatorial domain that does not satisfy the triple condition; however, it satisfies the weaker condition in Appendix C that we use to prove the result. A positive feature of our additional assumption therefore is that the triple condition or its weaker counterpart are often required to prove dictatorship or random dictatorship even in the two voter case. Domains that violate the weaker assumption such as the single-peaked domain (see Demange (1982), Danilov (1994)) are not dictatorial for any number of voters.

5 Random Dictatorship Results

In this section we provide two conditions that ensure that a domain is a random dictatorship domain. The first imposes a great degree of connectivity in the linked domain connectivity graph. The second strengthens the requirement for the connectedness of two alternatives but imposes a weaker requirement on the connectivity graph.

5.1 Linked Domains with Condition H

We impose the following condition on the connectedness structure of domains.

Definition 15 *A domain $\mathbb{D}$ satisfies Condition H if there exists $a \in A$ such that $b \sim a$, for all $b \in A \backslash \{a\}$.*

The alternative a that is connected to all other alternatives will be referred to as a *hub*. It is clear that both domain $\mathbb{D}_L$ (Figure 1 and Appendix A) in Section 3.1 and domain $\mathbb{D}_{NRD}$ in Section 3.2 violate Condition H. We provide examples of six connectivity graphs below to illustrate the relation between Condition H and linked domains.

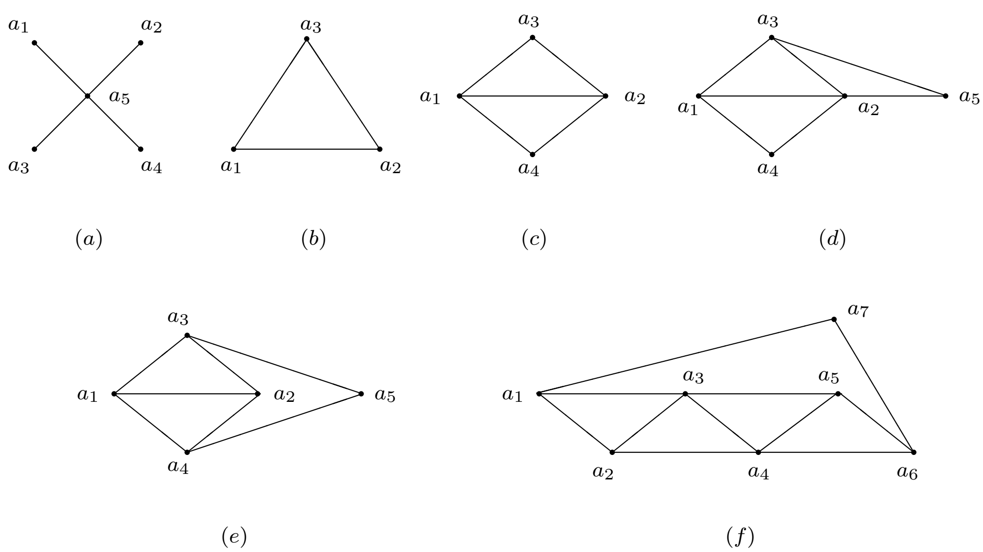


Figure 2: Connectivity Graphs

In diagram (a), the domain is not linked but satisfies Condition H. The domains whose connectivity graphs are shown in diagrams (b), (c), (d), (e) and (f), are linked. The domains corresponding to (b), (c) and (d) satisfy Condition H, while domains related to (e) and (f) violate it. In diagram (b), any alternative can be a hub; in diagram (c) it must be either a_1 or a_2, while in diagram (d) the only candidate for the hub is a_2. Observe that in diagrams (e) and (f) for any two alternatives, they are either connected or connected to a common alternative.

Our main result in this subsection is that the assumption of a linked domain in conjunction with Condition H ensures that the domain is a random dictatorship domain.

Theorem 4 *A linked domain satisfying Condition H is a random dictatorship domain.*

Proof: Let $\mathbb{D}$ be a linked domain satisfying Condition H. In particular, let a_k be a hub. Since a linked domain satisfies the triple property, Theorem 3 applies. In order to prove the theorem, it suffices therefore to show that every strategy-proof RSCF $\varphi : \mathbb{D}^2 \to \mathcal{L}(A)$ satisfying unanimity, is a random dictatorship. Assume without loss of generality that $I = \{i, j\}$.

Lemma 1 *There exists a function $\sigma : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$ for domain $\mathbb{D}$ as specified in Definition 10 with $a_k = a_{\sigma(1)}$.*

Proof: Suppose not, i.e., for every function $\hat{\sigma}$ satisfying Definition 10, we have $a_k \neq a_{\hat{\sigma}(1)}$. Suppose there exists σ satisfying Definition 10 with $a_k \in \{a_{\sigma(2)}, a_{\sigma(3)}\}$. Assume without loss of generality that $a_k = a_{\sigma(2)}$. Now permute elements in A such that $a_{\sigma(2)}$ is re-labeled $a_{\sigma(1)}$, while $a_{\sigma(1)}$ is re-labeled as $a_{\sigma(2)}$, while keeping all other labels intact. This creates another function σ' satisfying Definition 10 but $a_k = a_{\sigma'(1)}$. This contradicts our earlier assumption.

Now, suppose $\hat{\sigma}$ satisfying Definition 10 is such that $a_k = a_{\hat{\sigma}(k^*)}$ with $k^* > 3$. Define a new function: $\sigma(1) = \hat{\sigma}(k^*)$, $\sigma(s) = \hat{\sigma}(s-1)$, for all $2 \leq s \leq k^*$ and $\sigma(t) = \hat{\sigma}(t)$ for all $k^* + 1 \leq t \leq m$. According to Condition H, $a_{\sigma(2)} = a_{\hat{\sigma}(1)} \sim a_{\hat{\sigma}(k^*)} = a_{\sigma(1)}$ and $a_{\sigma(3)} = a_{\hat{\sigma}(2)} \sim a_{\hat{\sigma}(k^*)} = a_{\sigma(1)}$. Meanwhile, by $\hat{\sigma}$, $a_{\sigma(3)} = a_{\hat{\sigma}(2)} \sim a_{\hat{\sigma}(1)} = a_{\sigma(2)}$. Thus we have $a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}$ form a triple of connectedness. Next, according to $\hat{\sigma}$, we know that for all $3 \leq j \leq k^* - 1$, $a_{\hat{\sigma}(j)}$ is linked to $\{a_{\hat{\sigma}(1)}, \ldots, a_{\hat{\sigma}(j-1)}\}$. Meanwhile, since $a_{\hat{\sigma}(j)} = a_{\sigma(j+1)}$ and $\{a_{\hat{\sigma}(1)}, \ldots, a_{\hat{\sigma}(j-1)}\} = \{a_{\sigma(2)}, \ldots, a_{\sigma(j)}\} \subset \{a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(j)}\}$, we know that $a_{\sigma(j+1)}$ is linked to $\{a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(j)}\}$. Similarly, for all $k^* \leq j \leq m$, $a_{\hat{\sigma}(j+1)} = a_{\sigma(j+1)}$, $a_{\hat{\sigma}(j+1)}$ is linked to $\{a_{\hat{\sigma}(1)}, \ldots, a_{\hat{\sigma}(j)}\}$ and $\{a_{\hat{\sigma}(1)}, \ldots, a_{\hat{\sigma}(j)}\} = \{a_{\hat{\sigma}(1)}, \ldots, a_{\hat{\sigma}(k^*-1)}, a_{\hat{\sigma}(k^*)}, a_{\hat{\sigma}(k^*+1)}, \ldots, a_{\hat{\sigma}(j)}\} = \{a_{\sigma(2)}, \ldots, a_{\sigma(k^*)}, a_{\sigma(1)}, a_{\sigma(k^*+1)}, \ldots, a_{\sigma(j)}\} = \{a_{\sigma(1)}, \ldots, a_{\sigma(j)}\}$. Therefore, $a_{\sigma(j+1)}$ is linked to $\{a_{\sigma(1)}, \ldots, a_{\sigma(j)}\}$. Hence σ satisfies Definition 10, contradicting our initial assumption once again. ■

Assume for simplicity that the function in Definition 10 is the identity function. It follows from Lemma 1 that the hub is a_1. Define $S_l = \{a_1, \ldots, a_l\}$, $l = 3, \ldots, m$. Clearly, $a_1 \in S_l$ for all l. Our proof consists in establishing two steps.

Step 1. There exists $\varepsilon \in [0, 1]$ such that for all $P_i, P_j \in \mathbb{D}^{S_3}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$.

Step 2. If for all $P_i, P_j \in \mathbb{D}^{S_{l-1}}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$, then for all $P_i, P_j \in \mathbb{D}^{S_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$.

The following lemma establishes Step 1.

Lemma 2 *There exists $\varepsilon \in [0, 1]$ such that for all $P_i, P_j \in \mathbb{D}^{S_3}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$.*

Proof: Since a_1, a_2, a_3 form a triple of connectedness, the lemma follows from Theorem 2 in Sen (2011). ■

To verify Step 2, we use the following induction hypothesis: for all $P_i, P_j \in \mathbb{D}^{S_{l-1}}$, $l > 3$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$. We will show that for all $P_i, P_j \in \mathbb{D}^{S_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$. Since $\mathbb{D}$ is linked and a_1 is a hub, we know that there exists $a_k \in S_{l-1}$ such that $a_l \sim a_k$, $a_k \sim a_1$ and $a_l \sim a_1$, forming a triple of connectedness. The next 3 lemmas explain the verification of Step 2.

Lemma 3 *For all* $P_i, P_j \in \mathbb{D}^{\{a_1,a_k,a_l\}}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$.

Proof: Since a_l, a_k, a_1 form a triple of connectedness, applying Theorem 2 in Sen (2011), we infer that there exists $\beta \in [0,1]$ such that $\varphi(P_i, P_j) = \beta\, e_{b(r_1(P_i))} + (1-\beta)e_{b(r_1(P_j))}$ for all $P_i, P_j \in \mathbb{D}^{\{a_1,a_k,a_l\}}$.

By the induction hypothesis, we know that $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$ for all $P_i, P_j \in \mathbb{D}^{\{a_1,a_k\}}$. Therefore, $\varepsilon = \beta$. ■

For the next lemma, pick any $a_j \in S_{l-1}\backslash\{a_1, a_k\}$. Since $a_l \sim a_1$ and $a_j \sim a_1$ (by Condition H), we have $P_i^* \in \mathbb{D}^{a_l,a_1}$ and $P_j^* \in \mathbb{D}^{a_j,a_1}$.

Lemma 4 $\varphi_{a_l}(P_i^*, P_j^*) = \varepsilon$ *and* $\varphi_{a_j}(P_i^*, P_j^*) = 1-\varepsilon$.

Proof: We consider two cases.

Firstly, suppose that $\varphi_{a_l}(P_i^*, P_j^*) = \beta$ and $\varphi_{a_j}(P_i^*, P_j^*) = 1-\beta$. Since there exists $P_i' \in \mathbb{D}^{a_1,a_l}$ (recall $a_1 \sim a_l$), strategy-proofness and the induction hypothesis imply that $\beta = \varphi_{a_l}(P_i^*, P_j^*) = \varphi_{a_l}(P_i^*, P_j^*) + \varphi_{a_1}(P_i^*, P_j^*) = \varphi_{a_l}(P_i', P_j^*) + \varphi_{a_1}(P_i', P_j^*) = \varphi_{a_1}(P_i', P_j^*) = \varepsilon$.

Second, suppose that $\varphi_{a_l}(P_i^*, P_j^*) + \varphi_{a_j}(P_i^*, P_j^*) < 1$. Since there exist $P_i' \in \mathbb{D}^{a_1,a_l}$ and $P_j' \in \mathbb{D}^{a_1,a_j}$ (by Condition H), strategy-proofness, the induction hypothesis and Lemma 3 imply that $\varphi_{a_l}(P_i^*, P_j^*) + \varphi_{a_1}(P_i^*, P_j^*) = \varphi_{a_l}(P_i', P_j^*) + \varphi_{a_1}(P_i', P_j^*) = \varepsilon$ and $\varphi_{a_j}(P_i^*, P_j^*) + \varphi_{a_1}(P_i^*, P_j^*) = \varphi_{a_j}(P_i^*, P_j') + \varphi_{a_1}(P_i^*, P_j') = 1-\varepsilon$. Therefore, it must be the case that $\varphi_{a_1}(P_i^*, P_j^*) > 0$. Assume that $\varphi_{a_1}(P_i^*, P_j^*) = \alpha > 0$. Then, $\varphi_{a_l}(P_i^*, P_j^*) = \varepsilon - \alpha$, $\varphi_{a_j}(P_i^*, P_j^*) = 1 - \varepsilon - \alpha$ and $\sum_{a_t \notin \{a_1,a_j,a_l\}} \varphi_{a_t}(P_i^*, P_j^*) = \alpha$. This implies that there exists $a_i \in A\backslash\{a_1, a_j, a_l\}$ such that $\varphi_{a_i}(P_i^*, P_j^*) > 0$.

By Condition H, there exists $P_k \in \mathbb{D}^{a_1,a_i}$. Let s, s' be such that $a_l = r_s(P_k)$ and $a_j = r_{s'}(P_k)$. We need to consider two cases.

Case 1: $s < s'$.

Let $\bar{P}_i = P_k$. By the induction hypothesis, $\varphi(\bar{P}_i, P_j^*) = \varepsilon\, e_1 + (1-\varepsilon)e_j$. Then, $\sum_{k=1}^{s} \varphi_{r_k(\bar{P}_i)}(\bar{P}_i, P_j^*) = \varepsilon < \varphi_{a_1}(P_i^*, P_j^*) + \varphi_{a_l}(P_i^*, P_j^*) + \varphi_{a_i}(P_i^*, P_j^*) \le \sum_{k=1}^{s} \varphi_{r_k(\bar{P}_i)}(P_i^*, P_j^*)$. Therefore, voter i manipulates at $(\bar{P}_i, P_j^*)$ via P_i^*.

Case 2: $s > s'$.

Let $\bar{P}_j = P_k$. By Lemma 3, $\varphi(P_i^*, \bar{P}_j) = \varepsilon\, e_l + (1-\varepsilon)e_1$. Then, $\sum_{k=1}^{s'} \varphi_{r_k(\bar{P}_j)}(P_i^*, \bar{P}_j) = 1-\varepsilon < \varphi_{a_1}(P_i^*, P_j^*) + \varphi_{a_j}(P_i^*, P_j^*) + \varphi_{a_i}(P_i^*, P_j^*) \le \sum_{k=1}^{s'} \varphi_{r_k(\bar{P}_j)}(P_i^*, P_j^*)$. Therefore, voter j manipulates at $(P_i^*, \bar{P}_j)$ via P_j^*.

Summing up, we conclude that manipulation always occurs when $\alpha > 0$. This establishes the Lemma. ■

Lemma 5 *(i) For all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{S_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_l + (1-\varepsilon)e_{b(r_1(P_j))}$.*

(ii) For all $P_i \in \mathbb{D}^{S_l}$ and $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_l$.

Proof: We verify part (i) first. Let $a_j \in S_{l-1}\backslash\{a_1, a_k\}$, $P_j \in \mathbb{D}^{a_j}$, $P_i^* \in \mathbb{D}^{a_l,a_1}$ and $P_j^* \in \mathbb{D}^{a_j,a_1}$. Strategy-proofness and Lemma 4 imply $\varphi_{a_j}(P_i^*, P_j) = \varphi_{a_j}(P_i^*, P_j^*) = 1-\varepsilon$ and $1-\varepsilon = \varphi_{a_j}(P_i^*, P_j^*) + \varphi_{a_1}(P_i^*, P_j^*) \geq \varphi_{a_j}(P_i^*, P_j) + \varphi_{a_1}(P_i^*, P_j)$. Therefore, $\varphi_{a_1}(P_i^*, P_j) = 0$.

Next, consider $P_i \in \mathbb{D}^{a_l}$ and $P_i' \in \mathbb{D}^{a_1,a_l}$. By strategy-proofness and the induction hypothesis, $\varphi_{a_l}(P_i, P_j) = \varphi_{a_l}(P_i^*, P_j) = \varphi_{a_l}(P_i^*, P_j) + \varphi_{a_1}(P_i^*, P_j) = \varphi_{a_l}(P_i', P_j) + \varphi_{a_1}(P_i', P_j) = \varphi_{a_1}(P_i', P_j) = \varepsilon$.

Similarly, for all $P_i \in \mathbb{D}^{a_l}$, we have $\varphi_{a_1}(P_i, P_j^*) = 0$. Let $P_j' \in \mathbb{D}^{a_1,a_j}$. Strategy-proofness and Lemma 3 imply $\varphi_{a_j}(P_i, P_j) = \varphi_{a_j}(P_i, P_j^*) = \varphi_{a_j}(P_i, P_j^*) + \varphi_{a_1}(P_i, P_j^*) = \varphi_{a_j}(P_i, P_j') + \varphi_{a_1}(P_i, P_j') = \varphi_{a_1}(P_i, P_j') = 1-\varepsilon$.

Therefore, $\varphi(P_i, P_j) = \varepsilon e_l + (1-\varepsilon)e_j$ for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_j}$ where $a_j \in S_{l-1}\backslash\{a_1, a_k\}$. By unanimity and Lemma 3, we conclude that $\varphi(P_i, P_j) = \varepsilon\, e_l + (1-\varepsilon)e_{b(r_1(P_j))}$ for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{S_l}$.

The proof of part (ii) is the symmetric counterpart of the proof of part (i) and is therefore omitted. ■

Therefore, by the induction hypothesis, we have proved that $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$ for all $P_i, P_j \in \mathbb{D}^{S_l}$. This completes the verification of Step 2 and hence the proof of the theorem. ■

We observe that Condition H is a strong condition and imposing it on the connectivity graph of a linked domain constitutes a significant strengthening of the linked domain condition. We note some implications of Theorem 4 below.

Remark 4 The Free Pair at the Top domain (FPT domain) (Aswal et al. (2003)) in which every two alternatives are connected, is a linked domain satisfying Condition H (any alternative could be a hub) and is consequently a random dictatorship domain. This partially addresses an open question in Sen (2011): is the FPT domain a random dictatorship domain for any arbitrary number of voter greater than two?

Remark 5 It was noted in Aswal et al. (2003) that the minimal cardinality of a linked domain is $4m-6$. It is also possible to construct a linked domain satisfying Condition H of the same cardinality. This can be done as follows: $a_1 \sim a_2, a_2 \sim a_3, a_1 \sim a_3$ and $a_j \sim a_1$, $a_j \sim a_2$ for all $j = 4, \ldots, m$. We can therefore find "small" random dictatorship domains - those that grow linearly in the number of alternatives.

5.2 Strongly Linked Domains with Condition TS

In this subsection we provide another condition that ensures that a domain is a random dictatorship domain. Our approach here is to strengthen the notion of connectedness of alternatives along the lines initiated in Chatterji et al. (2010).

Definition 16 *A pair of alternatives a, b is strongly connected (denoted by* $a \approx b$*) if there exist* P_i, $P_i' \in \mathbb{D}$ *such that*

(i) $r_1(P_i) = a$ *and* $r_2(P_i) = b$*,*

(ii) $r_1(P_i') = b$ *and* $r_2(P_i') = a$*,*

(iii) $r_k(P_i) = r_k(P_i')$, $k = 3, \ldots, m$.

In other words, a and b are stongly connected if it is possible to find an ordering in the domain where a and b are first and second ranked and it is possible to flip a and b while keeping the position of all other alternatives fixed.

Definition 17 *A strongly linked domain is defined in exactly the same way as a linked domain except that the notion of connectedness is replaced by strong connectedness.*

A strongly linked domain has stronger restrictions embedded in it than a linked domain. However, they are not necessarily randomly dictatorial. This is easily seen from domain $\mathbb{D}_L$ in Section 3.1 - the domain described in Appendix A can be specified to be strongly linked. An additional condition needs to be imposed to make a strongly linked domain a random dictatorship domain.

Definition 18 *A domain satisfies Condition TS if for all* $a, b \in A$*, either* $a \approx b$*, or there exists* $c \in A$ *such that* $a \approx c$ *and* $b \approx c$*.*

In other words, every alternative is strongly connected to any other alternative in at most two steps. The counterpart of this condition for connectedness is clearly weaker than Condition H. If the graphs in Figure 2 (e) and (f) are interpreted in terms of strong connectedness, then they represent strongly linked domains satisfying Condition TS.

Theorem 5 *A strongly linked domain satisfying Condition TS is a random dictatorship domain.*

Proof: Let $\mathbb{D}$ be a strongly linked domain satisfying Condition TS. Since domain $\mathbb{D}$ satisfies the triple property (the strong connectedness implies the connectedness), it suffices as in Theorem 4 to show that every strategy-proof and unanimous RSCF $\varphi : \mathbb{D}^2 \to \mathcal{L}(A)$ is a random dictatorship. Let $I = \{i, j\}$. For notational simplicity, we assume that the function σ in Definition 17 is the identity function. Define $S_l = \{a_1, a_2, \ldots, a_l\}$, $l = 3, \ldots, m$. Our proof proceeds by establishing the same two steps as those in the proof of Theorem 4.

Step 1. There exists $\varepsilon \in [0, 1]$ such that for all $P_i, P_j \in \mathbb{D}^{S_3}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1 - \varepsilon) e_{b(r_1(P_j))}$.

Step 2. If for all $P_i, P_j \in \mathbb{D}^{S_{l-1}}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1 - \varepsilon) e_{b(r_1(P_j))}$, then for all $P_i, P_j \in \mathbb{D}^{S_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1 - \varepsilon) e_{b(r_1(P_j))}$.

The following lemma establishes Step 1.

Lemma 6 *There exists* $\varepsilon \in [0,1]$ *such that for all* $P_i, P_j \in \mathbb{D}^{S_3}$, $\varphi(P_i,P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$.

Proof: Since a_1, a_2, a_3 form a triple of strong connectedness, which implies the triple of connectedness, the lemma follows from Theorem 2 in Sen (2011). ■

To verify Step 2, we use the following induction hypothesis.

Induction Hypothesis Level 1: for all $P_i, P_j \in \mathbb{D}^{S_{l-1}}$, $l > 3$, $\varphi(P_i,P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$.

We will show that for all $P_i, P_j \in \mathbb{D}^{S_l}$, $\varphi(P_i,P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$.

Pick an arbitrary $l > 3$ and $a \in S_{l-1}$. We say that a_l is strongly connected to a by a *chain of length* t *located in* S_l if there exists a sequence $\{y_k\}_{k=1}^{t+2} \subset S_l$ of length $t+2$ such that $a_l = y_1$, $a = y_{t+2}$ and $y_k \approx y_{k+1}$, $k = 1, \ldots, t+1$. We let $T_t(a_l, S_l)$ denote the set of the alternatives $a \in S_{l-1}$ satisfying the following two properties: (i) a is strongly connected to a_l by a chain of length t located in S_l and (ii) there does not exist a chain of length strictly less than t located in S_l connecting a_l and a.[5]

It is evident that $T_s(a_l,S_l) \cap T_{s'}(a_l,S_l) = \emptyset$ whenever $s \neq s'$. Moreover, it also follows that (i) $T_s(a_l,S_l) = \emptyset$ implies that $T_{s'}(a_l,S_l) = \emptyset$ for all $s' > s$, (ii) $\cup_{t\geq 0} T_t(a_l,S_l) = S_{l-1}$ and (iii) if $a \in T_s(a_l,S_l)$ with $s > 0$, there exists $b \in T_{s-1}(a_l,S_l)$ such that $b \approx a$. The next lemma considers $T_0(a_l,S_l)$.

Lemma 7 *(i) For all* $P_i \in \mathbb{D}^{a_l}$ *and* $P_j \in \mathbb{D}^{T_0(a_l,S_l)}$, $\varphi(P_i,P_j) = \varepsilon\, e_l + (1-\varepsilon)e_{b(r_1(P_j))}$.
(ii) For all $P_i \in \mathbb{D}^{T_0(a_l,S_l)}$ *and* $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i,P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_l$.

Proof: We show part (i) first. Let $a_s \in T_0(a_l,S_l)$. Since $a_l \sim a_s$ (recall that $[a_l \approx a_s] \Rightarrow [a_l \sim a_s]$), Lemmas 1 and 2 in Sen (2011) imply that for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_s}$, there exists $\beta \in [0,1]$ such that $\varphi(P_i,P_j) = \beta\, e_l + (1-\beta)e_s$. Now, pick $a_t \in T_0(a_l,S_l)\backslash\{a_s\}$,[6] $\bar{P}_i \in \mathbb{D}^{a_l,a_t}$, $\bar{P}_i^* \in \mathbb{D}^{a_t,a_l}$ and $P_j \in \mathbb{D}^{a_s}$. Strategy-proofness and Level 1 induction hypothesis imply $\beta = \varphi_{a_l}(\bar{P}_i,P_j) = \varphi_{a_l}(\bar{P}_i,P_j) + \varphi_{a_t}(\bar{P}_i,P_j) = \varphi_{a_l}(\bar{P}_i^*,P_j) + \varphi_{a_t}(\bar{P}_i^*,P_j) = \varphi_{a_t}(\bar{P}_i^*,P_j) = \varepsilon$. By symmetric arguments, part (ii) also holds. ■

To exhaust all alternatives in S_{l-1}, we provide another induction hypothesis as follows.

Induction Hypothesis Level 2: Fix an arbitrary $l \leq m$. Suppose that for all $0 \leq t' < t$ and either $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{\cup_{k=0}^{t'} T_k(a_l,S_l)}$ or $P_i \in \mathbb{D}^{\cup_{k=0}^{t'} T_k(a_l,S_l)}$ and $P_j \in \mathbb{D}^{a_l}$, we have that $\varphi(P_i,P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$.

We will show that for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{T_t(a_l,S_l)}$, or $P_i \in \mathbb{D}^{T_t(a_l,S_l)}$ and $P_j \in \mathbb{D}^{a_l}$, we have $\varphi(P_i,P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$.

[5] We use an example to explain the chain. Let Figure 2 (e) denote a strong connectivity graph of a strongly linked domain. Let the one to one function σ be the identity function. Considering a_1 and a_5, then $\{a_5,a_3,a_2,a_4,a_1\}$, $\{a_5,a_3,a_2,a_1\}$ and $\{a_5,a_3,a_1\}$ are chains of length 3, 2 and 1 located in S_5 respectively. Meanwhile, $T_0(a_5,S_5) = \{a_3,a_4\}$, $T_1(a_5,S_5) = \{a_1,a_2\}$ and $T_t(a_5,S_5) = \emptyset$ for all $t \geq 2$.

[6] Definition of strongly linked domains (Definition 17) implies that $|T_0(a_l,S_l)| \geq 2$.

Lemma 8 *(i) For all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{T_t(a_l,S_l)}$, $\varphi(P_i,P_j) = \varepsilon\, e_l + (1-\varepsilon)e_{b(r_1(P_j))}$.*

(ii) For all $P_i \in \mathbb{D}^{T_t(a_l,S_l)}$ and $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i,P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_l$.

Proof: Pick $a_j \in T_t(a_l,S_l)$ with $t>0$. According to Condition TS, there exists $a_i \in A$ such that $a_i \approx a_l$ and $a_i \approx a_j$. There are two cases to consider: $a_i \in S_{l-1}$ and $a_i \notin S_{l-1}$.[7] The proof of Lemma 8 follows the following 6 claims. We verify part (i) first. Claim 1 below consider $a_i \in S_{l-1}$.

Claim 1: (i) For all $P_i \in \mathbb{D}^{\{a_l,a_j\}}$ [8] and $P_j \in \mathbb{D}^{a_i}$, $\varphi(P_i,P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_i$.

(ii) For all $P_i \in \mathbb{D}^{a_i}$ and $P_j \in \mathbb{D}^{\{a_l,a_j\}}$, $\varphi(P_i,P_j) = \varepsilon\, e_i + (1-\varepsilon)e_{b(r_1(P_j))}$.

Since $a_i \in S_{l-1}$, it must be the case that $a_i \in T_0(a_l,S_l)$ and $a_j \in T_1(a_l,S_l)$. The claim then follows from Lemma 7 and the Level 1 induction hypothesis. This completes the verification of Claim 1.

Next, we will show that the same conclusions hold when $a_i \notin S_{l-1}$. Now, it must be the case that $a_j \in T_t(a_l,S_l)$ where $t>1$. Since $a_i \notin S_{l-1}$, we can assume that $a_l \approx a_s$, where $a_s \in T_0(a_l,S_l)$ (by Definition 17) and $a_j \approx a_k$, where $a_k \in T_{t-1}(a_l,S_l)$, $t>1$ (by property (iii) of $T_t(a_l,S_l)$ above). Since $t>1$, it is evident that $a_s \neq a_k$. The next three claims assume that $a_i \notin S_{l-1}$.

Claim 2: (i) For *some* $\bar{P}_i \in \mathbb{D}^{a_l,a_s}$ and $\bar{P}_j \in \mathbb{D}^{a_j,a_k}$, $\varphi(\bar{P}_i,\bar{P}_j) = \varepsilon\, e_l + (1-\varepsilon)e_j$.

(ii) For *some* $\bar{P}_i \in \mathbb{D}^{a_j,a_k}$ and $\bar{P}_j \in \mathbb{D}^{a_l,a_s}$, $\varphi(\bar{P}_i,\bar{P}_j) = \varepsilon\, e_j + (1-\varepsilon)e_l$.

We first consider part (i). By strong connectedness, we can assume that there exist $P'_i \in \mathbb{D}^{a_s,a_l}$ and $P'_j \in \mathbb{D}^{a_k,a_j}$ such that $r_\nu(P'_i) = r_\nu(\bar{P}_i)$ and $r_\nu(P'_j) = r_\nu(\bar{P}_j)$, $\nu = 3,\ldots,m$. Now, since $a_s, a_j \in S_{l-1}$, by strategy-proofness and the Level 1 induction hypothesis, we have that $\varphi_{a_l}(\bar{P}_i,\bar{P}_j) + \varphi_{a_s}(\bar{P}_i,\bar{P}_j) = \varphi_{a_l}(P'_i,\bar{P}_j) + \varphi_{a_s}(P'_i,\bar{P}_j) = \varphi_{a_s}(P'_i,\bar{P}_j) = \varepsilon$. Similarly, since $a_k \in T_{t-1}(a_l,S_l)$, by strategy-proofness and the Level 2 induction hypothesis, we have that $\varphi_{a_j}(\bar{P}_i,\bar{P}_j) + \varphi_{a_k}(\bar{P}_i,\bar{P}_j) = \varphi_{a_j}(\bar{P}_i,P'_j) + \varphi_{a_k}(\bar{P}_i,P'_j) = \varphi_{a_j}(\bar{P}_i,P'_j) = 1-\varepsilon$. Therefore, for all $a \notin \{a_l,a_j,a_s,a_k\}$, $\varphi_a(\bar{P}_i,\bar{P}_j) = 0$.

Suppose $\varphi_{a_s}(\bar{P}_i,\bar{P}_j) = \alpha > 0$. Then, $\varphi_{a_l}(\bar{P}_i,\bar{P}_j) = \varepsilon - \alpha$. Assume $a_l = r_{k_1}(\bar{P}_j)$ and $a_s = r_{k_2}(\bar{P}_j)$. Then, $a_l = r_{k_1}(P'_j)$ and $a_s = r_{k_2}(P'_j)$. We have two cases.

Case 1: $k_1 < k_2$.

Fix $P_j \in \mathbb{D}^{a_l}$. By unanimity, $\varphi_{a_l}(\bar{P}_i,P_j) = 1$. Hence, $\sum_{\nu=1}^{k_1}\varphi_{r_\nu(\bar{P}_j)}(\bar{P}_i,\bar{P}_j) = \varphi_{a_j}(\bar{P}_i,\bar{P}_j) + \varphi_{a_k}(\bar{P}_i,\bar{P}_j) + \varphi_{a_l}(\bar{P}_i,\bar{P}_j) = 1-\alpha < \sum_{\nu=1}^{k_1}\varphi_{r_\nu(\bar{P}_j)}(\bar{P}_i,P_j)$. Then, voter j would manipulate at $(\bar{P}_i,\bar{P}_j)$ via P_j.

Case 2: $k_1 > k_2$.

[7] We provide an example to show both cases of $a_i \in S_{l-1}$ and $a_i \notin S_{l-1}$. Let Figure 2 (f) denote the strong connectivity graph of a strongly linked domain. Then, the domain satisfies Condition TS. Furthermore, it is true that for every one to one function $\sigma : \{1,\ldots,7\} \to \{1,\ldots,7\}$ satisfied by a domain in Definition 17, $a_7 = a_{\sigma(7)}$. Let function σ be the identity function. We first consider a_1, a_5 and S_4. We know that $a_1 \approx a_3$, $a_3 \approx a_5$ and $a_3 \in S_4$. Next, considering a_1, a_6 and S_5, we know that $a_1 \approx a_7$, $a_7 \approx a_6$ and $a_7 \notin S_5$.

[8] Recall that $\mathbb{D}^{\{a_l,a_j\}}$ is different from $\mathbb{D}^{a_l,a_j}$. In the following proof, we never use $\mathbb{D}^{a_l,a_j}$, for we have no idea on whether $\mathbb{D}^{a_l,a_j} = \emptyset$ or not.

By the Level 2 induction hypothesis, $\sum_{\nu=1}^{k_2} \varphi_{r_\nu(P'_j)}(\bar{P}_i, P'_j) = \varphi_{a_k}(\bar{P}_i, P'_j) = 1 - \varepsilon < 1 - \varepsilon + \alpha = \varphi_{a_j}(\bar{P}_i, \bar{P}_j) + \varphi_{a_k}(\bar{P}_i, \bar{P}_j) + \varphi_{a_s}(\bar{P}_i, \bar{P}_j) = \sum_{\nu=1}^{k_2} \varphi_{r_\nu(P'_j)}(\bar{P}_i, \bar{P}_j)$. Then, voter j would manipulate at $(\bar{P}_i, P'_j)$ via $\bar{P}_j$.

Now, $\varphi_{a_s}(\bar{P}_i, \bar{P}_j) = 0$. Next, suppose that $\varphi_{a_k}(\bar{P}_i, \bar{P}_j) = \alpha > 0$. Then, $\varphi_{a_j}(\bar{P}_i, \bar{P}_j) = 1 - \varepsilon - \alpha$. Assume $a_j = r_{t_1}(\bar{P}_i)$ and $a_k = r_{t_2}(\bar{P}_i)$. Then, $a_j = r_{t_1}(P'_i)$ and $a_k = r_{t_2}(P'_i)$. We have two cases.

Case 1: $t_1 < t_2$.

Fix $P_i \in \mathbb{D}^{a_j}$. By unanimity, $\varphi_{a_j}(P_i, \bar{P}_j) = 1$. Hence, $\sum_{\nu=1}^{t_1} \varphi_{r_\nu(\bar{P}_i)}(\bar{P}_i, \bar{P}_j) = \varphi_{a_l}(\bar{P}_i, \bar{P}_j) + \varphi_{a_s}(\bar{P}_i, \bar{P}_j) + \varphi_{a_j}(\bar{P}_i, \bar{P}_j) = 1 - \alpha < \sum_{\nu=1}^{t_1} \varphi_{r_\nu(\bar{P}_i)}(P_i, \bar{P}_j)$. Then, voter i would manipulate at $(\bar{P}_i, \bar{P}_j)$ via P_i.

Case 2: $t_1 > t_2$.

By the Level 1 induction hypothesis, $\sum_{\nu=1}^{t_2} \varphi_{r_\nu(P'_i)}(P'_i, \bar{P}_j) = \varphi_{a_s}(P'_i, \bar{P}_j) = \varepsilon < \varepsilon + \alpha = \varphi_{a_l}(\bar{P}_i, \bar{P}_j) + \varphi_{a_s}(\bar{P}_i, \bar{P}_j) + \varphi_{a_k}(\bar{P}_i, \bar{P}_j) = \sum_{\nu=1}^{t_2} \varphi_{r_\nu(P'_i)}(\bar{P}_i, \bar{P}_j)$. Then, voter i would manipulate at $(P'_i, \bar{P}_j)$ via $\bar{P}_i$.

Then, $\varphi_{a_k}(\bar{P}_i, \bar{P}_j) = 0$. Therefore, $\varphi(\bar{P}_i, \bar{P}_j) = \varepsilon\, e_l + (1 - \varepsilon)e_j$.

By symmetric arguments, part (ii) also holds. This completes the verification of Claim 2.

Claim 3: (i) For all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_i}$, $\varphi(P_i, P_j) = \varepsilon\, e_l + (1 - \varepsilon)e_i$.

(ii) For all $P_i \in \mathbb{D}^{a_i}$ and $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_i + (1 - \varepsilon)e_l$.

We first consider part (i). Since $a_l \sim a_i$ (recall $[a_l \approx a_i] \Rightarrow [a_l \sim a_i]$), Lemmas 1 and 2 in Sen (2011) imply that there exists $\beta \in [0, 1]$ such that for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_i}$, $\varphi(P_i, P_j) = \beta\, e_l + (1 - \beta)e_i$.

Next, fix $\bar{P}_i \in \mathbb{D}^{a_l, a_s}$, $\bar{P}_j \in \mathbb{D}^{a_j, a_k}$, where profile $(\bar{P}_i, \bar{P}_j)$ satisfies Claim 2 (i), $P^*_i \in \mathbb{D}^{a_s, a_l}$ and $P^*_j \in \mathbb{D}^{a_j, a_i}$. Since $a_s, a_j \in S_{l-1}$, by strategy-proofness and the Level 1 induction hypothesis, we have $\varphi_{a_l}(\bar{P}_i, P^*_j) + \varphi_{a_s}(\bar{P}_i, P^*_j) = \varphi_{a_l}(P^*_i, P^*_j) + \varphi_{a_s}(P^*_i, P^*_j) = \varphi_{a_s}(P^*_i, P^*_j) = \varepsilon$. Meanwhile, by strategy-proofness and Claim 2 (i), $\varphi_{a_j}(\bar{P}_i, P^*_j) = \varphi_{a_j}(\bar{P}_i, \bar{P}_j) = 1 - \varepsilon$. Therefore, $\varphi_{a_i}(\bar{P}_i, P^*_j) = 0$. Now, fix $\bar{P}^*_j \in \mathbb{D}^{a_i, a_j}$. Strategy-proofness implies $1 - \beta = \varphi_{a_i}(\bar{P}_i, \bar{P}^*_j) = \varphi_{a_i}(\bar{P}_i, \bar{P}^*_j) + \varphi_{a_j}(\bar{P}_i, \bar{P}^*_j) = \varphi_{a_i}(\bar{P}_i, P^*_j) + \varphi_{a_j}(\bar{P}_i, P^*_j) = \varphi_{a_j}(\bar{P}_i, P^*_j) = 1 - \varepsilon$. Therefore, $\beta = \varepsilon$.

In conclusion, for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_i}$, $\varphi(P_i, P_j) = \varepsilon\, e_l + (1 - \varepsilon)e_i$.

By symmetric arguments, we have that for all $P_i \in \mathbb{D}^{a_i}$ and $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_i + (1 - \varepsilon)e_l$. This completes the verification of Claim 3.

Claim 4: (i) For all $P_i \in \mathbb{D}^{a_j}$ and $P_j \in \mathbb{D}^{a_i}$, $\varphi(P_i, P_j) = \varepsilon\, e_j + (1 - \varepsilon)e_i$.

(ii) For all $P_i \in \mathbb{D}^{a_i}$ and $P_j \in \mathbb{D}^{a_j}$, $\varphi(P_i, P_j) = \varepsilon\, e_i + (1 - \varepsilon)e_j$.

This Claim is similar to Claim 3 but its proof follows from Claim 2 and the Level 2 induction hypothesis while the proof for Claim 3 follows from Claim 2 and the Level 1 induction hypothesis. This completes the verification of Claim 4.

We have shown that irrespective of whether $a_i \in S_{l-1}$ or $a_i \notin S_{l-1}$, $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1 - \varepsilon)e_{b(r_1(P_j))}$ holds for all $P_i \in \mathbb{D}^{\{a_l, a_j\}}$ and $P_j \in \mathbb{D}^{a_i}$ or $P_i \in \mathbb{D}^{a_i}$ and $P_j \in \mathbb{D}^{\{a_l, a_j\}}$.

For next claim, let $P_i^* \in \mathbb{D}^{a_l,a_i}$ and $P_j^* \in \mathbb{D}^{a_j,a_i}$.

Claim 5: $\varphi(P_i^*, P_j^*) = \varepsilon\, e_l + (1-\varepsilon)e_j$.

Suppose that the Claim is false. Similar to Lemma 4, we can assume that $\varphi_{a_i}(P_i^*, P_j^*) = \alpha > 0$. Since $a_l \approx a_i$ and $a_j \approx a_i$, we can assume that there exist $\bar{P}_i^* \in \mathbb{D}^{a_i,a_l}$ and $\bar{P}_j^* \in \mathbb{D}^{a_i,a_j}$ such that $r_k(\bar{P}_j^*) = r_k(P_j^*)$, $k = 3, \ldots, m$. Since Claims 1, 3 (i) and 4 (ii) imply that $\varphi_{a_i}(P_i^*, \bar{P}_j^*) + \varphi_{a_j}(P_i^*, \bar{P}_j^*) = 1-\varepsilon$ and $\varphi_{a_i}(\bar{P}_i^*, P_j^*) + \varphi_{a_l}(\bar{P}_i^*, P_j^*) = \varepsilon$, by strategy-proofness, we have that $\varphi_{a_j}(P_i^*, P_j^*) = 1-\varepsilon-\alpha$ and $\varphi_{a_l}(P_i^*, P_j^*) = \varepsilon - \alpha$. Assume $a_l = r_s(P_j^*)$. It is evident that $s \geq 3$. Then, strong connectedness implies that $a_l = r_s(\bar{P}_j^*)$. According to Claims 1 and 3 (i), $\sum_{k=1}^{s-1} \varphi_{r_k(\bar{P}_j^*)}(P_i^*, \bar{P}_j^*) = 1-\varepsilon$. Next, by strong connectedness, we know that $\{r_k(P_j^*)\}_{k=1}^{s-1} = \{r_k(\bar{P}_j^*)\}_{k=1}^{s-1}$. Hence, by strategy-proofness, we have that $\sum_{k=1}^{s-1} \varphi_{r_k(P_j^*)}(P_i^*, P_j^*) = \sum_{k=1}^{s-1} \varphi_{r_k(\bar{P}_j^*)}(P_i^*, \bar{P}_j^*) = 1-\varepsilon$. Therefore, $\sum_{k=1}^{s} \varphi_{r_k(P_j^*)}(P_i^*, P_j^*) = \sum_{k=1}^{s-1} \varphi_{r_k(P_j^*)}(P_i^*, P_j^*) + \varphi_{a_l}(P_i^*, P_j^*) = 1-\alpha$. Now, fix $P_j \in \mathbb{D}^{a_l}$. By unanimity, $\sum_{k=1}^{s} \varphi_{r_k(P_j^*)}(P_i^*, P_j^*) = 1-\alpha < 1 = \varphi_{a_l}(P_i^*, P_j) = \sum_{k=1}^{s} \varphi_{r_k(P_j^*)}(P_i^*, P_j)$. Therefore voter j manipulates at (P_i^*, P_j^*) via P_j. This completes the verification of Claim 5.

Claim 6: For all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_j}$, we have $\varphi(P_i, P_j) = \varepsilon\, e_l + (1-\varepsilon)e_j$.

The proof of this claim follows from Lemma 5.[9]

By symmetric arguments, it follows that $\varphi(P_i, P_j) = \varepsilon\, e_j + (1-\varepsilon)e_l$ for all $P_i \in \mathbb{D}^{a_j}$ and $P_j \in \mathbb{D}^{a_l}$. This completes the proof of Lemma 8. ■

We can now complete the proof of the Theorem. We have shown that under the Level 1 induction hypothesis, the Level 2 induction hypothesis is established. With unanimity, this implies that for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{S_l}$, or $P_i \in \mathbb{D}^{S_l}$ and $P_j \in \mathbb{D}^{a_l}$, we have $\varphi(P_i, P_j) = \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))}$ as required, to complete Step 2. ■

Remark 6 It is easy to construct domains that satisfy the conditions of Theorem 4 but not of Theorem 5. The key step in such a construction is to make the notions of connectedness and strong connectedness equivalent; i.e., whenever $a \sim b$, we also have $a \approx b$ for all $a, b \in A$. In this setting, Condition TS is weaker than Condition H as is illustrated by the connectivity graphs in Figure 2 (e) and (f).

6 Conclusion

In this paper, we have shown that dictatorial domains are not necessarily random dictatorship domains. We have provided additional conditions on a class of dictatorial domains to ensure that they are random dictatorship domains. These additional conditions are quite restrictive and they suggest that a large class of dictatorial domains admit strategy-proof random social choice functions satisfying unanimity that are not random dictatorships.

[9]In the verification of Claim 6, we only need the conditions that $a_l \sim a_i$ and $a_j \sim a_i$, which are implied by the strong connectedness. Furthermore, in the proof of Lemma 5, we do not apply the full property of the hub a_1 (the hub is connected to every alternative else). Therefore, we could apply the same argument of Lemma 5 to the verification of Claim 6.

References

Aswal, N., S. Chatterji, and A. Sen (2003): “Dictatorial Domains,” *Economic Theory*, 22, 45–62.

Chatterji, S., R. Sanver, and A. Sen (2010): “On Domains That Admit Well-behaved Strategy-proof Social Choice Functions,” Working Paper.

Chatterji, S. and A. Sen (2011): “Tops-only Domains,” *Economic Theory*, 46, 255–282.

Danilov, V. (1994): “The Structure of Non-manipulable Social Choice Rules on A Tree,” *Mathematical Social Sciences*, 27, 123–131.

Demange, G. (1982): “Single-peaked Orders on A Tree,” *Mathematical Social Sciences*, 3, 389–396.

Duggan, J. (1996): “A Geometric Proof of Gibbard’s Random Dictatorship Theorem,” *Economic Theory*, 7, 365–369.

Gibbard, A. (1973): “Manipulation of Voting Schemes: A General Result,” *Econometrica*, 41, 587–601.

——— (1977): “Manipulation of Schemes That Mix Voting with Chance,” *Econometrica*, 45, 665–681.

Kalai, E. and E. Muller (1977): “Characterization of Domains Admitting Non-dictatorial Social Welfare Functions and Non-manipulable Voting Procedures,” *Journal of Economic Theory*, 16, 457–469.

Kim, K. and F. Roush (1989): “Kelly’s Conjecture,” *Mathematical Social Sciences*, 17, 189–194.

Manelli, A. and D. Vincent (2007): “Multi-dimensional Mechanism Design: Revenue Maximization and the Multiple-good Monopoly,” *Journal of Economic Theory*, 137, 153–185.

Mehta, A. and V. Vazirani (2004): “Randomized Truthful Auctions of Digital Goods are Randomizations over Truthful Auctions,” in *ACM Transactions on Electronic Commerce*, ACM, 120–124.

Picot, J. and A. Sen (2011): “An Extreme Point Characterization of Random Strategy-proof Social Choice Functions: The Two Alternatives Case,” *Economics Letters*, 115, 49–52.

Sato, S. (2010): “Circular Domains,” *Review of Economic Design*, 14, 331–342.

Satterthwaite, M. (1975): "Strategy-proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," *Journal of Economic Theory*, 10, 187–217.

Sen, A. (2001): "Another Direct Proof of the Gibbard–Satterthwaite Theorem," *Economics Letters*, 70, 381–385.

——— (2011): "The Gibbard Random Dictatorship Theorem: A Generalization and A New Proof," *SERIEs: Journal of the Spanish Economic Association*, 2, 515–527.

Appendices

Appendix A: The Domain $\mathbb{D}_L$

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}
a_1	a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4	a_4	a_4	a_5	a_5	a_5	a_5	a_6	a_6	a_6	a_7	a_7	a_7	a_7
a_3	a_4	a_5	a_6	a_7	a_1	a_4	a_1	a_3	a_5	a_7	a_1	a_4	a_6	a_7	a_2	a_5	a_7	a_2	a_4	a_5	a_6
$\cdot$	a_2	a_2	$\cdot$	$\cdot$	$\cdot$	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	$\cdot$	a_2	a_2	$\cdot$	a_2	a_2	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	a_2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table 1: Domain $\mathbb{D}_L$

In the table above, dots on a particular ordering signify that alternatives not specified are arbitrarily ordered.

Appendix B: The Proof of Proposition 2

Proof: It suffices to construct a unanimous, strategy-proof and non-dictatorial RSCF $\varphi : \mathbb{D}^2_{NRD} \to \mathcal{L}(A)$. Let $I = \{i, j\}$. Similar to the RSCF in the proof of Proposition 1, consider the RSCF φ below:

$$\varphi(P_i, P_j) = \begin{cases} \varepsilon\, e_{b(x)} + \alpha\, e_{b(r_2(P_j))} + (1-\varepsilon-\alpha)e_{b(y)} & \text{if } P_i \in \mathbb{D}^x_{NRD} \text{ and } P_j \in \mathbb{D}^y_{NRD} \\ \varepsilon\, e_{b(r_1(P_i))} + (1-\varepsilon)e_{b(r_1(P_j))} & \text{otherwise.} \end{cases}$$

where $0 < \varepsilon < 1$ and $0 < \alpha \leq \min(\varepsilon, 1-\varepsilon)$.

It is evident that RSCF φ is unanimous and not a random dictatorship. To check the strategy-proofness of φ, it suffices to consider only the five cased below.

Case 1: The profile is (P_i, P_j) where $P_j \in \mathbb{D}^y_{NRD}$ and $P_i \notin \mathbb{D}^x_{NRD}$. Voter i considers a manipulation via $P'_i \in \mathbb{D}^x_{NRD}$.

Let $u \in \mathbb{U}(P_i)$. The loss from misrepresentation is $U_i(P_i, P_j) - U_i(P'_i, P_j) = \varepsilon u(r_1(P_i)) + \alpha u(y) - \varepsilon u(x) - \alpha u(r_2(P_j))$. If $y P_i r_2(P_j)$, then $U_i(P_i, P_j) - U_i(P'_i, P_j) \geq 0$. Suppose $r_2(P_j) P_i y$. Then $P_i \notin \mathbb{D}^y_{NRD}$, which implies that $P_i \in \mathbb{D}^{A\setminus\{x,y\}}_{NRD}$. Since $r_2(P_j) \in \mathcal{S}(y)$, applying the third restriction of the Condition NRD, we have $U_i(P_i, P_j) - U_i(P'_i, P_j) = \varepsilon\Big[u(r_1(P_i)) - u(r_2(P_j))\Big] + (\varepsilon - \alpha)\Big[u(r_2(P_j)) - u(y)\Big] - \varepsilon\Big[u(y) - u(x)\Big] \geq 0$.

Case 2: The profile is (P_i, P_j) where $P_j \in \mathbb{D}^y_{NRD}$ and $P_i \in \mathbb{D}^x_{NRD}$. Voter i considers a manipulation via $P'_i \notin \mathbb{D}^x_{NRD}$.

Let $u \in \mathbb{U}(P_i)$. Since $P_i \in \mathbb{D}^x_{NRD}$, $r_2(P_j) P_i y$ by the second restriction of the Condition NRD. Hence, the loss from misrepresentation is $U_i(P_i, P_j) - U_i(P'_i, P_j) = \varepsilon\Big[u(x) - u(r_1(P'_i))\Big] + \alpha\Big[u(r_2(P_j)) - u(y)\Big] \geq 0$.

Case 3: The profile is (P_i, P_j) where $P_i \in \mathbb{D}^x_{NRD}$ and $P_j \notin \mathbb{D}^y_{NRD}$. Voter j considers a manipulation via $P'_j \in \mathbb{D}^y_{NRD}$.

Let $u \in \mathbb{U}(P_j)$. The loss from misrepresentation is $U_j(P_i, P_j) - U_j(P_i, P'_j) = (1 - \varepsilon - \alpha)\Big[u(r_1(P_j)) - u(y)\Big] + \alpha\Big[u(r_1(P_j)) - u(r_2(P'_j))\Big] \geq 0$.

Case 4: The profile is (P_i, P_j) where $P_i \in \mathbb{D}^x_{NRD}$ and $P_j \in \mathbb{D}^y_{NRD}$. Voter j considers a manipulation via $P'_j \notin \mathbb{D}^y_{NRD}$.

Let $u \in \mathbb{U}(P_j)$. The loss from misrepresentation is $U_j(P_i, P_j) - U_j(P_i, P'_j) = (1 - \varepsilon - \alpha)\Big[u(y) - u(r_1(P'_j))\Big] + \alpha\Big[u(r_2(P_j)) - u(r_1(P'_j))\Big] \geq 0$.

Case 5: The profile is (P_i, P_j) where $P_i \in \mathbb{D}^x_{NRD}$ and $P_j \in \mathbb{D}^y_{NRD}$. Voter j considers a manipulation via $P'_j \in \mathbb{D}^y_{NRD}$ and $P'_j \neq P_j$.

Let $u \in \mathbb{U}(P_j)$. The loss from misrepresentation is $U_j(P_i, P_j) - U_j(P_i, P'_j) = \alpha\Big[u(r_2(P_j)) - u(r_2(P'_j))\Big] \geq 0$.

We conclude that RSCF φ is strategy-proof. ∎

Appendix C: The Proof of Theorem 3

In this section we provide a proof of Theorem 3. In fact, we prove a stronger version of the Theorem by using a weaker but slightly more complicated condition than the triple property. We begin by describing this condition.

For an ordering $P_i \in \mathbb{D}$ and $a \in A$, we let $B(P_i, a)$ denote the set of alternatives that are *strictly* better than a according to P_i, i.e., $[x \in B(P_i, a)] \Rightarrow [xP_ia]$, while $W(P_i, a)$ denotes the set of alternatives that are *strictly* worse than a according to P_i, i.e., $[x \in W(P_i, a)] \Rightarrow [aP_ix]$. For a profile $P \in \mathbb{D}^N$, let $\tau(P) = \{r_1(P_i)\}_{i=1}^N$. For any $i \in I$ and profile $P \in \mathbb{D}^N$, $\tau(P_{-i})$ denotes the set of alternatives that are first-ranked by all voters other than i. For a profile $P \in \mathbb{D}^N$ with $|\tau(P)| = N$, define $\overline{W}(P) = \cup_{i=1}^N W\Big(P_i, \max(P_i, \tau(P_{-i}))\Big)$.

Definition 19 *A domain $\mathbb{D}$ satisfies Richness Condition α if there exist $P_1, P_2, P_3 \in \mathbb{D}$ with $|\tau(P_1, P_2, P_3)| = 3$ such that $\overline{W}(P_1, P_2, P_3) = A$.*

If domain $\mathbb{D}$ satisfies the triple property, there exist alternatives a, b and c, and orderings P_1, P_2 and P_3 such that $r_1(P_1) = a$, $r_2(P_1) = b$, $r_1(P_2) = b$, $r_2(P_2) = c$, $r_1(P_3) = c$ and $r_2(P_3) = a$. The orderings P_1, P_2 and P_3 satisfy Richness Condition α.

Henceforth, we assume that domain $\mathbb{D}$ satisfies Minimal Richness (Definition 14) and Richness Condition α.

An additional piece of notation that we shall be using throughout the proof is the following: for all $a, b \in A$, $I(a, b)$ is the indicator function where $I(a, b) = 1$ if $a = b$ and $I(a, b) = 0$ if $a \neq b$.

The following definition serves as a critical bridge in the proof of Theorem 3.

Definition 20 *A unanimous and strategy-proof RSCF* $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ *is a quasi-random dictatorship, if there exists* $\{\varepsilon_k\}_{k=1}^N \geq 0$ *with* $\sum_{k=1}^N \varepsilon_k = 1$ *such that for all* $P \in \mathbb{D}^N$*, where there exist* $i, j \in I$ *that* $P_i = P_j$*,* $\varphi(P) = \sum_{k=1}^N \varepsilon_k \, e_{b(r_1(P_k))}$*.*

The random dictatorship is stronger than quasi-random dictatorship, for quasi-random dictatorship only considers those profiles of preferences with at least two voters sharing a same preference ordering and the outcome under such a profile of preferences is a convex combination of N (deterministic) dictatorial social choice functions with respect to an N-dimensional sequence $\{\varepsilon_k\}_{k=1}^N$.

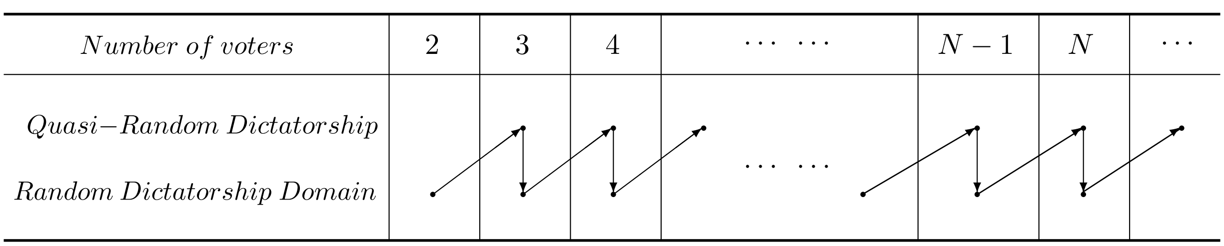
The proof of $(b) \Rightarrow (a)$ in Theorem 3 is trivial. We focus on showing $(a) \Rightarrow (b)$. The proof consists in establishing three steps. Following is the outline of the proof.

Step 1. Suppose domain $\mathbb{D}$ satisfies Richness Condition α. Every unanimous and strategy-proof RSCF $g : \mathbb{D}^2 \to \mathcal{L}(A)$ is a random dictatorship $\Rightarrow$ every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^3 \to \mathcal{L}(A)$ is a quasi-random dictatorship. This is shown in Proposition 4. □

Step 2. Every unanimous and strategy-proof RSCF $g : \mathbb{D}^{N-1} \to \mathcal{L}(A)$, $N > 3$, is a random dictatorship $\Rightarrow$ every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ is a quasi-random dictatorship. This is shown in Proposition 5. □

Step 3. Suppose for all $2 \leq t < N$, every unanimous and strategy-proof RSCF $g : \mathbb{D}^t \to \mathcal{L}(A)$, is a random dictatorship. A unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ is a quasi-random dictatorship $\Rightarrow$ φ is a random dictatorship. This is shown in Proposition 6. □

Note that we only use Richness Condition α in the verification of Step 1. Thus the induction problem is solved in the way shown by the arrows in the diagram below.



Proposition 4 *Let* $\mathbb{D}$ *be a minimally rich domain satisfying Richness Condition* α*. Suppose every unanimous and strategy-proof RSCF* $g : \mathbb{D}^2 \to \mathcal{L}(A)$ *is a random dictatorship. Then every unanimous and strategy-proof RSCF* $\varphi : \mathbb{D}^3 \to \mathcal{L}(A)$ *is a quasi-random dictatorship.*

Proof: Define three RSCF's as follows: $g^{(2,3)}(P_1, P_2) = \varphi(P_1, P_2, P_2)$, $g^{(1,3)}(P_1, P_2) = \varphi(P_1, P_2, P_1)$ and $g^{(1,2)}(P_1, P_3) = \varphi(P_1, P_1, P_3)$ for all $P_1, P_2, P_3 \in \mathbb{D}$. According to Lemma 3 in Sen (2011), we know that RSCF's $g^{(2,3)}$, $g^{(1,3)}$ and $g^{(1,2)}$ are random dictatorships. Then, there exist ε_1, ε_2, $\varepsilon_3 \geq 0$ such that for all $P_1, P_2, P_3 \in \mathbb{D}$,

$$\begin{aligned}
\varphi(P_1, P_2, P_2) &= \varepsilon_1\, e_{b(r_1(P_1))} + (1-\varepsilon_1) e_{b(r_1(P_2))} \\
\varphi(P_1, P_2, P_1) &= (1-\varepsilon_2) e_{b(r_1(P_1))} + \varepsilon_2\, e_{b(r_1(P_2))} \\
\varphi(P_1, P_1, P_3) &= (1-\varepsilon_3) e_{b(r_1(P_1))} + \varepsilon_3\, e_{b(r_1(P_3))}
\end{aligned}$$

To establish that φ is a quasi-random dictatorship, it suffices to show that $\varepsilon_1+\varepsilon_2+\varepsilon_3 = 1$. Now, fix a profile $P^* = (P_1^*, P_2^*, P_3^*)$, satisfying Richness Condition α. Since $\overline{W}(P^*) = A$ implies that $\tau(P^*) \subset \overline{W}(P^*)$, we could assume without loss of generality that $r_1(P_1^*) = a$, $r_1(P_2^*) = b$, $r_1(P_3^*) = c$, $b\, P_1^*\, c$, $c\, P_2^*\, a$ and $a\, P_3^*\, b$. Furthermore, assume $b = r_s(P_1^*)$ and $c = r_{s'}(P_1^*)$. Hence, $1 < s < s'$. By strategy-proofness, we know that for all $t \geq 1$, $\sum_{k=1}^{t} \varphi_{r_k(P_1^*)}(P_2^*, P_2^*, P_3^*) \leq \sum_{k=1}^{t} \varphi_{r_k(P_1^*)}(P^*) \leq \sum_{k=1}^{t} \varphi_{r_k(P_1^*)}(P_1^*, P_1^*, P_3^*)$. Since $\varphi(P_2^*, P_2^*, P_3^*) = g^{(1,2)}(P_2^*, P_3^*)$ and $\varphi(P_1^*, P_1^*, P_3^*) = g^{(1,2)}(P_1^*, P_3^*)$, we have that for all $t \geq 1$, $\sum_{k=1}^{t} g^{(1,2)}_{r_k(P_1^*)}(P_2^*, P_3^*) \leq \sum_{k=1}^{t} \varphi_{r_k(P_1^*)}(P^*) \leq \sum_{k=1}^{t} g^{(1,2)}_{r_k(P_1^*)}(P_1^*, P_3^*)$.

Next, since $g^{(1,2)}$ is a random dictatorship with respect to $\{1-\varepsilon_3, \varepsilon_3\}$, we have

$$\sum_{k=1}^{s} g^{(1,2)}_{r_k(P_1^*)}(P_2^*, P_3^*) = \sum_{k=1}^{s'-1} g^{(1,2)}_{r_k(P_1^*)}(P_2^*, P_3^*) = g^{(1,2)}_{b}(P_2^*, P_3^*) = 1-\varepsilon_3$$

$$\sum_{k=1}^{s} g^{(1,2)}_{r_k(P_1^*)}(P_1^*, P_3^*) = \sum_{k=1}^{s'-1} g^{(1,2)}_{r_k(P_1^*)}(P_1^*, P_3^*) = g^{(1,2)}_{a}(P_1^*, P_3^*) = 1-\varepsilon_3$$

$$\sum_{k=1}^{s'} g^{(1,2)}_{r_k(P_1^*)}(P_2^*, P_3^*) = g^{(1,2)}_{b}(P_2^*, P_3^*) + g^{(1,2)}_{c}(P_2^*, P_3^*) = 1$$

$$\sum_{k=1}^{s'} g^{(1,2)}_{r_k(P_1^*)}(P_1^*, P_3^*) = g^{(1,2)}_{a}(P_1^*, P_3^*) + g^{(1,2)}_{c}(P_1^*, P_3^*) = 1$$

Therefore, $\sum_{k=1}^{s} \varphi_{r_k(P_1^*)}(P^*) = \sum_{k=1}^{s'-1} \varphi_{r_k(P_1^*)}(P^*) = 1-\varepsilon_3$ and $\sum_{k=1}^{s'} \varphi_{r_k(P_1^*)}(P^*) = 1$. Hence, $\varphi_c(P^*) = \sum_{k=1}^{s'} \varphi_{r_k(P_1^*)}(P^*) - \sum_{k=1}^{s'-1} \varphi_{r_k(P_1^*)}(P^*) = \varepsilon_3$ and $\sum_{k=1}^{s} \varphi_{r_k(P_1^*)}(P^*) + \varphi_c(P^*) = 1$. Then, we know that for all $x \in W(P_1^*, b)\backslash\{c\}$, $\varphi_x(P^*) = 0$. Symmetrically, we can obtain $\varphi_a(P^*) = \varepsilon_1$, $\varphi_x(P^*) = 0$ for all $x \in W(P_2^*, c)\backslash\{a\}$; and $\varphi_b(P^*) = \varepsilon_2$, $\varphi_x(P^*) = 0$ for all $x \in W(P_3^*, a)\backslash\{b\}$. In conclusion, for all $x \in \overline{W}(P^*)\backslash\{a, b, c\}$, $\varphi_x(P^*) = 0$. Furthermore, since $\overline{W}(P^*) = A$, we have that $1 = \sum_{x \in A} \varphi_x(P^*) = \sum_{x \in \overline{W}(P^*)} \varphi_x(P^*) = \varphi_a(P^*) + \varphi_b(P^*) + \varphi_c(P^*) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. This completes the verification of Proposition 4. ■

Proposition 5 *Let $\mathbb{D}$ be a minimally rich domain. Suppose that every unanimous and strategy-proof RSCF $g : \mathbb{D}^{N-1} \to \mathcal{L}(A)$ is a random dictatorship for $N > 3$. Then every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ is a quasi-random dictatorship.*

Proof: This proposition holds when $m = 3$, since a domain with exact three alternatives is a random dictatorship domain for the case of $N-1$ voters iff it is the complete domain.[10] We therefore need to consider $m \geq 4$. The proof of the Proposition follows from Lemmas 9, 10, 11 and 12.

Let $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ be a unanimous and strategy-proof RSCF. Pick two arbitrary voters, say i and j. Define a RSCF $g^{(i,j)}$ as follows: for all $P_i \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, $g^{(i,j)}(P_i, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$.

Lemma 9 *The RSCF $g^{(i,j)}$ is a random dictatorship for all $i, j \in I$.*

Proof: This lemma follows from Lemma 3 in Sen (2011). According to the proof of Lemma 3 in Sen (2011), the unanimity and strategy-proofness of φ imply that $g^{(i,j)}$ is unanimous and strategy-proof. Then by the hypothesis of Proposition 5, we know that $g^{(i,j)}$ is a random dictatorship. ■

Fix $i, j \in I$. It follows from Lemma 9 above that there exist $\varepsilon^{(i,j)}$, $\varepsilon_k^{(i,j)} \geq 0$ for all $k \neq i, j$ such that $\varepsilon^{(i,j)} + \sum_{k \neq i,j} \varepsilon_k^{(i,j)} = 1$ and satisfying the following property: $\varphi(P_i, P_i, P_{-\{i,j\}}) = g^{(i,j)}(P_i, P_{-\{i,j\}}) = \varepsilon^{(i,j)} \, e_{b(r_1(P_i))} + \sum_{k \neq i,j} \varepsilon_k^{(i,j)} \, e_{b(r_1(P_k))}$ for all $P_i \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. The next lemma shows that we could split the probability $\varepsilon^{(i,j)}$ appropriately into two parts and together with all $\varepsilon_k^{(i,j)}$, $k \neq i, j$, construct a new N-dimensional sequence of probabilities, which are able to be applied to all profiles of preferences where voter i and j share a same preference ordering.

Lemma 10 *Pick $i, j \in I$. For all $P \in \mathbb{D}^N$ with $P_i = P_j$ there exists $\{\alpha_k^{(i,j;\,s,t)}\}_{k=1}^N \geq 0$ with $\sum_{k=1}^N \alpha_k^{(i,j;\,s,t)} = 1$, where $s, t \in I \backslash \{i, j\}$ and $s \neq t$, such that $\varphi(P) = \sum_{k=1}^N \alpha_k^{(i,j;\,s,t)} \, e_{b(r_1(P_k))}$.*

Proof: Now, i, j, s, t are mutually distinct. For every $l \neq i, j, s, t$, we consider a profile $P^{(l)} = (P_i, P_i, P_s, P_s, P_l, P_{-\{i,j,s,t,l\}})$ [11] where $r_1(P_i) = a$, $r_1(P_s) = b$, $r_1(P_l) = c$ and $\tau(P_{-\{i,j,s,t,l\}}) \cap \{a, b, c\} = \emptyset$ (recall that $m \geq 4$). Standard properties of $g^{(i,j)}$ imply that $\varphi_a(P^{(l)}) = \varepsilon^{(i,j)}$, $\varphi_b(P^{(l)}) = \varepsilon_s^{(i,j)} + \varepsilon_t^{(i,j)}$ and $\varphi_c(P^{(l)}) = \varepsilon_l^{(i,j)}$. Meanwhile, by $g^{(s,t)}$, $\varphi_a(P^{(l)}) = \varepsilon_i^{(s,t)} + \varepsilon_j^{(s,t)}$, $\varphi_b(P^{(l)}) = \varepsilon^{(s,t)}$ and $\varphi_c(P^{(l)}) = \varepsilon_l^{(s,t)}$. Therefore, $\varepsilon^{(i,j)} = \varepsilon_i^{(s,t)} + \varepsilon_j^{(s,t)}$, $\varepsilon^{(s,t)} = \varepsilon_s^{(i,j)} + \varepsilon_t^{(i,j)}$ and $\varepsilon_l^{(i,j)} = \varepsilon_l^{(s,t)}$ for all $l \neq i, j, s, t$. Since $\varepsilon^{(i,j)} + \sum_{k \neq i,j} \varepsilon_k^{(i,j)} = 1$ and $\varepsilon^{(s,t)} + \sum_{k \neq s,t} \varepsilon_k^{(s,t)} = 1$, we have $\varepsilon_i^{(s,t)} + \varepsilon_j^{(s,t)} + \sum_{k \neq i,j} \varepsilon_k^{(i,j)} = 1$ and $\varepsilon_s^{(i,j)} + \varepsilon_t^{(i,j)} + \sum_{k \neq s,t} \varepsilon_k^{(s,t)} = 1$.

Setting $\alpha_i^{(i,j;\,s,t)} = \varepsilon_i^{(s,t)}$, $\alpha_j^{(i,j;\,s,t)} = \varepsilon_j^{(s,t)}$, $\alpha_s^{(i,j;\,s,t)} = \varepsilon_s^{(i,j)}$, $\alpha_t^{(i,j;\,s,t)} = \varepsilon_t^{(i,j)}$ and $\alpha_l^{(i,j;\,s,t)} = \varepsilon_l^{(s,t)} = \varepsilon_l^{(i,j)}$ for all $l \neq i, j, s, t$, we have $\alpha_k^{(i,j;\,s,t)} \geq 0$, $k = 1, \ldots, N$ and $\sum_{k=1}^N \alpha_k^{(i,j;\,s,t)} = 1$.

Fix a profile $P = (P_i, P_j, P_{-\{i,j\}})$ with $P_i = P_j \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. It follows from properties of $g^{(i,j)}$ that $\varphi_{r_1(P_i)}(P) = \varepsilon^{(i,j)} + \sum_{k \neq i,j} \varepsilon_k^{(i,j)} \, I(r_1(P_k), r_1(P_i)) =$

[10] The sufficiency part is shown in Gibbard (1977), Duggan (1996) and Sen (2011). The unique seconds property in Aswal et al. (2003) implies the necessity. Let a domain satisfy the unique seconds property. Then this domain is not dictatorial and hence not randomly dictatorial. Furthermore, when $m = 3$, every domain other than the complete domain satisfies the unique seconds property.

[11] If $N = 4$, we let $P = (P_i, P_i, P_s, P_s)$ where $r_1(P_i) = a$ and $r_1(P_s) = b$.

$\sum_{k=1}^{N} \alpha_k^{(i,j;\,s,t)}\, I(r_1(P_k), r_1(P_i))$ and for all $x \in A\backslash\{r_1(P_i)\}$, $\varphi_x(P) = \sum_{k \neq i,j} \varepsilon_k^{(i,j)}\, I(r_1(P_k), x) = \sum_{k=1}^{N} \alpha_k^{(i,j;\,s,t)}\, I(r_1(P_k), x)$. ∎

Note that $\{\alpha_k^{(i,j;\,s,t)}\}_{k=1}^{N} = \{\alpha_k^{(s,t;\,i,j)}\}_{k=1}^{N}$, where i, j, s, t are mutually distinct. The next lemma shows that sequence $\{\alpha_k^{(i,j;\,s,t)}\}_{k=1}^{N}$ is independent of $\{s, t\}$ whenever $s, t \in I\backslash\{i, j\}$ and $s \neq t$.

Lemma 11 *Fix $i, j \in I$. For all $s, t, \bar{s}, \bar{t} \in I\backslash\{i, j\}$, where $s \neq t$ and $\bar{s} \neq \bar{t}$, we have $\{\alpha_k^{(i,j;\,s,t)}\}_{k=1}^{N} = \{\alpha_k^{(i,j;\,\bar{s},\bar{t})}\}_{k=1}^{N}$.*

Proof: According to Lemma 10, $\alpha_i^{(i,j;\,s,t)} = \varepsilon_i^{(s,t)}$, $\alpha_j^{(i,j;\,s,t)} = \varepsilon_j^{(s,t)}$, $\varepsilon_i^{(s,t)} + \varepsilon_j^{(s,t)} = \varepsilon^{(i,j)}$ and $\alpha_k^{(i,j;\,s,t)} = \varepsilon_k^{(i,j)}$ for all $k \neq i, j$. Meanwhile, $\alpha_i^{(i,j;\,\bar{s},\bar{t})} = \varepsilon_i^{(\bar{s},\bar{t})}$, $\alpha_j^{(i,j;\,\bar{s},\bar{t})} = \varepsilon_j^{(\bar{s},\bar{t})}$, $\varepsilon_i^{(\bar{s},\bar{t})} + \varepsilon_j^{(\bar{s},\bar{t})} = \varepsilon^{(i,j)}$ and $\alpha_k^{(i,j;\,\bar{s},\bar{t})} = \varepsilon_k^{(i,j)}$ for all $k \neq i, j$. Therefore, $\alpha_i^{(i,j;\,s,t)} + \alpha_j^{(i,j;\,s,t)} = \alpha_i^{(i,j;\,\bar{s},\bar{t})} + \alpha_j^{(i,j;\,\bar{s},\bar{t})}$ and $\alpha_k^{(i,j;\,s,t)} = \alpha_k^{(i,j;\,\bar{s},\bar{t})}$ for all $k \neq i, j$.

Next, given a profile $P = (P_i, P_{-i})$ where $r_1(P_i) = a$ and for all $k, l \in I\backslash\{i\}$, $P_k = P_l \notin \mathbb{D}^a$, then by both $g^{(s,t)}$ and $g^{(\bar{s},\bar{t})}$ respectively, we have $\varphi_a(P) = \varepsilon_i^{(s,t)}$ and $\varphi_a(P) = \varepsilon_i^{(\bar{s},\bar{t})}$. Then, $\varepsilon_i^{(s,t)} = \varepsilon_i^{(\bar{s},\bar{t})}$ and hence $\alpha_i^{(i,j;\,s,t)} = \alpha_i^{(i,j;\,\bar{s},\bar{t})}$. Consequently, $\alpha_j^{(i,j;\,s,t)} = \alpha_j^{(i,j;\,\bar{s},\bar{t})}$. ∎

Fix $i, j \in I$. We have the following: for all $P \in \mathbb{D}^N$ with $P_i = P_j$, there exists $\{\alpha_k^{(i,j)}\}_{k=1}^{N} \geq 0$ with $\sum_{k=1}^{N} \alpha_k^{(i,j)} = 1$ such that $\varphi(P) = \sum_{k=1}^{N} \alpha_k^{(i,j)}\, e_{b(r_1(P_k))}$. In addition, $\{\alpha_k^{(i,j)}\}_{k=1}^{N} = \{\alpha_k^{(j,i)}\}_{k=1}^{N}$. We next show that the sequence $\{\alpha_k^{(i,j)}\}_{k=1}^{N}$ is independent of $\{i, j\}$.

Lemma 12 *For all $i, j, s, t \in I$, where $i \neq j$ and $s \neq t$, $\{\alpha_k^{(i,j)}\}_{k=1}^{N} = \{\alpha_k^{(s,t)}\}_{k=1}^{N}$.*

Proof: It is evident that $|\{i, j\} \cap \{s, t\}| = 0, 1$ or 2. If $|\{i, j\} \cap \{s, t\}| = 0$, then i, j, s, t are mutually distinct. Hence, $\{\alpha_k^{(i,j)}\}_{k=1}^{N} = \{\alpha_k^{(i,j;\,s,t)}\}_{k=1}^{N} = \{\alpha_k^{(s,t;\,i,j)}\}_{k=1}^{N} = \{\alpha_k^{(s,t)}\}_{k=1}^{N}$. Next, if $|\{i, j\} \cap \{s, t\}| = 2$, then $\{i, j\} = \{s, t\}$, which implies $\{\alpha_k^{(i,j)}\}_{k=1}^{N} = \{\alpha_k^{(s,t)}\}_{k=1}^{N}$.

Now, we consider $|\{i, j\} \cap \{s, t\}| = 1$, We can therefore assume without loss of generality that $i = s$. Since $N > 3$, there exists another voter: voter $\bar{s}$ and $\bar{s} \notin \{i, j, t\}$.

For every $k \notin \{i, j, t\}$, we consider a profile $P^{(k)} = (P_k, P_{-k})$ where $P_k \in \mathbb{D}^a$ and for all $l, n \in I\backslash\{k\}$, $P_l = P_n \notin \mathbb{D}^a$. By Lemma 11, it follows that $\varphi_a(P^{(k)}) = \alpha_k^{(i,j)}$ and $\varphi_a(P^{(k)}) = \alpha_k^{(i,t)}$. Therefore, $\alpha_k^{(i,j)} = \alpha_k^{(i,t)}$ for all $k \notin \{i, j, t\}$.

From the case where $|\{i, j\} \cap \{\bar{s}, t\}| = 0$, we have $\alpha_j^{(i,j)} = \alpha_j^{(\bar{s},t)}$. Consider a profile $P = (P_j, P_{-j})$ where $P_j \in \mathbb{D}^a$ and for all $l, n \in I\backslash\{j\}$, $P_l = P_n \notin \mathbb{D}^a$. By Lemma 11, it follows that $\varphi_a(P) = \alpha_j^{(\bar{s},t)}$ and $\varphi_a(P) = \alpha_j^{(i,t)}$. Therefore, $\alpha_j^{(\bar{s},t)} = \alpha_j^{(i,t)}$. Then, $\alpha_j^{(i,j)} = \alpha_j^{(i,t)}$. Similarly, $\alpha_t^{(i,j)} = \alpha_t^{(i,t)}$.

Finally, it is evident that $\alpha_i^{(i,j)} = 1 - \sum_{k \neq i} \alpha_k^{(i,j)} = 1 - \sum_{k \neq i} \alpha_k^{(i,t)} = \alpha_i^{(i,t)}$. We therefore conclude that $\{\alpha_k^{(i,j)}\}_{k=1}^{N} = \{\alpha_k^{(s,t)}\}_{k=1}^{N}$. ∎

In conclusion, there exists $\{\varepsilon_k\}_{k=1}^{N} \geq 0$ such that $\sum_{k=1}^{N} \varepsilon_k = 1$ and satisfying the following property: for all $P \in \mathbb{D}^N$ such that $P_i = P_j$ for some $i, j \in I$, we have $\varphi(P) = \sum_{k=1}^{N} \varepsilon_k\, e_{b(r_1(P_k))}$. Therefore, φ is a quasi-random dictatorship. ∎

Proposition 6 *Let $\mathbb{D}$ be a minimally rich domain. Suppose that for all $2 \leq t < N$, every unanimous and strategy-proof RSCF $g : \mathbb{D}^t \to \mathcal{L}(A)$ is a random dictatorship. If a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \mathcal{L}(A)$ is a quasi-random dictatorship, then φ is a random dictatorship.*

Proof: The proof proceeds in a sequence of lemmas. Let $\{\varepsilon_k\}_{k=1}^N \geq 0$ with $\sum_{k=1}^N \varepsilon_k = 1$ be the sequence that φ satisfies in Definition 20.

Lemma 13 *For all $P \in \mathbb{D}^N$, if there exist $i, j \in I$ such that $r_1(P_i) = r_1(P_j)$, then $\varphi(P) = \sum_{k=1}^N \varepsilon_k \, e_{b(r_1(P_k))}$.*

Proof: Fix a profile $P = (P_i, P_j, P_{-\{i,j\}})$. Assume that $r_1(P_i) = r_1(P_j) = x_0$ and $\tau(P_{-\{i,j\}})\backslash\{x_0\} = \{x_k\}_{k=1}^l$ where $0 \leq l \leq N-2$ and all elements in $\{x_k\}_{k=1}^l$ are distinct. If $\tau(P_{-\{i,j\}})\backslash\{x_0\} = \emptyset$, then $\tau(P) = \{x_0\}$ and unanimity gives the result. We complete the proof by considering $\tau(P_{-\{i,j\}})\backslash\{x_0\} \neq \emptyset$. By strategy-proofness and quasi-random dictatorship, we have $\varphi_{x_0}(P) = \varphi_{x_0}(P_i, P_i, P_{-\{i,j\}}) = \varepsilon_i + \varepsilon_j + \sum_{k \neq i,j} \varepsilon_k \, I(r_1(P_k), x_0) = \sum_{k=1}^N \varepsilon_k \, I(r_1(P_k), x_0)$.

Next, for the relative rankings of all elements in $\{x_k\}_{k=1}^l$ in P_i, we could assume without loss of generality that $x_t = r_{k_t}(P_i)$, $t = 1, \ldots, l$ and $k_1 < k_2 < \cdots < k_l$. By strategy-proofness, for all $s \geq 2$, $\sum_{\nu=1}^s \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{\nu=1}^s \varphi_{r_\nu(P_i)}(P) \leq \sum_{\nu=1}^s \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}})$.

Next, according to quasi-random dictatorship, we have that for $t = 1, \ldots, l$,

$$\begin{aligned}\sum_{\nu=1}^{k_t-1} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) &= \sum_{\nu=1}^{k_t-1} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}}) \\ &= \varepsilon_i + \varepsilon_j + \sum_{k \neq i,j} \varepsilon_k \Big[\sum_{s=0}^{t-1} I(r_1(P_k), x_s) \Big]\end{aligned}$$

and

$$\begin{aligned}\sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) &= \sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}}) \\ &= \varepsilon_i + \varepsilon_j + \sum_{k \neq i,j} \varepsilon_k \Big[\sum_{s=0}^{t} I(r_1(P_k), x_s) \Big]\end{aligned}$$

Consequently, for $t = 1, \ldots, l$, $\sum_{\nu=1}^{k_t-1} \varphi_{r_\nu(P_i)}(P) = \varepsilon_i + \varepsilon_j + \sum_{k \neq i,j} \varepsilon_k \Big[\sum_{s=0}^{t-1} I(r_1(P_k), x_s) \Big]$ and $\sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P) = \varepsilon_i + \varepsilon_j + \sum_{k \neq i,j} \varepsilon_k \Big[\sum_{s=0}^{t} I(r_1(P_k), x_s) \Big]$. Hence, for $t = 1, \ldots, l$, $\varphi_{x_t}(P) = \sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P) - \sum_{\nu=1}^{k_t-1} \varphi_{r_\nu(P_i)}(P) = \sum_{k \neq i,j} \varepsilon_k \, I(r_1(P_k), x_t) = \sum_{k=1}^N \varepsilon_k \, I(r_1(P_k), x_t)$.

Therefore, $\sum_{x \in \tau(P)} \varphi_x(P) \equiv \sum_{i=0}^l \varphi_{x_i}(P) = \sum_{k=1}^N \varepsilon_k = 1$. Then, for all $x \notin \tau(P)$, $\varphi_x(P) = 0$. In conclusion, $\varphi(P) = \sum_{k=1}^N \varepsilon_k e_{b(r_1(P_k))}$. ■

If $m < N$, then for all $P \in \mathbb{D}^N$, there always exist at least two voters who share a common maximal alternative. Then, Lemma 13 implies that φ is a random dictatorship. We complete the proof by considering $m \geq N$. Given a profile $P \in \mathbb{D}^N$ with $|\tau(P)| = N$, recall $\overline{W}(P) = \cup_{k=1}^{N} W\Big(P_k, \max(P_k, \tau(P_{-k}))\Big)$.

Lemma 14 *For all $P \in \mathbb{D}^N$ with $|\tau(P)| = N$, we have $|\tau(P) \cap \overline{W}(P)| \geq N - 1$.*

Proof: This lemma asserts that for every profile $P \in \mathbb{D}^N$ with $|\tau(P)| = N$, $\tau(P)$ and $\overline{W}(P)$ have at least $N-1$ alternatives in common.

Suppose not. Then there exists $P \in \mathbb{D}^N$ with $|\tau(P)| = N$ such that $|\tau(P) \cap \overline{W}(P)| < N - 1$. Hence, there exist $a, b \in \tau(P) \backslash \overline{W}(P)$. Since $|\tau(P)| = N$ and $N \geq 3$, we know that there exists $P_i \in \mathbb{D}^c$ for some $i \in I$ such that $c \notin \{a, b\}$. Let $\max(P_i, \tau(P_{-i})) = x$. If $x \notin \{a, b\}$, we know that $\{a, b\} \subseteq W(P_i, x)$ which implies that $\{a, b\} \subseteq \overline{W}(P)$. If $x = a$, then $b \in W(P_i, x)$ which implies that $b \in \overline{W}(P)$. If $x = b$, then $a \in W(P_i, x)$ which implies that $a \in \overline{W}(P)$. We have a contradiction. ■

Lemma 15 *For all $P \in \mathbb{D}^N$ with $|\tau(P)| = N$ and $x \in \overline{W}(P)$, we have $\varphi_x(P) = \sum_{k=1}^{N} \varepsilon_k \, I(r_1(P_k), x)$.*

Proof: Fix voter i. Assume without loss of generality that $\tau(P_{-i}) = \{x_k\}_{k=1}^{N-1}$, $x_t = r_{k_t}(P_i)$, $t = 1, \ldots, N-1$, $k_1 < k_2 < \cdots < k_{N-1}$ and $x_1 = r_1(P_j)$ for some $j \in I \backslash \{i\}$. By strategy-proofness, we have that $\sum_{\nu=1}^{s} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{\nu=1}^{s} \varphi_{r_\nu(P_i)}(P) \leq \sum_{\nu=1}^{s} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}})$ for all $s \geq k_1$.

According to quasi-random dictatorship, we have the following:

$$\sum_{\nu=1}^{k_1} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) = \sum_{\nu=1}^{k_1} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}}) = \varepsilon_i + \varepsilon_j$$

and for $t = 2, \ldots, N-1$,

$$\begin{aligned} \sum_{\nu=1}^{k_t - 1} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) &= \sum_{\nu=1}^{k_t - 1} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}}) \\ &= \varepsilon_i + \varepsilon_j + \sum_{k \neq i,j} \varepsilon_k \Big[\sum_{s=2}^{t-1} I(r_1(P_k), x_s) \Big] \end{aligned}$$

and

$$\begin{aligned} \sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) &= \sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}}) \\ &= \varepsilon_i + \varepsilon_j + \sum_{k \neq i,j} \varepsilon_k \Big[\sum_{s=2}^{t} I(r_1(P_k), x_s) \Big] \end{aligned}$$

Then, similar to the proof of Lemma 13, we have $\sum_{\nu=1}^{k_1} \varphi_{r_\nu(P_i)}(P) = \varepsilon_i + \varepsilon_j$ and $\varphi_{x_t}(P) = \sum_{k=1}^{N} \varepsilon_k \, I(r_1(P_k), x_t)$, for $t = 2, \ldots, N-1$. Since $|\tau(P)| = N$ and $\tau(P_{-\{i,j\}}) =$

$\{x_t\}_{t=2}^{N-1}$, we know that $\varphi_{r_1(P_k)}(P) = \varepsilon_k$ for all $k \neq i, j$. Then, $\sum_{\nu=1}^{k_1} \varphi_{r_\nu(P_i)}(P) + \sum_{k \neq i,j} \varphi_{r_1(P_k)}(P) = \sum_{k=1}^{N} \varepsilon_k = 1$. Therefore, for all $x \in W(P_i, x_1) \backslash \{x_t\}_{t=2}^{N-1}$, $\varphi_x(P) = 0$. In conclusion, for all $x \in W(P_i, x_1)$, $\varphi_x(P) = \sum_{k=1}^{N} \varepsilon_k \, I(r_1(P_k), x)$.

Applying the same argument to all other voters, we have $\varphi_x(P) = \sum_{k=1}^{N} \varepsilon_k \, I(r_1(P_k), x)$ for all $x \in \overline{W}(P)$. ■

From Lemma 15, we can infer that for all $P \in \mathbb{D}^N$ with $|\tau(P)| = N$, if $\tau(P) \subseteq \overline{W}(P)$, then $\varphi(P) = \sum_{k=1}^{N} \varepsilon_k \, e_{b(r_1(P_k))}$. By Lemmas 14 and 15, we know that for every $P \in \mathbb{D}^N$ with $|\tau(P)| = N$, the probabilities over at least $N-1$ elements of $\tau(P)$ in $\varphi(P)$ are revealed.

In the next lemma, we will identify properties that a profile P and $\varphi(P)$ must satisfy if $\varphi(P) \neq \sum_{k=1}^{N} \varepsilon_k \, e_{b(r_1(P_k))}$. Given a profile $P \in \mathbb{D}^N$ with $|\tau(P)| = N$, let $\bar{B}_i(P) = B\Big(P_i, \max(P_i, \tau(P_{-i}))\Big) \Big\backslash \{r_1(P_i)\}$, $i \in I$ and $\bar{B}(P) = \cap_{i=1}^{N} \bar{B}_i(P)$.

Lemma 16 *Let $P \in \mathbb{D}^N$ be a profile. If $\varphi(P) \neq \sum_{k=1}^{N} \varepsilon_k \, e_{b(r_1(P_k))}$, then the following conditions must be satisfied:*

- *(i)* $|\tau(P)| = N$.
- *(ii)* *There exists $i \in I$ such that $\varphi_{r_1(P_i)}(P) < \varepsilon_i$ and $\varphi_{r_1(P_k)}(P) = \varepsilon_k$ for all $k \neq i$.*
- *(iii)* $r_1(P_i) = \max(P_k, \tau(P_{-k}))$ *for all $k \neq i$.*
- *(iv)* $\varphi_{r_1(P_i)}(P) + \sum_{x \in \bar{B}(P)} \varphi_x(P) = \varepsilon_i$.
- *(v)* $\bar{B}(P) \neq \emptyset$. *Furthermore, there exists $x \in \bar{B}(P)$ such that $\varphi_x(P) > 0$.*

Proof: (i) Since $\varphi(P) \neq \sum_{k=1}^{N} \varepsilon_k \, e_{b(r_1(P_k))}$, Lemma 13 implies that $|\tau(P)| = N$.

(ii) According to Lemmas 14 and 15 and the hypothesis $\varphi(P) \neq \sum_{k=1}^{N} \varepsilon_k \, e_{b(r_1(P_k))}$, it must be true that $|\tau(P) \cap \overline{W}(P)| = N - 1$. Assume without loss of generality that $r_1(P_i) \notin \overline{W}(P)$. Then, by Lemma 15, we have that for all $k \neq i$, $\varphi_{r_1(P_k)}(P) = \varepsilon_k$. Consequently, $\varphi_{r_1(P_i)}(P) \leq 1 - \sum_{k \neq i} \varphi_{r_1(P_k)}(P) = \varepsilon_i$. This implies that $\varphi_{r_1(P_i)}(P) < \varepsilon_i$, otherwise $\varphi(P) = \sum_{k=1}^{N} \varepsilon_k e_{b(r_1(P_k))}$.

(iii) The proof of statement (ii) shows that $r_1(P_i) \notin \overline{W}(P)$, which implies that $r_1(P_i) = \max(P_k, \tau(P_{-k}))$ for all $k \neq i$.

(iv) Assume without loss of generality that $\max(P_i, \tau(P_{-i})) = r_1(P_j)$ for some $j \in I \backslash \{i\}$ and let $r_1(P_j) = r_s(P_i)$. Then, as we showed in the proof of Lemma 15, $\varepsilon_i + \varepsilon_j = \sum_{k=1}^{s} \varphi_{r_k(P_i)}(P) = \sum_{k=1}^{s-1} \varphi_{r_k(P_i)}(P) + \varphi_{r_1(P_j)}(P) = \varphi_{r_1(P_i)}(P) + \sum_{x \in \bar{B}_i(P)} \varphi_x(P) + \varphi_{r_1(P_j)}(P)$. Furthermore, statement (ii) implies that $\varphi_{r_1(P_j)}(P) = \varepsilon_j$. Hence, $\varphi_{r_1(P_i)}(P) + \sum_{x \in \bar{B}_i(P)} \varphi_x(P) = \varepsilon_i$. Next, since $\bar{B}_i(P) \backslash \bar{B}(P) \subset \overline{W}(P)$ and $\bar{B}(P) \subseteq \bar{B}_i(P)$, we have $\varphi_x(P) = 0$ for all $x \in \bar{B}_i(P) \backslash \bar{B}(P)$ by Lemma 15 and $\varphi_{r_1(P_i)}(P) + \sum_{x \in \bar{B}(P)} \varphi_x(P) = \varepsilon_i$.

(v) By statements (ii) and (iv), we know that $\sum_{x \in \bar{B}(P)} \varphi_x(P) > 0$, which implies that $\bar{B}(P) \neq \emptyset$ and furthermore, there exists $x \in \bar{B}(P)$ such that $\varphi_x(P) > 0$. ■

The voter i specified in statement (ii) of Lemma 16 is called the special voter of P. As we showed in the proof of statement (ii) of Lemma 16, we know that the peak of the special voter of P does not belong to $\overline{W}(P)$. It is evident that in a profile P with $\varphi(P) \neq \sum_{k=1}^{N} \varepsilon_k e_{b(r_1(P_k))}$, there exists a unique special voter.

We next show what property the sequence $\{\varepsilon_k\}_{k=1}^{N}$ must satisfy, when there exists a profile P^* such that $\varphi(P^*) \neq \sum_{k=1}^{N} \varepsilon_k e_{b(r_1(P_k^*))}$.

Lemma 17 *If there exists $P^* \in \mathbb{D}^N$ such that $\varphi(P^*) \neq \sum_{k=1}^{N} \varepsilon_k \, e_{b(r_1(P_k^*))}$, then $0 < \varepsilon_k < 1$, $k = 1, \ldots, N$.*

Proof: Suppose there exists $\varepsilon_k = 0$. Fix P_k^* (the k_{th} element of P^*). Define a RSCF: $g(P_{-k}) = \varphi(P_k^*, P_{-k})$ for all $P_{-k} \in \mathbb{D}^{N-1}$. The strategy-proofness of φ implies that g is strategy-proof. Next, Lemma 13 implies that g is unanimous. Furthermore, according to Lemma 16 (v), we know that there exists $x \notin \tau(P^*)$ such that $\varphi_x(P^*) > 0$. Therefore, $g_x(P_{-k}^*) = \varphi_x(P_k^*, P_{-k}^*) > 0$ where $x \notin \tau(P_{-k}^*)$, which implies that RSCF g is not a random dictatorship. This is a contradiction to the hypothesis of Proposition 6.

Next, suppose that there exists $\varepsilon_k = 1$. Then, there exists $j \neq k$ such that $\varepsilon_j = 0$, which would lead to the same contradiction. ∎

In the next lemma, we show it is true that for all $P \in \mathbb{D}^N$ with $|\tau(P)| = N$, $\varphi(P) = \sum_{k=1}^{N} \varepsilon_k e_{b(r_1(P_k))}$ by contradiction. Suppose φ is not a random dictatorship. Then we construct a RSCF $h : \mathbb{D}^2 \to \mathcal{L}(A)$ and show it is unanimous and strategy-proof and not a random dictatorship, which hence contradicts the hypothesis of Proposition 6.

Lemma 18 *For all $P \in \mathbb{D}^N$ with $|\tau(P)| = N$, we have $\varphi(P) = \sum_{k=1}^{N} \varepsilon_k \, e_{b(r_1(P_k))}$.*

Proof: Suppose RSCF φ is not a random dictatorship with respect to $\{\varepsilon_k\}_{k=1}^{N}$. Then, there exists $P^* \in \mathbb{D}^N$ such that $\varphi(P^*) \neq \sum_{k=1}^{N} \varepsilon_k e_{b(r_1(P_k^*))}$. By Lemma 16 ($ii$) and ($v$), we know that there exist a special voter of P^* and $y \notin \tau(P^*)$ such that $\varphi_y(P^*) > 0$. Assume without loss of generality that voter 1 be the special voter of P^*. Next, pick arbitrarily another voter, i.e., voter 2 and fix $P^*_{-\{1,2\}}$ (elements in P^* other than P_1^* and P_2^*). By Lemma 17, we can construct the following function: for all $P_1, P_2 \in \mathbb{D}$,

$$h(P_1, P_2) = \begin{cases} \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} e_{b(r_1(P_1))} + \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} e_{b(r_1(P_2))} & \text{if } \varphi_{r_1(P_1)}(P_1, P_2, P^*_{-\{1,2\}}) \geq \varepsilon_1 \\ & \text{and } \varphi_{r_1(P_2)}(P_1, P_2, P^*_{-\{1,2\}}) \geq \varepsilon_2 \\ \frac{1}{\varepsilon_1+\varepsilon_2} \Big[\varphi(P_1, P_2, P^*_{-\{1,2\}}) - \sum_{k=3}^{N} \varepsilon_k e_{b(r_1(P_k^*))} \Big] & \text{otherwise} \end{cases}$$

Note that Lemma 16 (ii) implies that it is impossible that $\varphi_{r_1(P_1)}(P_1, P_2, P^*_{-\{1,2\}}) < \varepsilon_1$ and $\varphi_{r_1(P_2)}(P_1, P_2, P^*_{-\{1,2\}}) < \varepsilon_2$ simultaneously. Therefore, given $P = (P_1, P_2, P^*_{-\{1,2\}})$, by Lemma 16 ($ii$) and ($iv$), when either $\varphi_{r_1(P_1)}(P) < \varepsilon_1$ or $\varphi_{r_1(P_2)}(P) < \varepsilon_2$, $h(P_1, P_2)$ must be specified as below:

if $\varphi_{r_1(P_1)}(P) < \varepsilon_1$, then

$$h(P_1, P_2) = \frac{1}{\varepsilon_1 + \varepsilon_2} \Big[\varphi_{r_1(P_1)}(P) \, e_{b(r_1(P_1))} + \sum_{x \in \bar{B}(P)} \varphi_x(P) \, e_{b(x)} + \varepsilon_2 \, e_{b(r_1(P_2))} \Big] \qquad (1)$$

where $\varphi_{r_1(P_1)}(P) + \sum_{x \in \bar{B}(P)} \varphi_x(P) = \varepsilon_1$; and if $\varphi_{r_1(P_2)}(P) < \varepsilon_2$, then

$$h(P_1,P_2)=\frac{1}{\varepsilon_1+\varepsilon_2}\Big[\varepsilon_1\, e_{b(r_1(P_1))}+\varphi_{r_1(P_2)}(P)\, e_{b(r_1(P_2))}+\sum_{x\in\bar{B}(P)}\varphi_x(P)\, e_{b(x)}\Big] \qquad (2)$$

where $\varphi_{r_1(P_2)}(P)+\sum_{x\in\bar{B}(P)}\varphi_x(P)=\varepsilon_2$.

Next, we will show that h is a unanimous and strategy-proof RSCF. Furthermore, to complete the proof of Lemma 18, we also show that h is not a random dictatorship which contradicts the hypothesis of Proposition 6.

Claim 1: Function h is a RSCF.

Firstly, if $\varphi_{r_1(P_1)}(P_1,P_2,P^*_{-\{1,2\}})\geq\varepsilon_1$ and $\varphi_{r_1(P_2)}(P_1,P_2,P^*_{-\{1,2\}})\geq\varepsilon_2$, it is evident that $h_x(P_1,P_2)\geq 0$ for all $x\in A$ and $\sum_{x\in A}h_x(P_1,P_2)=1$. Secondly, if either $\varphi_{r_1(P_1)}(P_1,P_2,P^*_{-\{1,2\}})<\varepsilon_1$ or $\varphi_{r_1(P_2)}(P_1,P_2,P^*_{-\{1,2\}})<\varepsilon_2$, either equation (1) or (2) above implies that $h_x(P_1,P_2)\geq 0$ for all $x\in A$ and $\sum_{x\in A}h_x(P_1,P_2)=1$. This completes the verification of Claim 1.

Claim 2: RSCF h is unanimous.

Let $r_1(P_1)=r_1(P_2)=a$. Then, by Lemma 13, we know that $\varphi_a(P_1,P_2,P^*_{-\{1,2\}})=\varepsilon_1+\varepsilon_2+\sum_{k=3}^N\varepsilon_k I(a,r_1(P^*_k))$. Hence, $\varphi_{r_1(P_1)}(P_1,P_2,P^*_{-\{1,2\}})=\varphi_a(P_1,P_2,P^*_{-\{1,2\}})\geq\varepsilon_1$ and $\varphi_{r_1(P_2)}(P_1,P_2,P^*_{-\{1,2\}})=\varphi_a(P_1,P_2,P^*_{-\{1,2\}})\geq\varepsilon_2$. Consequently, $h_a(P_1,P_2)=\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}+\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}=1$. This completes the verification of Claim 2.

Claim 3: RSCF h is not a random dictatorship.

Since we have assumed that voter 1 is the special voter of P^*, it is true that $\varphi_{r_1(P^*_1)}(P^*)<\varepsilon_1$ by Lemma 16 (*ii*). Consequently, $h(P^*_1,P^*_2)$ follows from equation (1). Next, since we have assumed that $\varphi_y(P^*)>0$ where $y\notin\tau(P^*)$ in the beginning proof of Lemma 18, we have that $h_y(P^*_1,P^*_2)>0$ and $y\notin\tau(P^*_1,P^*_2)$, which implies that h is not a random dictatorship. This completes the verification of Claim 3.

Claim 4: RSCF h is strategy-proof.

We consider the possible manipulation of voter 1 in h. Firstly, it is evident that the manipulation only occurs at (P_1,P_2) via P'_1 where either $h(P_1,P_2)=\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}e_{b(r_1(P_1))}+\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}e_{b(r_1(P_2))}$ and $h(P'_1,P_2)=\frac{1}{\varepsilon_1+\varepsilon_2}\Big[\varphi(P'_1,P_2,P^*_{-\{1,2\}})-\sum_{k=3}^N\varepsilon_k e_{b(r_1(P^*_k))}\Big]$, or $h(P_1,P_2)=\frac{1}{\varepsilon_1+\varepsilon_2}\Big[\varphi(P_1,P_2,P^*_{-\{1,2\}})-\sum_{k=3}^N\varepsilon_k e_{b(r_1(P^*_k))}\Big]$ and $h(P'_1,P_2)=\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}e_{b(r_1(P'_1))}+\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}e_{b(r_1(P_2))}$.

Secondly, if $\varphi(P_1,P_2,P^*_{-\{1,2\}})=\varepsilon_1 e_{b(r_1(P_1))}+\varepsilon_2 e_{b(r_1(P_2))}+\sum_{k=3}^N\varepsilon_k e_{b(r_1(P^*_k))}$, then $h(P_1,P_2)=\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}e_{b(r_1(P_1))}+\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}e_{b(r_1(P_2))}=\frac{1}{\varepsilon_1+\varepsilon_2}\Big[\varphi(P_1,P_2,P^*_{-\{1,2\}})-\sum_{k=3}^N\varepsilon_k e_{b(r_1(P^*_k))}\Big]$, which implies that there exists no manipulation at (P_1,P_2) via P'_1 or at (P'_1,P_2) via P_1.

Therefore, given two profiles $P=(P_1,P_2,P^*_{-\{1,2\}})$ and $P'=(P'_1,P_2,P^*_{-\{1,2\}})$ such that $\varphi(P)\neq\varepsilon_1 e_{b(r_1(P_1))}+\varepsilon_2 e_{b(r_1(P_2))}+\sum_{k=3}^N\varepsilon_k e_{b(r_1(P^*_k))}$ and $\varphi(P')\neq\varepsilon_1 e_{b(r_1(P'_1))}+\varepsilon_2 e_{b(r_1(P_2))}+\sum_{k=3}^N\varepsilon_k e_{b(r_1(P^*_k))}$, the manipulation at (P_1,P_2) via P'_1 may occur in following 4 cases. [12]

[12]Since $\varphi(P)\neq\varepsilon_1 e_{b(r_1(P_1))}+\varepsilon_2 e_{b(r_1(P_2))}+\sum_{k=3}^N\varepsilon_k e_{b(r_1(P^*_k))}$ and $\varphi(P')\neq\varepsilon_1 e_{b(r_1(P'_1))}+\varepsilon_2 e_{b(r_1(P_2))}+\sum_{k=3}^N\varepsilon_k e_{b(r_1(P^*_k))}$, we could apply Lemma 16 to P and P' in the analysis of the following 4 cases.

Case 1: (i) $\varphi_{r_1(P_1)}(P) \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P) \geq \varepsilon_2$, and (ii) $\varphi_{r_1(P_1')}(P') < \varepsilon_1$.

Now, $h(P_1', P_2)$ follows from equation (1). Then, given $u \in \mathbb{U}(P_1)$, the loss from misrepresentation in h is $U_1(P_1, P_2) - U_1(P_1', P_2) = \frac{1}{\varepsilon_1+\varepsilon_2}\Big[\varepsilon_1 u(r_1(P_1)) - \varphi_{r_1(P_1')}(P')u(r_1(P_1')) - \sum_{x \in \bar{B}(P')} \varphi_x(P')u(x)\Big] \geq 0$. This completes the verification of Case 1.

Case 2: (i) $\varphi_{r_1(P_1)}(P) \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P) \geq \varepsilon_2$, and (ii) $\varphi_{r_1(P_2)}(P') < \varepsilon_2$.

We first claim that this case only occurs when $N = 3$. Suppose not, i.e., $N \geq 4$. Since $\varphi_{r_1(P_1)}(P) \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P) \geq \varepsilon_2$, by Lemma 16 (*ii*), we assume without loss of generality that voter i, where $i \in \{3, \ldots, N\}$, is the special voter of P. Next, since $N \geq 4$, there must exist another voter, i.e., voter j such that $j \notin \{1, 2, i\}$. Furthermore, applying Lemma 16 (*iii*) to P, we know that $r_1(P_i)\, P_j\, r_1(P_2)$. In the other hand, $\varphi_{r_1(P_2)}(P') < \varepsilon_2$ indicates that voter 2 is the special voter of P'. Therefore, applying Lemma 16 (*iii*) to P', we have that $r_1(P_2)\, P_j\, r_1(P_i)$. Contradiction!

Now, by Lemma 16 (*i*), to simplify the notation, we can assume that $r_1(P_1) = a$, $r_1(P_2) = c$, $r_1(P_3^*) = f$ and $r_1(P_1') = d$, where a, c, f are mutually distinct and d, c, f are mutually distinct. (it is possible that $a = d$) Furthermore, $h(P_1', P_2)$ follows from equation (2). Therefore, given $u \in \mathbb{U}(P_1)$, the loss from misrepresentation in h is

$$U_1(P_1, P_2) - U_1(P_1', P_2) = \frac{1}{\varepsilon_1 + \varepsilon_2}\Big[\varepsilon_1 u(a) + \varepsilon_2 u(c) - \varepsilon_1 u(d) - \varphi_c(P')u(c) - \sum_{x \in \bar{B}(P')} \varphi_x(P')u(x)\Big]$$

where $\varepsilon_2 = \varphi_c(P') + \sum_{x \in \bar{B}(P')} \varphi_x(P')$.

To show that $U_1(P_1, P_2) - U_1(P_1', P_2) \geq 0$, We will consider the following 2 situations: dP_1c and cP_1d.

Firstly, we claim that if dP_1c then $U_1(P_1, P_2) - U_1(P_1', P_2) \geq 0$. Since either $a = d$ or aP_1d, to verify the claim, we only need to show that cP_1x for all $x \in \bar{B}(P')$ with $\varphi_x(P') > 0$. Suppose not, i.e., there exists $x^* \in \bar{B}(P')$ such that $\varphi_{x^*}(P') > 0$ and x^*P_1c. In profile P, since $\varphi_a(P) \geq \varepsilon_1$, $\varphi_c(P) \geq \varepsilon_2$ and $N = 3$, by Lemma 16 (*ii*) and (*iii*), we know that voter 3 is the special voter of P and fP_1c. Let $x' = \min(P_1, \{x^*, d, f\})$. Hence $x'P_1c$. Assume $x' = r_s(P_1)$. As we showed in the proof of Lemma 15, $\sum_{k=1}^{s} \varphi_{r_k(P_1)}(P) = \varepsilon_1 + \varepsilon_3$. Meanwhile, Lemma 16 (*ii*) implies that $\varphi_d(P') = \varepsilon_1$ and $\varphi_f(P') = \varepsilon_3$. Then, $\varphi_{x^*}(P') > 0$ implies that $\sum_{k=1}^{s} \varphi_{r_k(P_1)}(P) < \varepsilon_1 + \varepsilon_3 + \varphi_{x^*}(P') = \varphi_d(P') + \varphi_f(P') + \varphi_{x^*}(P') \leq \sum_{k=1}^{s} \varphi_{r_k(P_1)}(P')$. Therefore, voter 1 manipulates at P via P_1' in φ - a contradiction.

Next, we claim that that if cP_1d then $U_1(P_1, P_2) - U_1(P_1', P_2) \geq 0$. Now, it is evident that $a \neq d$. Since $c \notin \bar{B}(P')$, we separate $\bar{B}(P')$ into two parts S and T as follows: for all $x \in S$, xP_1c and for all $z \in T$, cP_1z. If $S = \emptyset$, then for all $x \in \bar{B}(P')$, cP_1x. Therefore, it is true that $U_1(P_1, P_2) - U_1(P_1', P_2) = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}\Big[u(a) - u(d)\Big] + \frac{1}{\varepsilon_1+\varepsilon_2}\sum_{x \in \bar{B}(P')} \varphi_x(P')\Big[u(c) - u(x)\Big] \geq 0$.

Next, consider $S \neq \emptyset$. Let $x^* = \max(P_1, S)$. Then it is true that (i) either aP_1x^* or $a = x^*$, (ii) x^*P_1c, (iii) cP_1d and (iv) cP_1z for all $z \in T$ (if $T \neq \emptyset$). Furthermore,

$U_1(P_1, P_2) - U_1(P_1', P_2)$ could be modified as follows:

$$\begin{aligned}
& U_1(P_1, P_2) - U_1(P_1', P_2) \\
= \; & \frac{1}{\varepsilon_1 + \varepsilon_2}\Big[\varepsilon_1 u(a) + \varepsilon_2 u(c) - \varepsilon_1 u(d) - \varphi_c(P')u(c) - \sum_{x \in S} \varphi_x(P')u(x) - \sum_{z \in T} \varphi_z(P')u(z)\Big] \\
\geq \; & \frac{1}{\varepsilon_1 + \varepsilon_2}\Big[\varepsilon_1 u(a) + \varepsilon_2 u(c) - \varepsilon_1 u(d) - \varphi_c(P')u(c) - u(x^*)\sum_{x \in S} \varphi_x(P') - \sum_{z \in T} \varphi_z(P')u(z)\Big] \\
= \; & \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\Big[u(a) - u(x^*)\Big] + \frac{\varepsilon_1 - \sum\limits_{x \in S} \varphi_x(P')}{\varepsilon_1 + \varepsilon_2}\Big[u(x^*) - u(c)\Big] + \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\Big[u(c) - u(d)\Big] \\
& + \frac{1}{\varepsilon_1 + \varepsilon_2}\sum_{z \in T} \varphi_z(P')\Big[u(c) - u(z)\Big]
\end{aligned}$$

Therefore, according to the relative rankings in P_1 specified above, to show that $U_1(P_1, P_2) - U_1(P_1', P_2) \geq 0$, it suffices to show $\varepsilon_1 \geq \sum_{x \in S} \varphi_x(P')$.

Assume $\min(P_1, S) = y^*$ and let $z^* = \min(P_1, \{f, y^*\})$. Assume $z^* = r_s(P_1)$. Hence, $\{r_k(P_1)\}_{k=1}^{s} = B(P_1, z^*) \cup \{z^*\}$. In profile P, since $\varphi_a(P) \geq \varepsilon_1$, $\varphi_c(P) \geq \varepsilon_2$ and $N = 3$, by Lemma 16 (*ii*) and (*iii*), we know that voter 3 is the special voter of P and fP_1c. Hence, z^*P_1c. Therefore, as we showed in the proof of Lemma 15, we have that $\sum_{k=1}^{s} \varphi_{r_k(P_1)}(P) = \varepsilon_1 + \varepsilon_3$. Next, in profile P', by Lemma 16 (*ii*) and (*iv*), we know that for all $z \notin \{d, c, f\} \cup \bar{B}(P')$, $\varphi_z(P') = 0$. Furthermore, since $[B(P_1, z^*) \cup \{z^*\}] \cap \{d, c, f\} = \{f\}$ and $[B(P_1, z^*) \cup \{z^*\}] \cap \bar{B}(P') = S$, we have that $\sum_{k=1}^{s} \varphi_{r_k(P_1)}(P') \equiv \sum_{x \in B(P_1, z^*) \cup \{z^*\}} \varphi_x(P') = \varphi_f(P') + \sum_{x \in S} \varphi_x(P') = \varepsilon_3 + \sum_{x \in S} \varphi_x(P')$ (Lemma 16 (*ii*) implies that $\varphi_f(P') = \varepsilon_3$). Then, the strategy-proofness of φ implies that $\varepsilon_1 \geq \sum_{x \in S} \varphi_x(P')$. This completes the verification of Case 2.

Case 3: (i) $\varphi_{r_1(P_1)}(P) < \varepsilon_1$, and (ii) $\varphi_{r_1(P_1')}(P') \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P') \geq \varepsilon_2$.

Now, $h(P_1, P_2)$ follows from equation (1). Then, given $u \in \mathbb{U}(P_1)$, the loss from misrepresentation in h is

$$U_1(P_1, P_2) - U_1(P_1', P_2) = \frac{1}{\varepsilon_1 + \varepsilon_2}\Big[\varphi_{r_1(P_1)}(P)u(r_1(P_1)) + \sum_{x \in \bar{B}(P)} \varphi_x(P)u(x) - \varepsilon_1 u(r_1(P_1'))\Big]$$

where $\varphi_{r_1(P_1)}(P) + \sum_{x \in \bar{B}(P)} \varphi_x(P) = \varepsilon_1$.

Firstly, since $\varphi_{r_1(P_1)}(P) < \varepsilon_1$ and $\varphi_{r_1(P_1')}(P') \geq \varepsilon_1$, strategy-proofness implies that $r_1(P_1) \neq r_1(P_1')$. Next, it is evident that $r_1(P_1)\, P_1\, r_1(P_1')$. Therefore, to show $U_1(P_1, P_2) - U_1(P_1', P_2) \geq 0$, it suffices to show that for all $x \in \bar{B}(P)$ with $\varphi_x(P) > 0$ and $x \neq r_1(P_1')$, $xP_1r_1(P_1')$.

Now, suppose there exists $z' \in \bar{B}(P)$ such that $\varphi_{z'}(P) > 0$ and $r_1(P_1')P_1z'$. Firstly, $\bar{B}(P) \subseteq \bar{B}_1(P)$ implies that $z' \in \bar{B}_1(P)$. Let s_1 and s_2 be such that $r_1(P_1') = r_{s_1}(P_1)$ and $z' = r_{s_2}(P_1)$. Hence, $1 < s_1 < s_2$. As we showed in the proof of Lemma 16 (*iv*), $\varphi_{r_1(P_1)}(P) + \sum_{x \in \bar{B}_1(P)} \varphi_x(P) = \varepsilon_1$. Then, $\varphi_{z'}(P) > 0$ and $z' \in \bar{B}_1(P)$ imply that $\sum_{k=1}^{s_1} \varphi_{r_k(P_1)}(P) < \sum_{k=1}^{s_2} \varphi_{r_k(P_1)}(P) \leq \varepsilon_1 = \varphi_{r_1(P_1')}(P') \leq \sum_{k=1}^{s_1} \varphi_{r_k(P_1)}(P')$.

Therefore, voter 1 manipulates at P via P_1' in φ - a contradiction. This complete the verification of Case 3.

Case 4: (i) $\varphi_{r_1(P_2)}(P) < \varepsilon_2$, and (ii) $\varphi_{r_1(P_1')}(P') \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P') \geq \varepsilon_2$.

As in Case 2, we can claim that this case only occur when $N = 3$. Now, $h(P_1, P_2)$ follows from equation (2). Since $\varphi_{r_1(P_2)}(P) < \varepsilon_2$, we know that voter 2 is the special voter of P by Lemma 16 (*ii*). Hence, for all $x \in \bar{B}(P)$, $xP_1r_1(P_2)$ by Lemma 16 (*iii*). Then, given $u \in \mathbb{U}(P_1)$, the loss from manipulation in h is $U_1(P_1, P_2) - U_1(P_1', P_2) = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}\Big[u(r_1(P_1)) - u(r_1(P_1'))\Big] + \frac{1}{\varepsilon_1+\varepsilon_2}\sum_{x\in\bar{B}(P)}\varphi_x(P)\Big[u(x) - u(r_1(P_2))\Big] \geq 0$. This completes the verification of Case 4.

Finally, using symmetric arguments for voter 2, we conclude that h is strategy-proof. This completes the verification of Claim 4 and the proof of Lemma 18. ■

This concludes the proof of Proposition 6 and Theorem 3. ■