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Contracting over Prices*

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Abstract

We define a solution concept, perfectly contracted equilibrium, for an intertemporal exchange economy where agents are simultaneously price takers in spot commodity markets while engaging in efficient, non-Walrasian contracting over future prices. Without requiring that agents have perfect foresight, we show that perfectly contracted equilibrium outcomes are a subset of Pareto optimal allocations. It is a robust possibility for perfectly contracted equilibrium outcomes to differ from Arrow-Debreu equilibrium outcomes. We show that both centralized banking and retrading with bilateral contracting can lead to perfectly contracted equilibria.

Keywords: equilibrium, future prices, uncertainty, contracts.

JEL classification: D5, D70, D72, D84.

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1 Introduction

In an Arrow-Debreu economy (Arrow [1953], Debreu [1959], Arrow and Debreu [1954]), when agents trade Arrow-Debreu contingent contracts, equilibrium resolves all uncertainty (i.e. uncertainty over future states of the world and uncertainty over future prices). In the formulation of an equilibrium of a sequence economy where securities replace markets for contingent claims, Arrow [1953] and Radner [1972] adopt the notion of a rational expectations equilibrium where agents are assumed to know at each date the map between future realized states and future equilibrium commodity and asset prices. Thus, in such an equilibrium with securities the realized state must resolve all uncertainty including the uncertainty of future prices.

This paper examines trade in a simple two period economy without uncertainty over states of the world. The Radner formulation alluded to above specializes to the notion of a perfect foresight equilibrium and requires agents to trade a security competitively and have perfect foresight over the spot market prices in the second period. Under these conditions, the allocations arising out of Radner equilibria coincide with Arrow-Debreu allocations. We study the model without insisting that intertemporal transfers be organized via a security market where agents necessarily trade competitively and, importantly, we do not impose perfect foresight.[1] Agents operate under uncertainty over future prices, they need not agree on the distribution of future prices, and are compelled to make decisions on the basis of their beliefs over future prices.

In our paper, the trading of such price uncertainty is accomplished by supplementing spot trade in commodities with contracts over future prices that belong to a price set rather than by Arrow-Debreu forward state contingent contracts. Traditionally, adverse selection and moral hazard problems have been analyzed by introducing contracts in general equilibrium models. In our model there is a somewhat more primitive role for introducing contracts and it is tied to the fact that agents need not have perfect foresight of future prices. Under our contracts, an agent who receives wealth in period one, commits to a delivery of wealth in period two that is *contingent* on the spot prices prevailing in the period two market. We require these contracts to be individually rational and efficient.

We define a notion of equilibrium, *perfectly contracted* equilibrium, for an exchange economy where commodities are traded competitively in spot markets and agents contract over

[1] That the assumption of perfect foresight is extraordinarily strong is a view expressed by various scholars. A case in point is Radner's own critique of perfect foresight (Radner [1972], page 142). Exploring the possibilities for dynamic resource allocation without this assumption as we do in this paper appears to be an avenue worth exploring. The Radner equilibrium arises as a special case of our formulation where agents are subjectively certain about the prices that support the Arrow-Debreu allocation and the asset is traded competitively.

future prices that belong to a price set. We show that a subset of Pareto optimal allocations, called *attainable* allocations, can be obtained as equilibrium allocations. For an attainable allocation to emerge as an equilibrium allocation, we require that agents contract over a price set such that the agents who give up wealth in period one attach sufficiently high probability to this price set and furthermore, this price set contains the price vector that supports the attainable allocation. A consequence of our formulation of individual rationality is that there are no restrictions on the price forecasts of agents who receive wealth in the first period. Ours is thus a model of trade where the forecasts of agents may be rather disparate. In particular, agents need not even agree on the sets of prices that receive positive probability under their respective expectations. By example, we show that it is a robust possibility for (i) perfectly contracted equilibrium outcomes to differ from Arrow-Debreu equilibrium outcomes; (ii) perfectly contracted equilibria exist even when the unique Arrow-Debreu price does not belong to the commonly forecasted price set.

An integral feature of the model we propose is that agents behave simultaneously as Walrasian agents in the spot markets, but in a non-Walrasian manner in the contracting process.[2] In particular, our formulation of the individual rationality condition reflects this feature since agents believe that they can consume their Walrasian demands for each possible period two price that belongs to the set of prices that agents contract over. If a second period price belongs to this set, it reflects the fact that agents believe that this price might arise in the second period spot market, and if it does, they can trade their demand at this price. The agents disregard feasibility issues as agents in a Walrasian setting ought to in order to be price takers. In the contracting process, agents are assumed to interact with other spot market price takers in a non-Walrasian setting and here the contracts they exchange are constrained by their endowments. This model reflects a feature of trade in market environments where agents act as price takers in many markets but do exercise market power in bilateral negotiations or negotiations within small groups of agents. These instances arise in negotiations between employers and worker, or in transactions involving real estate etc. But these transctions appear not to impinge on these agents acting as price takers in other markets. In fact the prevailing prices at these markets where the agents act as price takers affects their dealings in the markets where they exercise market power since these prices affect the value of their endowments and hence their reservation utilities etc. The market clearing prices in the markets where the agents act as price takers are in turn affected by these non-Walrasian exchanges. Our equilibrium concept takes cognizance of this interaction between the contracting process and the markets where agents act as Walrasian agents, and requires market clearing while requiring the contracts emerging from the non-Walrasian interaction to be individually rational and efficient.

[2]Ghosal and Polemarchakis [1997] introduce a related class of models which study a general equilibrium setting where agents are price takers but also simultaneously choose strategies.

The assumption that the contracts over prices written by agents are efficient and individually rational requires economy-wide coordination between agents. We first show that these contracts are readily computed by a centralized institution (central bank) by maximizing the weighted sum of indirect utilities subject to individual rationality constraints. It bears mentioning that the contracts generating attainable allocations are chosen even though the bank is not constrained to choosing contracts that generate feasible demands.[3] This feature affirms further the relevance of attainable allocations in our intertemporal resource allocation problem. We next show that a decentralized process of bilateral contracting with an element of retrading can also achieve the required level of coordination.

Our model shows that for Pareto optimal allocations to result from intertemporal trade, one does not need perfect foresight; what is needed is that agents be able to write contracts that exhaust gains to trade conditional on particular second period prices arising in the future. This in our setting is accomplished via perfect contracting in price contingent contracts.[4] The problems in achieving perfect contracting are perhaps better understood vis a vis problems faced in ensuring perfect foresight. Indeed, there is a large and evolving literature on difficulties posed by issues of incentives, search, matching etc in achieving what we term perfect contracting. Our paper sets up a framework where our understanding of these issues can be brought to bear on the problem of efficient intertemporal allocation of resources and help in formalizing a theory of the second best. On the other hand, the informational and computational requirements, the level of coordination on expectations etc that underlie a perfect foresight equilibrium, are very different and in our view present considerably greater conceptual challenges.

Our formulation of contracts over future prices is distinct from, and complements the analysis of "Endogenous Uncertainty" (Kurz [1974] introduced the term to describe it). To deal with the issue of price uncertainty, Kurz [1974] proposed that agents trade this uncertainty using Price Contingent Contracts (PCC). Further work along this line includes Svensson [1981], Henrotte [1996], Kurz [1994], Kurz and Wu [1996][5]. The key differences

[3]We note that our formulation of the maximization problem that the central bank solves is distinct from the maximization problem of a conventional central planner whose objective is to obtain a Pareto optimal allocation by maximizing a weighted sum of utilities. This is because in our model after the central bank selects a contract, agents act as Walrasian price takers and the central bank, unlike a conventional central planner, ignores the issues of feasibility that would arise when agents post their Walrasian demands after the contracts are executed.

[4]Contracts over future prices are a common feature of a variety of market transactions such as the use of options and other such contracts in the planning of projects involving large-scale investment to hedge against the risk of fluctuating demand and price risk by natural resource companies, manufacturers and football clubs. Evidently, such contracts are redundant in a scenario where markets are complete and all agents know the map from time (and states of the world) to prices.

[5]Kurz and Wu [1996] study a dynamic stochastic overlapping generations economy with complete com-

between our work and this literature is that in our paper contracts over future prices are not themselves traded in competitive markets and moreover, such contracts are limited to only those prices that belong to a price set.

The rest of the paper is organized as follows. The next section describes the formal model, states and proves the key result of the paper, and examines the distinction between a perfectly contracted equilibrium and an Arrow-Debreu equilibrium in the context of a simple but robust example. Section 3 studies aspects of contracting processes that lead to perfectly contracted equilibria. Section 4 contains results on centralized contracting and bilateral contracting. The last section concludes. All proofs of the results reported in the main text are gathered in the Appendix.

2 The model and some results

2.1 A two period exchange economy

We analyze a two period economy with a finite number of agents and finitely many commodities in each of the two periods. The set of commodities in period one is $M = \{1, ..., M\}$, indexed by $m = 1, ..., M$, $M \geq 1$; the set of commodities in period two is $N = \{1, ..., N\}$, indexed by $n = 1, ..., N$, $N \geq 2$; the set of agents is $I = \{1, ..., I\}$, indexed by $i = 1, ..., I$. A consumption plan for agent i is a vector $x^i = (x^i_1, x^i_2) = ((x^i_{11}, ...x^i_{1M}), (x^i_{21}, ...x^i_{2N})) \in \Re^{M+N}_+$, $\forall i \in I$. The endowment of each agent i is represented as $\omega^i = (\omega^i_1, \omega^i_2) \in \Re^{M+N}_+$, $\forall i \in I$. An allocation is a vector $(x_1, x_2) = ((x^i_1)_{i\in I}, (x^i_2)_{i\in I})$. The endowment vector of the economy is given by $(\omega_1, \omega_2) = ((\omega^i_{11}, ...\omega^i_{1M})_{i\in I}, (\omega^i_{21}, ...\omega^i_{2N})_{i\in I})$.

We make the following assumptions and these will be maintained throughout the paper.

A 1 The preferences of each agent i is represented by a utility function $U^i(x^i_1, x^i_2) = u^i_1(x^i_1) + u^i_2(x^i_2)$, $\forall i \in I$, where at each $t = 1, 2$, $u^i_t(\cdot)$ is strictly increasing, strictly quasi-concave and at each $x^i_t \gg 0$, is twice continuously differentiable.

A 2 The aggregate endowment vector of the economy satisfies $\sum_{i\in I} \omega^i \gg 0$ with $\omega^i > 0$, $\forall i \in I$.

petitive markets for trading price uncertainty. They draw a connection between Rational beliefs equilibrium (a weaker notion than rational expectations), and a particular notion of Pareto optimality. Our work is motivated by similar concerns, but our formal models and hypotheses are very different. Beliefs are in our setting price forecasts and we impose no consistency condition on beliefs of the sort used in the aforementioned paper.

The assumption of time separable utilities simplifies the exposition considerably. It is possible to extend the analysis of the paper to the case where this assumption does not hold.

A spot price in period one is an M-vector $p_1 > 0$ that belongs to $\mathcal{P}_1 = \{l \in \Re^M_+ | l_M = 1\}$ where the price of the last commodity is normalized to one and analogously, a spot price in period two is an N-vector $p_2 > 0$ that belongs to $\mathcal{P}_2 = \{l \in \Re^N_+ | l_N = 1\}$. Each agent i is endowed with an expectation function $\phi^i(p_1)$ which maps $\mathcal{P}_1$ into the space of probability distributions on $\mathcal{P}_2$. The support of this probability distribution will be denoted $\text{supp}\phi^i(p_1)$, $\forall i \in I$.[6]

2.2 Contracts and perfectly contracted equilibria

Before proceeding to a description of contracts, we note that in the absence of an asset or contracts, at a given period one spot price $p_1 \in \mathcal{P}_1$, an agent solves in period one the problem Max $u^i_1(x^i_1)$ subject to $p_1 \cdot x^i_1 \leq p_1 \cdot \omega^i_1$. Analogously, in period two, given a price $p_2 \in \mathcal{P}_2$, the agent solves the problem Max $u^i_2(x^i_2)$ subject to $p_2 \cdot x^i_2 \leq p_2 \cdot \omega^i_2$. The respective solutions to these problems are denoted $\overline{\xi}^i_1(p_1, p_1 \cdot \omega^i_1)$ and $\overline{\xi}^i_2(p_2, p_2 \cdot \omega^i_2)$. We will employ the same notation for the solutions to these problems when p_1 and p_2 are non-negative price vectors in $\Re^M_+$ and $\Re^N_+$ which are not normalized to lie in $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively.

We model a scenario where at a current price $p_1 \in \mathcal{P}_1$, agents contract over a set of period two spot prices Π that at least one agent deems is likely to occur, i.e., $\Pi \cap \text{supp}\phi^i(p_1) \neq \varnothing$ for some $i \in I$.

We first introduce the notion of a price contingent allocation. For a given $\Pi \subset \mathcal{P}_2$, a price contingent allocation with respect to Π specifies $(z_1, z_2(p_2))$ where $z^i_1 \in \Re^M_+$ is a vector of period one commodities for each $i \in I$, and for each $p_2 \in \Pi$, $z^i_2(p_2) \in \Re^N_+$ is vector of period two commodities for each $i \in I$. The following is our notion of a contract.

Definition 1 *Given a set of second period prices $\Pi \subset \mathcal{P}_2$, a contract, denoted s_Π, is a price contingent allocation $(z_1, z_2(p_2))$ with respect to Π such that*

(i) $\sum_{i\in I} z^i_1 = \sum_{i\in I} \omega^i_1$;

(ii) $\sum_{i\in I} z^i_2(p_2) = \sum_{i\in I} \omega^i_2$, $\forall p_2 \in \Pi$;

(iii) $z^i_2(p_2) = \omega^i_2$, $\forall i \in I$ *if* $p_2 \notin \Pi$.

[6]Here, "supp" denotes the support of a probability distribution.

In our formulation, the process of contracting over prices is embedded in the Walrasian adjustment process as follows. Given a period one price p_1, the profile of expectation functions $(\phi^i)_{i\in I}$ determines what prices agents expect might prevail in the period two spot markets. Before the agents announce their demands at p_1, they explore the possibility of writing price contingent contracts over some set $\Pi \subset \mathcal{P}_2$ of period two prices with other agents in order to transfer wealth across the two dates in a mutually beneficial manner given their expectations over future prices. A contract specifies a reallocation of the first period endowments and a price contingent reallocation of second period endowments among the agents. We do not address here the specific story underlying how such a contract is reached.[7] Our intention here is to specify conditions on the contracts that emerge out of this contracting process that ensure Pareto optimality under the equilibrium concept that we propose for this model. At the end of this contracting phase, the first period endowments are exchanged so that z_1 becomes the endowment vector in period one (if some agent i does not enter into any contract, then $z_1^i = \omega_1^i$). Given this new endowment, agents announce their utility maximizing demands at the price p_1. Of course, p_1 need not induce market clearing at these demands, in which case the auctioneer announces a different period one price and the process repeats itself. In period two, if a second period spot price $p_2 \in \Pi$ is realized, then the agents exchange endowments in accordance with $z_2(p_2)$ and then place their utility maximizing demands. If $p_2 \notin \Pi$, each agent $i \in I$ announces her utility maximizing demand with endowment fixed at ω_2^i.

We now turn to a description of what agents believe they can consume if they agree on a contract s_Π in period one given a spot price $p_1 \in \mathcal{P}_1$. At a period one price $p_1 \in \mathcal{P}_1$, for each $p_2 \in \mathcal{P}_2$, we associate to a contract s_Π, the price contingent vector $\xi^i(p, s_\Pi) = (\xi_1^i(p_1, s_\Pi), \xi_2^i(p_2, s_\Pi))$, $\forall i \in I$, which is constructed as follows. For each $i \in I$, let $\xi_1^i(p_1, s_\Pi)$ be the solution to Max $u_1^i(x_1^i)$ subject to $p_1 \cdot x_1^i \leq p_1 \cdot z_1^i$. Given $p_2 \in \mathcal{P}_2$, if (i) $p_2 \in \Pi$, define $\xi_2^i(p_2, s_\Pi)$ as the solution to Max $u_2^i(x_2^i)$ subject to $p_2 \cdot x_2^i \leq p_2 \cdot z_2^i(p_2)$; (ii) else define $\xi_2^i(p_2, s_\Pi)$ as the solution to Max $u_2^i(x_2^i)$ subject to $p_2 \cdot x_2^i \leq p_2 \cdot \omega_2^i$.

For a particular contract to be agreed upon at a period one price p_1, it must satisfy some form of individual rationality for each agent. We now turn to our formulation of individual rationality. The first part of our individual rationality requirement is that for *each* $p_2 \in \Pi$, the indirect utility of accepting s_Π for each agent exceeds the indirect utility of not accepting the contract. The second part of the individual rationality requirement pertains to the subset of agents who give up wealth in period one for a price contingent amount in the second period.[8] If a contract s_Π such that $z_1 \neq \omega_1$ is accepted by all agents at the price p_1, there is a subset of agents, denoted $I^-(p_1, z_1)$, for whom one has $p_1 \cdot \omega_1^i > p_1 \cdot z_1^i$, $\forall i \in I^-(p_1, z_1)$ However, if the second period price does not belong to Π they have contracted over, they receive no

[7] This may be the result of some non-cooperative procedure or some cooperative process. We defer a discussion of such processes to Sections 3 and 4 below.

[8] We discuss an alternative notion of individual rationality in Subsection 2.5 below.

compensation for the wealth they give up in period one. For these agents to want to accept such a contract, it must be that Π receives "enough" probability under their beliefs so that in expected utility terms, their payoff of accepting the contract is no less than their utility of not doing so.

Definition **2** *The contract s_Π is individually rational at p_1 if*

(*i*) $u_1^i(\xi_1^i(p_1, s_\Pi)) + u_2^i(\xi_2^i(p_2, s_\Pi)) \geq u_1^i(\overline{\xi}_1^i(p_1, p_1 \cdot \omega_1^i))) + u_2^i(\overline{\xi}_2^i(p_2, p_2 \cdot \omega_2^i))$, $\forall p_2 \in \Pi$ *and* $\forall i \in I$.

(*ii*) $u_1^i(\xi_1^i(p_1, s_\Pi)) + \int_{\mathcal{P}_2} u_2^i(\xi_2^i(p_2, s_\Pi)) d\phi^i(p_1) \geq u_1^i(\overline{\xi}_1^i(p_1, p_1 \cdot \omega_1^i))) + \int_{\mathcal{P}_2} u_2^i(\overline{\xi}_2^i(p_2, p_2 \cdot \omega_2^i)) d\phi^i(p_1)$, $\forall i \in I^-(p_1, z_1)$.

Observation **1** We note that the individually rationality requirement implies that s_Π must be such that $\Pi \cap \text{supp}\phi^i(p_1) \neq \varnothing$, $\forall i \in I^-(p_1, z_1)$, i.e., the set of period two prices the agents contract over have non-empty intersection with the set of prices that get positive probability under the expectations of all agents belonging to $I^-(p_1, z_1)$. If one assumes that for all agents in $I^-(p_1, z_1)$, the support of the expectations functions are identical and are given by Π, then one can dispense with Condition (*ii*) of the individual rationality requirement. Our proof of the Theorem that follows and subsequent examples will exploit this fact and verify only Condition (*i*) of Definition 2 in order to establish the individual rationality of contracts. ■

Our final requirement is that contracting be efficient. This states that given a period one price $p_1 \in \mathcal{P}_1$, the contract should be such that for any pair of agents i, j, for every price pair $p = (p_1, p_2)$, $p_2 \in \Pi$, they have contracted over, their intertemporal marginal rates of substitution for some pair of commodities, should be equalized when evaluated at the bundles they anticipate they will consume at that price pair. Thus the marginal rates of substitution are evaluated at the bundles $\xi^i(p, s_\Pi)$ that the agents anticipate they will consume if they accept the contract and if the price pair p is realized.

Definition **3** *A contract s_Π is efficient at p_1 if for any $i, j \in I$ and $\forall p_2 \in \Pi$, we have $MRS^i_{q,r}(\xi^i(p, s_\Pi)) = MRS^j_{q,r}(\xi^j(p, s_\Pi))$ for some $q \in M$ and $r \in N$.*

These consumption plans $\xi^i(p, s_\Pi)$ are merely Walrasian perceptions of what agents believe they can purchase at the spot price pair $p = (p_1, p_2)$ if they agree on the contract s_Π. These plans may not of course be feasible. They will however be feasible at the equilibrium that we will propose for this model. For our notion of an equilibrium in this formulation, we work with a price pair $p = (p_1, p_2)$ and a contract that satisfies the individual rationality and efficiency requirements. A price pair and an individually rational, efficient contract constitute an equilibrium provided that p_1 clears markets when agents period one demands are given by $\xi_1^i(p_1, s_\Pi)$, p_2 clears second period markets when the period two demands are given by $\xi_2^i(p_2, s_\Pi)$, and importantly, p_2 actually belongs to set Π that the agents had contracted over.

Definition **4** *A perfectly contracted equilibrium is a triple (p^*, x^*, s_Π) comprising a price vector $p^* = (p_1^*, p_2^*)$, an allocation x^* and a contract $s_\Pi = (z_1, z_2(p_2))$ that is an efficient and individually rational contract at p_1^*, such that*

(*i*) $p_2^* \in \Pi$.

(*ii*) $x_1^{*i} = \xi_1^i(p_1^*, s_\Pi)$, $\forall i \in I$ *and* $\sum_{i\in I} x_1^{*i} = \sum_{i\in I} \omega_1^i$.

(*iii*) $x_2^{*i} = \xi_2^i(p_2^*, s_\Pi)$, $\forall i \in I$ *and* $\sum_{i\in I} x_2^{*i} = \sum_{i\in I} \omega_2^i$.

An allocation x^* that is part of a perfectly contracted equilibrium will be referred to as a perfectly contracted allocation.

We conclude this subsection by defining attainable allocations; these are a subset of Pareto optimal allocations. By the assumption that each agents utility is separable over time, we can define the period t economy, $t = 1, 2$, as the static economy where the endowment vector is ω_t and each agent's utility is $u_t^i(x_t^i)$, $\forall i \in I$. Consider a Pareto optimal allocation $\overline{x} = (\overline{x}_1, \overline{x}_2) \gg 0$. The subvector $\overline{x}_t$, $t = 1, 2$, is Pareto optimal in the period t economy. Note that in each period t economy, given that preferences are assumed to be convex and $\overline{x}_t \gg 0, t = 1, 2$, there exists a price vector $\overline{p}_t(\overline{x}) \in \mathcal{P}_t$ that supports the subvector $\overline{x}_t$ as a Walrasian allocation in the period t economy provided the endowment vector ω_t is suitably redistributed.

Given the Pareto optimal allocation $\overline{x} = (\overline{x}_1, \overline{x}_2) \gg 0$, interpret the supporting price subvectors $\overline{p}_1(\overline{x})$ and $\overline{p}_2(\overline{x})$ as spot prices and recall that $\overline{\xi}_1^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \omega_1^i)$ is the solution to Max $u_1^i(x_1^i)$ subject to $\overline{p}_1(\overline{x}) \cdot x_1^i \leq \overline{p}_1(\overline{x}) \cdot \omega_1^i$ and $\overline{\xi}_2^i(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \omega_2^i)$ is the solution to Max $u_2^i(x_2^i)$ subject to $\overline{p}_2(\overline{x}) \cdot x_2^i \leq \overline{p}_2(\overline{x}) \cdot \omega_2^i$. Finally, let $\mathcal{U}_{\overline{x}}^i = u_1^i(\overline{\xi}_1^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \omega_1^i)) + u_2^i(\overline{\xi}_2^i(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \omega_2^i))$. We now have the following definition.

Definition **5** *The Pareto optimal allocation $\overline{x} \gg 0$ is attainable if $U^i(\overline{x}^i) \geq \mathcal{U}_{\overline{x}}^i$, $\forall i \in I$. It is strongly attainable if the inequality is strict for each individual $i \in I$.*

Observation **2** The restriction of attainability rules out some Pareto optimal allocations that are individually rational in the "usual" sense, in that they give at least as much utility as the utility associated with the endowment point of each agent. In Subsection 3.1.1, we provide an example of a Pareto optimal allocation that is individually rational in the usual sense but is not attainable. ■

In Subsection 2.3 below we present examples of perfectly contracted allocations. In Subsection 2.4, we provide a characterization of perfectly contracted allocations.

2.3 An example

We illustrate the concepts introduced in the previous Subsection using a simple example. We show examples of Pareto optimal allocations other than the unique Arrow-Debreu allocation of the economy that arise as perfectly contracted allocations. In particular we show an example where the economy admits a perfectly contracted equilibrium, but the Arrow-Debreu allocation does not arise as a perfectly contracted allocation because expectations are not suitably configured.

The economy we study here is as follows. The set of agents is $I = \{1, 2\}$. There is one commodity in period one and two in period two. Thus $M = \{x_{11}\}$ and $N = \{x_{21}, x_{22}\}$. The endowments are $\omega_1^1 = \omega_1^2 = 1$, $\omega_2^1 = (1, 0)$, $\omega_2^2 = (0, 1)$; accordingly, $\omega^1 = (1, 1, 0)$; $\omega^2 = (1, 0, 1)$. The utilities of the two agents are $U^1(x_{11}^1, x_{21}^1, x_{22}^1) = \ln x_{11}^1 + \alpha \ln x_{21}^1 + (1-\alpha) \ln x_{22}^1$, and $U^2(x_{11}^2, x_{21}^2, x_{22}^2) = \ln x_{11}^2 + \beta \ln x_{21}^2 + (1-\beta) \ln x_{22}^2$ respectively, where $0 < \alpha, \beta < 1$.

We first compute the Arrow-Debreu solution by postulating that markets are complete. There are three prices. Normalizing by p_{11}, these prices are $1, \theta_{21}, \theta_{22}$. The Walrasian demands of agent 1 are $x_{11}^{1d} = \frac{1}{2}(1 + \theta_{21}); x_{21}^{1d} = \frac{\alpha}{2}(1 + \frac{1}{\theta_{21}}); x_{22}^{1d} = \frac{1-\alpha}{2}\frac{1+\theta_{21}}{\theta_{22}}$ with similar expressions for the demands of agent 2. Imposing market clearing and solving gives the Arrow-Debreu equilibrium prices as $\theta_{21} = \frac{\alpha+3\beta}{2+\beta-\alpha}; \theta_{22} = \frac{4-3\alpha-\beta}{2+\beta-\alpha}$. The Arrow-Debreu allocation is $x^{1AD} = (\frac{1+2\beta}{2+\beta-\alpha}, \frac{\alpha(1+2\beta)}{\alpha+3\beta}, \frac{(1-\alpha)(1+2\beta)}{4-3\alpha-\beta})$ and $x^{2AD} = (\frac{3-2\alpha}{2+\beta-\alpha}, \frac{\beta(3-2\alpha)}{\alpha+3\beta}, \frac{(1-\beta)(3-2\alpha)}{4-3\alpha-\beta})$. Since $0 < \alpha, \beta < 1$, one has in particular that $0 < x_{11}^{2AD} < 1; x_{11}^{1AD} > 1$ whenever $\alpha + \beta > 1$.

We now turn to a description of the set of *attainable* allocations. Recall that a Pareto optimal allocation $\overline{x} \gg 0$ is *attainable* if $U^i(\overline{x}) \geq \mathcal{U}^i_{\overline{x}}, i = 1, 2$. At a Pareto optimal allocation, $\overline{x}^1 = (\overline{x}_{11}^1, \overline{x}_{21}^1, \overline{x}_{22}^1) \gg 0$ and $\overline{x}^2 = (2 - \overline{x}_{11}^1, 1 - \overline{x}_{21}^1, 1 - \overline{x}_{22}^1) \gg 0$ satisfy $\frac{\overline{x}_{11}^1}{2-\overline{x}_{11}^1} = \frac{\beta\overline{x}_{21}^1}{\alpha(1-\overline{x}_{21}^1)} = \frac{(1-\beta)\overline{x}_{22}^1}{(1-\alpha)(1-\overline{x}_{22}^1)}$. For attainability, such an allocation must additionally satisfy the two inequalites $\ln \overline{x}_{11}^1 + \ln \overline{x}_{21}^1 \geq \ln \alpha$ and $\ln(2 - \overline{x}_{11}^1) + \ln(1 - \overline{x}_{21}^1) + \ln \alpha\overline{x}_{22}^1 - \ln(1-\alpha)\overline{x}_{21}^1 \geq \ln \beta$.

We now study perfectly contracted equilibria for this economy. Since there is only one commodity in period one, we set $p_1 = 1$. The set of second period prices is $\mathcal{P}_2 = \{(q, 1) | q \geq 0\}$. In this framework, Π will be an interval of values of q. We specify $\Pi = [\underline{q}, \overline{q}]$ such that $\underline{q} > 1$ and assume that $\text{supp}\phi^i(p_1) = \Pi, i = 1, 2$. This will allow us to dispense with Condition (ii) in the definition of individual rationality (Definition 2) and simplify the exposition.

Since there is only one good in period one, in the absence of contracting, the agent consumes $\omega_{11}^i, i = 1, 2$. Thus $\overline{\xi}_1^i(1, \omega_1^i) = \omega_{11}^i, i = 1, 2$. As before, $\overline{\xi}_2^i((q, 1), (q, 1) \cdot \omega_2^i)$ denotes the solution to Max $u_2^i(x_2^i)$ subject to $(q, 1) \cdot x_2^i \leq (q, 1) \cdot \omega_2^i$, $\forall (q, 1) \in \mathcal{P}_2, i = 1, 2$. These are $\overline{\xi}_2^1((q, 1), (q, 1) \cdot \omega_2^1) = (\alpha, q(1-\alpha))$ and $\overline{\xi}_2^2((q, 1), (q, 1) \cdot \omega_2^2) = (\beta/q, 1-\beta)$. These are the consumption bundles the agent expects to be consume if the price in period two is q. (As remarked before, these plans will of course for an arbitrary q *not* be feasible.)

Observe that for a given $q \in \Pi = [\underline{q}, \overline{q}]$, $MRS^1_{x_{11}x_{21}}(\omega^1_{11}, \overline{\xi}^1_2((q,1),(q,1)\cdot\omega^1_2)) = 1$ while $MRS^2_{x_{11}x_{21}}(\omega^2_{11}, \overline{\xi}^2_2((q,1),(q,1)\cdot\omega^2_2)) = 1/q$. Thus if $q \in \Pi$, then $q > 1$, and so

$$MRS^1_{x_{11}x_{21}}(\omega^1_{11}, \overline{\xi}^1_2((q,1),(q,1)\cdot\omega^1_2)) > MRS^2_{x_{11}x_{21}}(\omega^2_{11}, \overline{\xi}^2_2((q,1),(q,1)\cdot\omega^2_2)),$$

and the agents can improve over the no contract utility level using a contract under which agent 1 gets some x_{11} in period one and gives up some (price contingent) x_{21} in period two.

We therefore study very simple contracts where agent one receives a certain amount of period one consumption (denoted ε) against a price contingent quantity of x_{21} (denoted $\delta^\varepsilon(q)$). Thus the contracts $(z_1, z_2(q))$ we study will be of the form $[(1+\varepsilon, 1-\varepsilon), ((1-\delta^\varepsilon(q), 0), (\delta^\varepsilon(q), 1))]$ for q in some interval Π, where

(1) $(\varepsilon, \delta^\varepsilon(q))$ satisfy $0 < \varepsilon < 1$; $0 < \delta^\varepsilon(q) < 1$, $\forall q \in \Pi$.

Following our formulation in the general case in the previous section, we associate to a contract s_Π the price contingent vector (for p_1 fixed at 1),

$$\begin{aligned} \xi^1((1,q,1), s_\Pi) &= (1+\varepsilon, [\alpha(1-\delta^\varepsilon(q)), (1-\alpha)q(1-\delta^\varepsilon(q))]) \text{ and} \\ \xi^2((1,q,1), s_\Pi) &= (1-\varepsilon, [\frac{\beta}{q} + \beta\delta^\varepsilon(q), (1-\beta) + q(1-\beta)\delta^\varepsilon(q)]), \ \forall (q,1) \in \mathcal{P}_2. \end{aligned}$$

For an efficient contract, it suffices to equate the marginal rates of substitution across the two agents between x_{11} and x_{21} evaluated at the price contingent vector $\xi^i((1,q,1), s_\Pi)$, $i = 1, 2$ associated with the contract s_Π. The efficiency condition

$$MRS^1_{x_{11}x_{21}}(\xi^1((1,q,1), s_\Pi)) = MRS^2_{x_{11}x_{21}}(\xi^2((1,q,1), s_\Pi))$$

holds under the condition $\frac{1-\delta^\varepsilon(q)}{1+\varepsilon} = \frac{\frac{1}{q}+\delta^\varepsilon(q)}{1-\varepsilon}$. Thus for efficiency one solves this to get

(2) $\delta^\varepsilon(q) = \frac{k(\varepsilon)-\frac{1}{q}}{1+k(\varepsilon)}$ with $k(\varepsilon) \equiv \frac{1-\varepsilon}{1+\varepsilon}$.

Notice that for $q > 1$, there exist solutions for ε sufficiently small. Thus there exist efficient contracts for this configuration.

We now turn to the individual rationality requirement that a contract has to satisfy in a perfectly contracted equilibrium. In the absence of a contract, the perceived utility level of agent 1 is $u^1_1(\omega^1_{11}) + u^1_2(\overline{\xi}^1_2((q,1),(q,1)\cdot\omega^1_2))$ which equals $\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)q$ for each $q \in \Pi$. If the contract s_Π is accepted by agent 1, her utility is $U^1(\xi^1((1,q,1), s_\Pi)) =$

$\ln(1+\varepsilon) + \alpha \ln \alpha(1-\delta^\varepsilon(q)) + (1-\alpha)\ln(1-\alpha)q(1-\delta^\varepsilon(q))$ for each $q \in \Pi$. The individual rationality requirement for agent 1 becomes therefore

(3.1) $\quad (1+\varepsilon)^2(1+1/q) \geq 2.$

An analogous computation for agent 2 yields the condition

(3.2) $\quad (1-\varepsilon)^2(1+q) \geq 2$

Finally, we turn to the market clearing condition embodied in the definition of a perfectly contracted equilibrium. For a second period endowment vector of the form $(1-\delta^\varepsilon(q), 0), (\delta^\varepsilon(q), 1)$ for agents 1 and 2 respectively, the equilibrium spot price is $q = \frac{\beta}{1-\alpha+\delta^\varepsilon(q)(\alpha-\beta)}$.

Substituting the formula for $\delta^\varepsilon(q)$ given in (2) into the formula above and solving gives

(4) $\quad q^* = \frac{[k(\varepsilon)+1]\beta+(\alpha-\beta)}{(1-\alpha)[k(\varepsilon)+1]+(\alpha-\beta)k(\varepsilon)}$

In this simplified framework, a perfectly contracted equilibrium is a contract $s_\Pi = (\varepsilon, \delta^\varepsilon(q))$ for a given interval $\Pi = [\underline{q}, \overline{q}]$ such that $\underline{q} > 1$, a price $q^* \in \Pi$, that together satisfy (1) – (4), and an allocation x^* such that $x^{*i} = \xi^i((1, q^*, 1), s_\Pi)$, $i = 1, 2$. We now examine two parametric configurations of this formulation.

Case A. Here we set $\alpha = \beta = 0.8$. In this case, $q^* = 4$ independently of $(\varepsilon, \delta^\varepsilon(q))$. Thus for an interval of prices Π to be part of a perfectly contracted equilibrium, it must necessarily contain $q^* = 4$. Furthermore, every perfectly contracted equilibrium price equals $q^* = 4$. We specify here

(i) $\Pi = [\underline{q}, \overline{q}] = [3.1, 5.4]$ so that $q^* \in \Pi$.

(ii) $\varepsilon = 0.3$. Accordingly, $k(\varepsilon) = 0.538$. The condition $0.538 > 1/\underline{q} > 1/\overline{q}$ holds so that the numerator of $\delta^\varepsilon(q)$ as specified in (2) is positive and accordingly we have that $0 < \delta^\varepsilon(q) < 1$, $\forall q \in \Pi$.

(iii) It can be verified that $(1+\underline{q})(1-0.3)^2 > 2$ and $(1+1/\overline{q})(1+0.3)^2 > 2$ so that (3.1) and (3.2) hold.

Thus $s_\Pi = (\varepsilon, \delta^\varepsilon(q))$ where $\varepsilon = 0.3$ and $\delta^\varepsilon(q)$ is specified by (2), with $q^* = 4$ constitutes a perfectly contracted equilibrium. The allocation it generates is $x^{*1} = (1.3, (0.65, 0.65))$ and $x^{*2} = (0.7, (0.35, 0.35))$. The allocation $x^{*1} = (1.3, (0.65, 0.65))$; $x^{*2} = (0.7, (0.35, 0.35))$ is indeed the Arrow-Debreu allocation for this economy.

The Arrow-Debreu allocation however is not the only allocation that can arise as a perfectly contracted equilibrium for this economy. The two inequalities $\ln \overline{x}^1_{11} + \ln \overline{x}^1_{21} \geq \ln \alpha$ and $\ln(2-\overline{x}^1_{11}) + \ln(1-\overline{x}^1_{21}) + \ln \alpha \overline{x}^1_{22} - \ln(1-\alpha)\overline{x}^1_{21} \geq \ln \beta$ that are required for a Pareto optimal allocation to be attainable reduce here (after setting $q^* = 4$) to $\frac{(\overline{x}^1_{11})^2}{2} \geq 0.8$ and $(2 - \overline{x}^1_{11})^2 \geq 0.4$ respectively.

Indeed any value of $\overline{x}^1_{11}$ that satisfies the two inequalities generates an attainable allocation via the Pareto optimality equalities $\frac{\overline{x}^1_{11}}{2-\overline{x}^1_{11}} = \frac{\overline{x}^1_{21}}{1-\overline{x}^1_{21}} = \frac{\overline{x}^1_{22}}{1-\overline{x}^1_{22}}$. In particular, values of $\overline{x}^1_{11}$ satisfying $1.265 \leq \overline{x}^1_{11} \leq 1.365$ generates attainable allocations. Any such attainable allocation can arise as a perfectly contracted equilibrium with respect to $\Pi = [\underline{q}, \overline{q}] = [3.1, 5.4]$ and $q^* = 4$. ■

As is well-known, the Arrow-Debreu allocation of the economy arises as a sequential Radner equilibrium where there is a competitive trade in an asset in period one, that pays in period two commodities followed by competitive trade in period two in the spot markets for the two commodities. This conclusion however requires that agents have perfect foresight of the market clearing spot price in period two. In the parameter configuration of Case A, this market clearing spot price ratio is always 4 which coincides with the second period price market clearing price under any perfectly contracted equilibrium. The resemblance is of course superficial, since contracts in our framework are not competitively traded in period one and furthermore agents are not required to anticipate correctly the unique market clearing price ratio in period two. In Case B below we show that there exist perfectly contracted equilibrium allocations that are distinct from the Arrow-Debreu allocation and where the market clearing price ratio in period two at such an equilibrium is distinct from the price ratio that clears markets in the perfect foresight Radner equilibrium of this economy.

Case B. We set $\alpha = 0.8; \beta = 0.75$. With these parameter values, $x^{AD}_{11} = 1.282$ and the Radner equilibrium spot price ratio that supports the Arrow-Debreu allocation, denoted $p^R \equiv \frac{\theta_{21}}{\theta_{22}}$ is equal to 3.5882.... As before, set $\varepsilon = 0.3$. Accordingly $k(\varepsilon) = 0.538$. Here too we set $\Pi = [\underline{q}, \overline{q}] = [3.1, 5.4]$. We observe that

(i) as before, the condition $0.538 > 1/\underline{q} > 1/\overline{q}$ so that the numerator of $\delta^\varepsilon(q)$ as specified in (2) is positive, so that $0 < \delta^\varepsilon(q) < 1$, $\forall q \in \Pi$.

(ii) furthermore $(1 + \underline{q})(1 - 0.3)^2 > 2$ and $(1 + 1/\overline{q})(1 + 0.3)^2 > 2$ so that (3.1) and (3.2) hold.

(iii) by computation, $q^* = 3.5979... \in \Pi$.

Thus $s_\Pi = (\varepsilon, \delta^\varepsilon(p))$ where $\varepsilon = 0.3$ and $\delta^\varepsilon(q)$ is specified by (2), with $q^* = 3.5979...$ constitutes a perfectly contracted equilibrium.

The perfectly contracted allocation x^{*1} and x^{*2} is not the Arrow-Debreu allocation for this economy since we have shown that the $x^{AD}_{11} = 1.282 < x^{*1}_{11} = 1.3$. Also $q^* = 3.5979... > 3.5882... = p^R$, where the equilibrium relative price in the period two spot market is p^R. Furthermore, if one sets $\Pi = [\underline{q}, \widehat{q}] = [3.59, 5.4]$, one gets a perfectly contracted allocation where the Radner equilibrium relative price does not belong to the forecasted price set Π. ■

2.4 Perfectly contracted equilibria and Pareto optimal allocations

In this Subsection we provide a characterization of perfectly contracted allocations. Recall that an Arrow-Debreu equilibrium is a pair (p', x'), $p' \in \Re_+^{M+N}$ such that (i) for each agent $i \in I$, x'^i solves the problem Max $u_1^i(x_1^i) + u_2^i(x_2^i)$ subject to $p_1' \cdot x_1^i + p_2' \cdot x_2^i \leq p_1' \cdot \omega_1^i + p_2' \cdot \omega_2^i$, and (ii) markets clear in all $M + N$ commodities. We show that every interior Arrow-Debreu allocation is a perfectly contracted allocation and additionally, there are other Pareto optimal allocations that are perfectly contracted allocations provided they are interior. The following is our characterization result.

Theorem *Assume that A1 and A2 hold. Every perfectly contracted allocation x^* such that $x^* \gg 0$ is an attainable Pareto optimal allocation.*

Let $\overline{x} \gg 0$ be an attainable Pareto optimal allocation. Then $\overline{x}$ is a perfectly contracted allocation for some specification of $(\phi^i)_{i \in I}$. Every Arrow-Debreu equilibrium allocation $x' \gg 0$ is attainable.

The Theorem establishes that the set of attainable allocations is the set of allocations that can emerge as perfectly contracted allocations. The proof of the first part of the theorem is straightforward. In the proof of the second part of the Theorem, we set the support of $\phi^i(\overline{p}_1(\overline{x}))$ to exactly the price $\overline{p}_2(\overline{x})$ for each $i \in I$. This is a knife-edge specification, which is undesirable as it requires that expectations be coordinated on a particular second period price. This was done for convenience as our interest lay in characterizing allocations that emerge as perfectly contracted allocations and not in specifying the most general conditions under which these would obtain. An important reason why perfectly contracted equilibria are of interest to us is precisely that this requirement on expectations can be relaxed substantially as we now show.

Given an attainable allocation $\overline{x}$, we now define $\Pi_{\overline{x}}$ as the set of second period prices to which expectations of a subset of agents has to be restricted to for $\overline{x}$ to emerge as a perfectly contracted allocation. First, by a slight abuse of notation, let $I^-(\overline{x})$ be the set of agents for whom one has $\overline{p}_1(\overline{x}) \cdot \omega_1^i > \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i$. Let $\Pi_{\overline{x}}$ be a set of second period prices such that (i) $\overline{p}_2(\overline{x}) \in \Pi_{\overline{x}}$, (ii) the contract $s_{\Pi_{\overline{x}}} = (\overline{x}_1, z_2(p_2)), p_2 \in \Pi_{\overline{x}}$ with $z_2(\overline{p}_2(\overline{x})) = \overline{x}_2$, satisfies Condition (i) of Definition 2 and is efficient at $\overline{p}_1(\overline{x})$, and lastly, (iii) $\Pi_{\overline{x}} \supseteq \Pi$ for all Π such that $\overline{p}_2(\overline{x}) \in \Pi$, and the contract $s_\Pi = (\overline{x}_1, z_2(p_2)), p_2 \in \Pi$ with $z_2(\overline{p}_2(\overline{x})) = \overline{x}_2$ satisfies Condition (i) of Definition 2 and is efficient at $\overline{p}_1(\overline{x})$.

Thus $\Pi_{\overline{x}}$ is the set of second period prices such that there exists a contract that renders $\overline{x}$ a perfectly contracted allocation provided Condition (ii) of Definition 2 is satisfied. Condition (ii) of Definition 2 is satisfied and the attainable allocation $\overline{x}$ arises as a perfectly contracted allocation whenever at the period one price $\overline{p}_1(\overline{x})$, each $i \in I^-(\overline{x})$ attaches enough probability to $\Pi_{\overline{x}}$. Furthermore, it follows from the discussion in Subsection 4.3 below that for a strongly attainable allocation $\overline{x}$, $\Pi_{\overline{x}}$ contains $\overline{p}_2(\overline{x})$ as an interior point so that we are able to avoid the knife-edge specification whereby expectations have to give enough probability to the specific price $\overline{p}_2(\overline{x})$.

Corollary *Let $\overline{x}$ be a strongly attainable allocation. Then there exists a set $\Pi_{\overline{x}}$ of second period prices containing $\overline{p}_2(\overline{x})$ as an interior point such that $\overline{x}$ is a perfectly contracted allocation whenever for each $i \in I^-(\overline{x})$, $\phi^i(\overline{p}_1(\overline{x}))(\Pi_{\overline{x}})$ is large enough.*

We note that in the statement above, there is *no* restriction on the expectations of agents in $I \backslash I^-(\overline{x})$ at the period one price $\overline{p}_1(\overline{x})$.

Extensions. We briefly mention some extensions of the model studied above.

1. We briefly consider two alternative versions of the individual rationality requirement. Our notion of individual rationality requires agents to be at least as well off after agreeing to a contract as by not contracting, at the prevailing price in period one for *each* possible configuration of the price in period two that they contract over. This puts restrictions on how large the set of second period prices Π they contract over can be. An alternative notion of individual rationality would dispense with Condition (i) of Definition 2 and instead impose just Condition (ii) of Definition 2 but require it to hold for all agents and not just the ones giving up wealth in period one. Thus a contract would then be required to keep an agent at least as well off as by not contracting in expected utility terms. One may analogously define a perfectly contracted equilibrium with such a version of individual rationality. Such a version of individual rationality would give greater flexibility in specifying the set of prices Π that agents contract over.

A second variant would be one where agents receiving wealth in period one are forced to transfer a fixed amount to agents who give up wealth in period one if $p_2 \notin \Pi$. One would then require Condition (ii) of Definition 2 to hold for all agents and not just agents in $I^-(p_1, z_1)$. This alternate formulation would leave the definition of perfectly contracted equilibrium and the material in the following Subsection unchanged. It would however affect the examples we present later as some aspects of these would have to be recomputed with the new notion of individual rationality.

2. It is evident that given a perfectly contracted equilibrium, even if the efficiency criterion requiring equality of marginal rates of substitution is violated for prices other than p_2^* in Π, one still gets Pareto optimality. It would appear that one can work with a weaker notion of efficiency. However, in a decentralized setting, agents cannot be expected to know what the "right" price p_2^* is, so this is not really a meaningful weakening of the requirement of efficient contracting. In the setting of decentralized contracting (Subsection 4.2 below), it may be more appropriate to treat efficient contracting as a behavioral rule (or as a restriction on the outcome of the contracting process) and require it to hold for all prices in Π.

3. Finally, our model and notion of a perfectly contracted equilibrium can be extended to the case where there are finitely many states in period two. Perfectly contracted equilibrium allocations will then be ex post Pareto optimal.

3 Contracts and perfectly contracted equilibria

The previous section showed that every attainable allocation can be realized as a perfectly contracted allocation provided one is free to specify the expectation functions and choose appropriate individually rational, efficient contracts. It is evident that for a particular attainable allocation $\overline{x}$ to emerge as a perfectly contracted equilibrium, the condition on expectations specified in the Corollary in Subsection 2.4, namely that for each $i \in I^{-}(\overline{x})$, $\phi^i(\overline{p}_1(\overline{x}))(\Pi)$ be large enough where $\overline{p}_2(\overline{x}) \in \Pi$, is also necessary. Here we assume that this condition on expectations is satisfied. The sufficiency of this condition, as stated in the Corollary, presupposes that there is an underlying contracting process that delivers the attainable allocation as a perfectly contracted allocation. In this section we focus on the contracting process.

In our formulation, the process of contracting over prices is embedded in the Walrasian adjustment process as follows. Given a period one price p_1, the profile of expectation functions $(\phi^i)_{i\in I}$ determines what prices agents expect might prevail in the period two spot markets. We say the triple $(p_1, \Pi, (\phi^i)_{i\in I})$ forms a *contracting problem* provided the configuration is such that there exist transfers of numeraire across the two periods that benefit all agents (with at least one agent doing strictly better) and exhaust potential gains from trade. Specifically, we say the triple $\mathcal{C} \equiv (p_1, \Pi, (\phi^i)_{i\in I})$ forms a contracting problem provided there exists a non-empty set of contracts that are individually rational (with one agent satisfying the individually rational allocations with strict inequalities) and efficient at p_1. This set of contracts will be referred to as the set of feasible contracts $\mathcal{F}(\mathcal{C}) = \{s_\Pi | s_\Pi$ is individually rational and efficient at $p_1\}$ for $\mathcal{C}$. We have shown in Subsection 2.4 that we can associate to each attainable allocation a contracting problem. There remains the issue of specifying which particular feasible contract is chosen. This would depend on the precise institutional details of the contracting process.[9] We take this choice as being summarized by a solution correspondence $g(\mathcal{C})$ which picks a subset of $\mathcal{F}(\mathcal{C})$. This is a reduced form formulation of the particular contracting process that is used by the agents.

[9]In the case where the contract is the outcome of a cooperative process, the contract chosen would depend on the solution concept used. In the case where a noncooperative method is used to determine the contract, this would depend on factors like which group of agents move first with a proposal etc.

In Subsection 3.1 below, we generalize our example from Subsection 2.3 to illustrate some aspects related to the choice of the solution correspondence. We first observe that an arbitrary solution correspondence $g(\mathcal{C})$ need not deliver an attainable allocation as a perfectly contracted equilibrium. We subsequently consider an example of a perfectly contracted equilibrium where the solution correspondence is generated using Nash Bargaining.

3.1 The example generalized

In this Subsection, we take as given a solution correspondence $g(\mathcal{C})$ and study the additional restrictions that must be satisfied for an attainable allocation $\overline{x}$ to emerge as a perfectly contracted allocation. It is immediate that we need that $\overline{p}_1(\overline{x})$ be the period one price vector, that $\overline{p}_2(\overline{x})$ belong to Π, and that $\mathcal{C} \equiv(\overline{p}_1(\overline{x}), \Pi, (\phi^i)_{i\in I})$ form a contracting problem. We highlight here the requirement that $\exists \overline{s}_\pi \in g(\mathcal{C})$ such that $\xi_1^i(\overline{p}_1(\overline{x}), \overline{s}_\pi) = \overline{x}_1^i$, $\forall i \in I$. Indeed for an arbitrary $\overline{s}_\pi \in \mathcal{F}(\mathcal{C})$, it will not be the case that $\overline{p}_1(\overline{x})$ will be a Walrasian market clearing price for the period one economy where the demands are $\xi_1^i(p_1, \overline{s}_\pi)$, $\forall i \in I$. This problem of market clearing did not appear in the examples studied in the previous section since there was only one good in period one and no relative price to contend with.

We next generalize the example studied in Subsection 2.3 to highlight the aforementioned point. We consider the case where $M = \{x_{11}, x_{12}\}$, so that there are two goods in period one as well and therefore a relative price in period one that will play a role in the analysis.

The set of agents is $I = \{1, 2\}$. There are two commodities in period one and two in period two. Thus $M = \{x_{11}, x_{12}\}$ and $N = \{x_{21}, x_{22}\}$. The endowments are $\omega_1^1 = (0, 1)$, $\omega_1^2 = (1, 0)$; $\omega_2^1 = (1, 0)$, $\omega_2^2 = (0, 1)$; accordingly, $\omega^1 = ((0, 1), (1, 0))$; $\omega^2 = ((1, 0), (0, 1))$. The utilities of the two agents are

$$\begin{aligned} U^1(x_{11}^1, x_{12}^1, x_{21}^1, x_{22}^1) &= \alpha_1 \ln x_{11}^1 + (1-\alpha_1) \ln x_{12}^1 + \alpha_2 \ln x_{21}^1 + (1-\alpha_2) \ln x_{22}^1 \\ U^2(x_{11}^2, x_{12}^2, x_{21}^2, x_{22}^2) &= \beta_1 \ln x_{11}^2 + (1-\beta_1) \ln x_{12}^2 + \beta_2 \ln x_{21}^2 + (1-\beta_2) \ln x_{22}^2. \end{aligned}$$

where $0 < \alpha_1, \alpha_2, \beta_1, \beta_2 < 1$. The set of period one prices is $\mathcal{P}_1 = \{(p, 1)|p \geq 0\}$ and the set of second period prices is, as before, $\mathcal{P}_2 = \{(q, 1)|q \geq 0\}$. Here too, Π will be an interval of values of q.

We work with a period one price p, and an interval of second period prices $\Pi = [\underline{q},\overline{q}]$ such that $p > 1/\underline{q} > 1/\overline{q}$, assuming that $\mathrm{supp}\phi^i(p) = \Pi$, $i = 1, 2$. Here too as before, the agents can improve over the no contract utility level using a contract under which agent one gets some x_{11} in period one and gives up some (a price contingent) x_{21} in period two. We therefore continue to study simple contracts where agent one receives a certain amount of period one consumption (denoted ε) against a price contingent quantity of x_{21} (denoted $\delta^\varepsilon(q)$). Thus the contracts $(z_1, z_2(q))$ we study will be of the form $[((\varepsilon, 1), (1-\varepsilon, 0)), ((1-\delta^{\varepsilon}(q), 0), (\delta^{\varepsilon}(q), 1))]$ for each q in the interval Π, where $(\varepsilon, \delta^{\varepsilon}(q))$ satisfy $0 < \varepsilon < 1$; $0 < \delta^{\varepsilon}(q) < 1$, $\forall q \in \Pi$.

As before, we impose efficiency by equating the marginal rates of substitution between x_{11} and x_{21} across the two agents and solve for $\delta^{\varepsilon}(q)$ to get $\delta^{\varepsilon}(q) = \frac{pq(1-\varepsilon)-p\varepsilon-1}{q(p+1)}$.

It is evident that there exist solutions $(\varepsilon, \delta^{\varepsilon}(q))$ which satisfy $0 < \varepsilon < 1$; $0 < \delta^{\varepsilon}(q) < 1$, $\forall q \in \Pi$.

We ignore the individual rationality requirement for the moment and move directly to the market clearing conditions that give us a formula for p and another one for q. These are $q = \frac{\beta_2}{1-\alpha_2+\delta^{\varepsilon}(q)(\alpha_2-\beta_2)}$ and $p = \frac{\alpha_1}{1-\beta_1+\varepsilon(\beta_1-\alpha_1)}$. It is evident that unless $\alpha_2 = \beta_2$, q depends on p via $\delta^{\varepsilon}(q)$.

3.1.1 Contract selection

We consider the parametric configuration where $\alpha_2 = \beta_2 = 0.8$ as in Case A earlier and obtain $q = 4$ independently of p. We set $\alpha_1 = 0.9$ and $\beta_1 = 0.8$

First, as an example of the possibility mentioned in Observation 2, we consider a Pareto optimal allocation $\overline{x} = ((\overline{x}_{11}^1, \overline{x}_{12}^1), (\overline{x}_{11}^2, \overline{x}_{12}^2), (\overline{x}_{21}^1, \overline{x}_{22}^1), (\overline{x}_{21}^2, \overline{x}_{22}^2)) =$ $((7/25, 14/95), (18/25, 81/95), (28/109, 28/109), (81/109, 81/109))$. This allocation is individually rational in the "usual" sense since it improves over autarchy. However, it is not attainable as it violates the requirement of Definition 5 since $U^1(\overline{x}^1) = \ln(0.06746) < \mathcal{U}^1_{\overline{x}} = \ln(0.1425)$.

Now we consider a Pareto optimal allocation that can emerge as a perfectly contracted allocation provided a suitable contract is chosen. Take the Pareto optimal allocation $\overline{x} = ((\overline{x}_{11}^1, \overline{x}_{12}^1), (\overline{x}_{11}^2, \overline{x}_{12}^2), (\overline{x}_{21}^1, \overline{x}_{22}^1), (\overline{x}_{21}^2, \overline{x}_{22}^2)) = ((3/5, 2/5), (2/5, 3/5), (4/7, 4/7), (3/7, 3/7))$. Here $U^1(\overline{x}) = \ln(0.3292)$ and $U^2(\overline{x}) = \ln(0.1859)$. For this allocation $p_1(\overline{x}) = (6, 1)$, and by computation $\mathcal{U}^1_{\overline{x}} = \ln(0.1152)$ and $\mathcal{U}^2_{\overline{x}} = \ln(0.1735)$. Since $U^i(\overline{x}^i) > \mathcal{U}^i_{\overline{x}}, i = 1, 2$, this allocation is attainable. A contract that supports this allocation is $\varepsilon = 1/2$ and $\delta^{\varepsilon}(4) = 2/7$.

For sake of illustration assume that $\phi^i(p_1(\overline{x}))$, $\forall i \in I$, gives probability one to $\Pi = \{(4, 1)\}$. It can be verified by computation that $\mathcal{C} = (\overline{p}_1(\overline{x}), \Pi, (\phi^i)_{i\in I})$ forms a contracting problem where $\mathcal{F}(\mathcal{C}) = \{(\varepsilon, \delta(4)) | 0.25547 \leq \varepsilon \leq 0.516422;\ \delta^{\varepsilon}(4) = \frac{23-30\varepsilon}{28}\}$ (The set $\mathcal{F}(\mathcal{C})$ has been restricted to simple contracts). We now turn to the restriction on the solution $g(\mathcal{C})$ that we need for $\overline{x}$ to emerge as a perfectly contracted equilibrium. Indeed for this to happen, it must be that $p_1(\overline{x}) = (6, 1)$ be market clearing for the period one demand resulting from the choice of contract. This requires that $g(\mathcal{C})$ admits the contract where $\varepsilon = 1/2$ and $\delta^{\varepsilon}(4) = 2/7$ from $\mathcal{F}(\mathcal{C})$. This configuration of $\mathcal{C} = (\overline{p}_1(\overline{x}), \Pi, (\phi^i)_{i\in I})$ and $g(\mathcal{C})$ verifies the conditions needed for $\overline{x}$ to emerge as a perfectly contracted allocation.

Remark 1 We have focussed on contracting problems which generate attainable allocations. There remains the possibility that we have a contracting problem $\mathcal{C}$ where p_1 is a Walrasian market clearing price for the period one economy where the demands are $(\xi_1^i(p_1, s_\Pi), s_\Pi)_{i\in I} \in g(\mathcal{C})$, but that the set of period two prices that clear the market in period two when the demands are in the set $\xi_2^i(p_2, s_\Pi)$, $s_\Pi \in g(\mathcal{C})$ for each $i \in I$, has an empty intersection with Π. This would violate Condition (i) in the definition of a perfectly contracted equilibrium and would typically lead to a non Pareto optimal allocation. It would be natural to term such a situation a temporary contracting equilibrium. Formally, a *temporary-contracting equilibrium* is a triple (p^*, x^*, s_Π) comprising a price vector $p^* = (p_1^*, p_2^*)$, an allocation x^* and a contract $s_\Pi \in g(\mathcal{C})$ such that (i) $x_1^{*i} = \xi_1^i(p_1^*, s_\Pi)$, $\forall i \in I$ and $\sum_{i\in I} x_1^{*i} = \sum_{i\in I} \omega_1^i$, and (ii) $x_2^{*i} = \xi_2^i(p_2^*, s_\Pi)$, $\forall i \in I$ and $\sum_{i\in I} x_2^{*i} = \sum_{i\in I} \omega_2^i$ hold. A perfectly contracted equilibrium is a special case of a temporary-contracting equilibrium where $p_2^* \in \Pi$. It is straightforward to generate an example of a temporary-contracting equilibrium using Case A of the example presented in the previous section. If one specifies an interval Π contained in $[\underline{q}, \overline{q}] = [3.1, 5.4]$ but which excludes the value $q = 4$, one can rework the example to generate a temporary contracted equilibrium where the period two market price will not belong to the set of prices agents had contracted over. We omit the details.

3.1.2 Nash Bargaining

We present an example of a perfectly contracted equilibrium where the solution $g(\mathcal{C})$ is derived using Nash Bargaining.

In order to simplify the computations we set $\alpha_2 = \beta_2 = 0.8$. This particular parameter configuration, as remarked earlier, gives $q = 4$, independently of p. As a further simplification we set $\alpha_1 = \beta_1 = 0.9$ to obtain $p = 9$ for market clearing in period one. We note that the Arrow-Debreu allocation for this parameter configuration is $x^{1AD} = ((9/20, 9/20), (9/20, 9/20))$; $x^{2AD} = ((11/20, 11/20), (11/20, 11/20))$.

We consider a contracting problem $\mathcal{C} = (9, \Pi, (\phi^i)_{i\in I})$, where $4 \in \Pi$, and as before, we continue to study simple contracts where agent one receives a certain amount of period one consumption (denoted $\varepsilon, 0 \leq \varepsilon \leq 1$) against a price contingent quantity of x_{21} (denoted $\delta^\varepsilon(q), 0 \leq \delta^\varepsilon(q) \leq 1$). Denote the expected utility of selecting the contract defined by a value ε for the agents as $\mathcal{U}^1(\varepsilon)$ and $\mathcal{U}^2(\varepsilon)$. The expected utility of not contracting and therefore using the initial endowments ω to generate their Walrasian demands in each of the two periods are denoted $\overline{\mathcal{U}}^1$, $\overline{\mathcal{U}}^2$. This pair constitutes the threat point of the bargaining problem. In order to select the contract using Nash Bargaining problem, we formulate the following problem whose solution defines the Nash Bargaining contract

$$\max_{\varepsilon\in[0,1]} \; [\mathcal{U}^1(\varepsilon) - \overline{\mathcal{U}}^1] \cdot [\mathcal{U}^2(\varepsilon) - \overline{\mathcal{U}}^2]$$

We consider below two specifications of this contracting problem $\mathcal{C}$ formulated above. The computational details are relegated to Appendix F.

Case A. We consider the case $\Pi = \{4\}$ which is the simplest to analyze and assume that each $\phi^i(9)(\Pi) = 1, i = 1, 2$. The efficiency of the contract is requires the condition $\delta^\varepsilon(4) = \frac{7-9\varepsilon}{8}$. By computation, $\mathcal{F}(\mathcal{C})$ =$\{(\varepsilon, \delta^\varepsilon(4))|0 \le \varepsilon \le 1; \frac{1}{8}(1+9\varepsilon)^2 \ge 1; \frac{9}{2}(1-\varepsilon)^2 \ge 1; \delta^\varepsilon(4) = \frac{7-9\varepsilon}{8}\}$.

We now turn to the Nash Bargaining solution. To calculate the utility possibility frontier, we impose the efficiency of the contract, that is, $\delta^\varepsilon(4) = \frac{7-9\varepsilon}{8}$. The indirect utility of an efficient contract is given by $\mathcal{U}^1(\varepsilon) = 2\ln\frac{9}{10}(\varepsilon + \frac{1}{9})$ and $\mathcal{U}^2(\varepsilon) = 2\ln\frac{9}{10}(1-\varepsilon)$. This gives the frontier of the utility possibility set as $\{(\mathcal{U}^1, \mathcal{U}^2) \in \Re^2 | e^{\frac{\mathcal{U}^1}{2}} + e^{\frac{\mathcal{U}^2}{2}} = 1\}$. This shows that the utility possibility set is convex and so the Nash Bargaining solution can be motivated by the usual axioms. The threat points are computed to be $\overline{\mathcal{U}}^1 = U^1((1/10, 1/10), (4/5, 4/5))$ and $\overline{\mathcal{U}}^2 = U^2((9/10, 9/10), (1/5, 1/5))$ respectively.

Finally, the FOC for the Nash Bargaining problem is

$$\frac{9}{1+9\varepsilon}[\ln\frac{9}{2} + 2\ln(1-\varepsilon)] - \frac{1}{1-\varepsilon}[\ln\frac{1}{8} + 2\ln(1+9\varepsilon)] = 0.$$

Solving, we get $\varepsilon^{NB} = 0.361828...$ (details are in Case A of Appendix F).

Under this Nash Bargaining solution, the perfectly contracted equilibrium allocation is
$\xi^1((p, 1), (q, 1), (\varepsilon, \delta^\varepsilon(q)) = [(0.425645..., 0.425645...), (0.425645..., 0.425645...)]$ and
$\xi^2((p, 1), (q, 1), (\varepsilon, \delta^\varepsilon(q)) = [(1-0.425645..., 1-0.425645...), (1-0.425645..., 1-0.425645...)]$.

We note that the allocation obtained via Nash Bargaining is distinct from the Arrow-Debreu allocation. This Nash Bargaining solution applies to a continuum economy with equal masses of the two types of the agents where each agent of type one is matched with another of type two. It is worth emphasizing that our model admits solutions that remain different from the Arrow-Debreu model of trade even when there are a continuum of agents, since going to the continuum does not ensure that our solutions converge to the Arrow-Debreu solution. This distinction is valid in spite of the fact that agents are subjectively certain about the spot prices in the period two economy that support the Arrow-Debreu solution. ■

Case B. The previous case dealt with a very restrictive setting where expectations were concentrated on the singleton set $\Pi = \{4\}$. We now turn to the case where agents expectations give probability one to an interval of the form $\Pi = [4-\sigma, 4+\sigma]$, $4 > \sigma > 0$. Specifically, we assume that $\phi^i(9)$, $i = 1, 2$, prescribes the probability measure associated with the uniform distribution over $\Pi = [4-\sigma, 4+\sigma]$.

By computation, $\mathcal{F}(\mathcal{C})=\{(\varepsilon,\delta^{\varepsilon}(q)),q\in\Pi|0\leq\varepsilon\leq 1;\ \frac{1}{9}[\frac{10(4+\sigma)}{5+\sigma}]^{\frac{1}{2}}-\frac{1}{9}\leq\varepsilon\leq 1-\frac{1}{3}(\frac{10}{5-\sigma})^{\frac{1}{2}};\ \delta^{\varepsilon}(q)=\frac{9-9\varepsilon(1+\frac{1}{q})-\frac{1}{q}}{10}\}$. Then, the Nash Bargaining solution is the maximizer of the following maximization problem:

$$\max_{\varepsilon\in\mathcal{F}(\mathcal{C})}[\mathcal{U}^1(\varepsilon)-\bar{\mathcal{U}}^1]\cdot[\mathcal{U}^2(\varepsilon)-\bar{\mathcal{U}}^2]$$

where the details of the expected utilities and the disagreement pair of utilities of agents are in Case B of Appendix F. Furthermore, one can verify that the utility possibility set is convex.

We now solve for the Nash Bargaining solution numerically (details are in Case B of Appendix F). We set $\sigma=2.25$. Then $\mathcal{F}(\mathcal{C})=\{(\varepsilon,\delta^{\varepsilon}(q)),q\in\Pi|0.2151\leq\varepsilon\leq 0.3644;\ \delta^{\varepsilon}(q)=\frac{9-9\varepsilon(1+\frac{1}{q})-\frac{1}{q}}{10}\}$. In particular we observe that the value of ε that sustains the Arrow-Debreu allocation is $\varepsilon^{AD}=7/18$, and it does not belong to $\mathcal{F}(\mathcal{C})$. Solving numerically, we get $\varepsilon^{NB}\approx 0.3556$ and we obtain the perfectly contracted Nash Bargaining allocation $x^{NB}=((0.42004,0.42004),(0.57996,0.57996),(0.42004,0.42004),(0.57996,0.57996))$. ■

4 A role for financial institutions

Our specification of a solution correspondence $g(\mathcal{C})$ which picks a subset of $\mathcal{F}(\mathcal{C})$ assumes that agents can write individually rational and efficient contracts over the set of prices Π. This assumption on the contracts over prices implicitly presupposes that there is an economy-wide coordination mechanism between agents at play. We examine two instances of financial institutions that may facilitate this coordination and lead to the formation of efficient and individually rational contracts. The first is a centralized institution which computes efficient and individually rational contracts for the agents, while the second is a financial institution which decentralizes the formation of contracts by conducting a trading protocol wherein agents interact pairwise.

In this section, as in Sections 2 and 3, we assume that spot prices $p_1\in\Re^M_+$ and spot prices $p_2\in\Re^N_+$. Consider a price pair $(p_1,p_2)\gg 0$ such that there exist individually rational and efficient contracts when agents are subjectively certain that the next period price is p_2 given the current price p_1. Such a price pair is termed contractable. More formally, consider a price pair such that $\mathcal{F}(\widetilde{\mathcal{C}})\neq\varnothing$ where $\widetilde{\mathcal{C}}\equiv(p_1,\widetilde{\Pi},(\widetilde{\phi}^i)_{i\in I})$, $\widetilde{\Pi}\equiv\{p_2\}$ and $\widetilde{\phi}^i(p_1)(\widetilde{\Pi})=1$, $\forall i\in I$. We term a price pair (p_1,p_2) *contractable* if $\mathcal{F}(\widetilde{\mathcal{C}})\neq\varnothing$, which is equivalent to requiring that $\widetilde{\mathcal{C}}\equiv(p_1,\widetilde{\Pi},(\widetilde{\phi}^i)_{i\in I})$ is a contracting problem. A contract $s_\Pi\in\mathcal{F}(\tilde{\mathcal{C}})$ can be identified naturally with a wealth vector $(\tau^i_1,\tau^i_2)_{i\in I}$ which desribes the wealth levels associated with the contract evaluated at the price pair (p_1,p_2).

The formation of contracts is done in two steps. Given a contractable price pair, we first determine a contract that is efficient and individually rational. Two instances of this process are presented below in Subsections 4.1 and 4.2. Next in Subsection 4.3, we extend the contract to a set of second period prices Π that contains p_2 while preserving the efficiency and individual rationality of the contract.

4.1 Centralized banking

The efficient and individually rational contracts used in the paper are readily computed by a centralized financial institution (central bank)[10] which elicits the preferences and endowments of member agents that subscribe to its services and computes efficient and individually rational contracts for its members by simply maximizing the weighted sum of utilities subject to the endowment and individually rationality constraints.

We now postulate that a contract is specified by choosing a wealth vector $(\tau_1^{*i}, \tau_2^{*i})_{i\in I}$ which solves the following maximization problem (denoted $\mathcal{CB}$) where (p_1, p_2) is a contractable price pair:

$$\begin{aligned}
\max_{\{\tau_1^i,\tau_2^i\}_{i\in I}} \quad & \sum_{i\in I} \theta^i[u_1^i(\xi_1^i(p_1,\tau_1^i)) + u_2^i(\xi_2^i(p_2,\tau_2^i))] \\
s.t. \quad & \sum_{i\in I} \tau_1^i \le \sum_{i\in I} p_1\cdot\omega_1^i \\
& \sum_{i\in I} \tau_2^i \le \sum_{i\in I} p_2\cdot\omega_2^i \\
& u_1^i(\overline{\xi}_1^i(p_1, p_1\cdot\omega_1^i)) + u_2^i(\overline{\xi}_2^i(p_2, p_2\cdot\omega_2^i)) \le u_1^i(\xi_1^i(p_1,\tau_1^i)) + u_2^i(\xi_2^i(p_2,\tau_2^i)), \forall i\in I
\end{aligned}$$

where $\{\theta^i\}_{i\in I} \gg 0$.

Recall that an attainable allocation $\overline{x} = (\overline{x}_1, \overline{x}_2) \gg 0$ is *strongly attainable* if $U^i(\overline{x}^i) > \mathcal{U}_{\overline{x}}^i$ for each $i \in I$. Given a strongly attainable allocation $\overline{x} = (\overline{x}_1, \overline{x}_2) \gg 0$, and the supporting price subvectors $\overline{p}_1(\overline{x})$ and $\overline{p}_2(\overline{x})$ (interpreted as spot prices)[11], note that we must have $\overline{x}_t^i = \overline{\xi}_t^i(\overline{p}_t(\overline{x}), \overline{p}_t(\overline{x})\cdot\overline{x}_t^i)$, $t = 1, 2$.

Consider the attainable allocation $\overline{x} \gg 0$ and supporting prices $\overline{p}_1(\overline{x}) \gg 0$ and $\overline{p}_2(\overline{x}) \gg 0$. For each $i \in I$, let $\overline{\lambda}_t^i > 0$ denote the marginal utility of money in period t, $t = 1, 2$. Then, $\frac{\partial u_1^i(\overline{x}_1^i)}{\partial x_{1,q}} - \overline{\lambda}_1^i\overline{p}_{1,q}(\overline{x}) = 0$ and $\frac{\partial u_2^i(\overline{x}_2^i)}{\partial x_{2,r}} - \overline{\lambda}_2^i\overline{p}_{2,r}(\overline{x}) = 0$, so that $\overline{\lambda}_1^i = \frac{\frac{\partial u_1^i(\overline{x}_1^i)}{\partial x_{1,q}}}{\overline{p}_{1,q}(\overline{x})}$ and $\bar{\lambda}_2^i = \frac{\frac{\partial u_2^i(\overline{x}_2^i)}{\partial x_{2,r}}}{\overline{p}_{2,r}(\overline{x})}$, $\forall i \in I$, $\forall q \in N$ and $\forall r \in M$. As, at an attainable allocation, $\frac{\partial u_2^i(\overline{x}_2^i)}{\partial x_{2,r}} / \frac{\partial u_1^i(\overline{x}_1^i)}{\partial x_{1,q}} = \frac{\overline{p}_{2,r}(\overline{x})}{\overline{p}_{1,q}(\overline{x})}$, $\forall i \in I$,

[10] For simplicity, we have assumed that all agents in the economy subscribe to the same bank. In future work we will extend the model to study the interaction between more than one such financial institution where each has as its subscribers some partition of agents of the economy.

[11] $(\overline{p}_1(\overline{x}), \overline{p}_2(\overline{x})) \in \Re_+^{M+N}$ is the supporting price vector for $\overline{x}$.

$\forall q \in N$ and $\forall r \in M$, it follows that $\overline{\lambda}_2^i = \frac{\frac{\partial u_2^i(\overline{x}_2^i)}{\partial x_{2,r}}}{\overline{p}_{2,r}(\overline{x})} = \frac{\frac{\partial u_1^i(\overline{x}_1^i)}{\partial x_{1,q}}}{\overline{p}_{1,q}(\overline{x})} = \overline{\lambda}_1^i$ as required. Therefore, we let $\overline{\lambda}^i > 0$ denote the corresponding marginal utility of income for each individual $i \in I$ at an attainable allocation.

Proposition 1 *Suppose that in addition to A1 and A2, the preferences over consumption in each period admits a strictly concave utility function representation for each individual $i \in I$. Given a strongly attainable allocation $\overline{x} \gg 0$, its supporting prices $\overline{p}_1(\overline{x})$, $\overline{p}_2(\overline{x})$ and marginal utility of wealth $(\overline{\lambda}^i)_{i\in I}$, there exists an interior solution $(\tau_1^{*i}, \tau_2^{*i})_{i\in I}$ to $\mathcal{CB}$ when $\theta^i = \frac{1}{\overline{\lambda}^i}$, $\forall i \in I$. Let the contract s_Π be such that its associated wealth vector is $(\tau_1^{*i}, \tau_2^{*i})_{i\in I}$. Then, the outcome of such a contract is the strongly attainable allocation $\overline{x}$.*

The proposition shows that a strongly attainable allocation can be obtained as the outcome of an efficient, individually rational contract computed by centralized financial institution which chooses the weights attached to the preferences appropriately provided all agents subscribe to its services. Clearly the result ignores any incentive problems that may arise in the implementation of such a contract. Moreover, the contracting scenario leaves agents with little control over the specific contract chosen; one way of addressing this issue could be via endogenizing the $(\theta^i)_{i\in I} \gg 0$ vector. We do not pursue these possibilities here. Instead, we examine next a scenario where agents retrade with each other via successive bilateral matches and thereby exercise control over the final contract chosen.

4.2 Decentralization via bilateral contracting

We now consider a financial institution that restricts itself to conducting a bilateral trading protocol for its members. Agents are repeatedly matched with one another and this bilateral contracting process is shown to lead to efficient contracts over prices.

Building on Feldman [1973], we specify the bilateral contracting process as follows. At the price pair (p_1, p_2), an allocation (x_1, x_2) can be *bilaterally blocked* by the pair of agents $\{i, j\}$ if there exists an allocation (y_1, y_2) such that $U^k(\xi_1^k(p_1, p_1 \cdot y_1^k), \xi_2^k(p_2, p_2 \cdot y_2^k)) \geq U^k(\xi_1^k(p_1, p_1 \cdot x_1^k), \xi_2^k(p_2, p_2 \cdot x_2^k))$, $\forall k \in \{i, j\}$, with a strict inequality for some $k \in \{i, j\}$, where $\sum_{k\in\{i,j\}}(y_1^k, y_2^k) = \sum_{k\in\{i,j\}}(x_1^k, x_2^k)$.

At the price pair (p_1, p_2), a *bilateral move* from allocation (x_1, x_2) to allocation (y_1, y_2) satisfies the conditions: (i) no agent is worse-off i.e., $U^k(\xi_1^k(p_1, p_1 \cdot y_1^k), \xi_2^k(p_2, p_2 \cdot y_2^k)) \geq U^k(\xi_1^k(p_1, p_1 \cdot x_1^k), \xi_2^k(p_2, p_2 \cdot x_2^k))$, $\forall k \in I$; (ii) $(x_1^k, x_2^k) = (y_1^k, y_2^k)$ for all agents but at most two. At the price pair (p_1, p_2) and allocation (x_1, x_2), a bilateral move to allocation (y_1, y_2) is *desirable* for the pair of agents $\{i, j\}$ if $\{i, j\}$ blocks (x_1, x_2) with (y_1, y_2).

A *bilateral rotating contracting pattern* is one where agent 1 trades with agent 2, then with agent 3,..., then with agent I, agent 2 trades with agent 3, then with agent 4, ..., then with agent I, and so on until agent $I-1$ trades with agent I, after the round of trades is repeated ad infinitum. At the price pair (p_1, p_2), a bilateral rotating contracting pattern is *admissible* if a desirable bilateral move is made whenever such a move exists.

A contract $s_\Pi = (z_1, z_2(\cdot))$ is *bilaterally unimprovable* at the price p_1 if for some $p_2 \in \Pi$, the allocation $(z_1, z_2(p_2))$ is the limit point of an admissible bilateral contracting pattern at the price pair (p_1, p_2) starting from endowments ω. Given a contracting problem $\mathcal{C}$ such that $\mathcal{F}(\mathcal{C}) \neq \varnothing$, the *bilateral contracting correspondence* $b^C(\mathcal{C})$ associates to the contracting problem $\mathcal{C}$ those elements of $\mathcal{F}(\mathcal{C})$ which are bilaterally unimprovable.

We show below that the bilateral rotating contracting pattern applied to a contractable price pair leads to efficient and individually rational contracts.

Proposition **2** *Suppose that in addition to A1 and A2, we have (i) $\omega^i \gg 0$, $\forall i \in I$ and (ii) the indifference curves of each individual through $\Re^{M+N}_{++}$ do not intersect the boundary of $\Re^{M+N}_{+}$. If the price pair (p_1, p_2) is contractable, then bilateral contracting leads to an efficient, individually rational contract.*

Proposition 2 demonstrates that a process of bilateral contracting with an element of retrading can achieve the required level of coordination for the emergence of efficient, individually rational contracts from any arbitrary configuration of contractable prices.

It also describes an explicit contracting process that ensures the non-emptiness of $g(\mathcal{C})$ (simply set $g(\mathcal{C}) = b^C(\mathcal{C})$).

In the example in the previous subsection, we have shown that for a particular attainable allocation to emerge as a perfectly contracted allocation, a suitable contract that generates the perfectly contracted allocation must belong to the solution correspondence $g(\mathcal{C})$ being used in the economy. One possibility is that the solution correspondence is obtained as $g(\mathcal{C}) = b^C(\mathcal{C})$. It remains to ensure that $b^C(\mathcal{C})$ contains the contract that is needed to implement the attainable allocation.

We take as given a strongly attainable allocation $\overline{x} \gg 0$; its supporting prices $\overline{p}_1(\overline{x}) \in \Re^M_+$, $\overline{p}_2(\overline{x}) \in \Re^N_+$ are a contractable price pair. We show below that there exists a bilateral rotating contracting pattern applied to a contractable price pair supporting a strongly attainable allocation that leads to the same strongly attainable allocation.

Proposition **3** *Suppose that in addition to A1 and A2, we have (i) $\omega^i \gg 0$, $\forall i \in I$, (ii) the indifference curves of each individual through $\Re^{M+N}_{++}$ do not intersect the boundary of $\Re^{M+N}_{+}$, and (iii) the preferences over consumption in each period admits a strictly concave utility function representation for each individual $i \in I$. Given a strongly attainable allocation $\overline{x} \gg 0$, at the supporting prices $\overline{p}_1(\overline{x})$, $\overline{p}_2(\overline{x})$, the outcome of bilateral contracting contains a contract that leads to the strongly attainable allocation $\overline{x}$.*

Remark 2 Although, for concreteness, we assumed that the bilateral contracting pattern is rotating, it is worth noting that our results would go through if agents were randomly matched at each bilateral bargaining round provided, conditional on any history of previous matches, any pair of agents would be matched with probability one in some subsequent round: such a property would be satisfied if matching process was i.i.d. across rounds with each pair of agents having an equal probability of being matched.

4.3 Extension of contract to Π

Propositions 1-3 establish that two different contracting procedures, centralized banking and decentralized bilateral bargaining, lead to attainable allocations. These results, as well as the Theorem, rely on an undesirable knife-edge specification of expectations which requires that expectations be coordinated on a particular second period price. Here we extend these results by relaxing this requirement on expectations substantially provided we strengthen the condition of attainability to strong attainability.

In the proof of the first part of the Theorem and in Propositions 1-3, we set the support of $\phi^i(\overline{p}_1)$ to exactly the price $\overline{p}_2$ for each $i \in I$. By continuity, the requirement (i) of individual rationality (Definition 2) will hold at a first period price of $\overline{p}_1(\overline{x})$, for a set Π of second period prices that is close enough to and includes $\overline{p}_2(\overline{x})$.

Specifically, given a Pareto optimal allocation $\overline{x}$, let $B(\overline{p}_2(\overline{x}), \varepsilon) = \{p_2 \in \mathcal{P}_2 | \|p_2 - \overline{p}_2(\overline{x})\| < \varepsilon\}$. We are now in a position to state the following lemma:

Lemma *Suppose that in addition to* $A1$ *and* $A2$, *we have (i)* $\omega^i \gg 0$, $\forall i \in I$; *(ii) the indifference curves of each individual through* $\Re^{M+N}_{++}$ *do not intersect the boundary of* $\Re^{M+N}_{+}$ *and (iii) the utility function of each individual has a non-zero Gaussian curvature. For some* $\varepsilon > 0$ *but small enough, given a strongly attainable allocation* $(\overline{x}_1, \overline{x}_2) \gg 0$, *it is possible to construct a contract* $s_\Pi = (\overline{x}_1, z_2(\cdot))$ *which is efficient at each price pair* $(\overline{p}_1(\overline{x}), p_2)$, $p_2 \in B(\overline{p}_2(\overline{x}), \varepsilon)$ *and satisfies* $z_2(\overline{p}_2(\overline{x})) = \overline{x}_2$.

Recall that $I^-(\overline{x})$ is the set of agents for whom one has $\overline{p}_1(\overline{x}) \cdot \omega^i_1 > \overline{p}_1(\overline{x}) \cdot \overline{x}^i_1$. If $\overline{x}$ is strongly attainable, then by continuity of the marginal utilities and the continuity of demand functions, there exists $\varepsilon > 0$ such that $I^-(\overline{x}) = I^-(\overline{p}_1(\overline{x}), \overline{x}_1)$ and furthermore, the Condition (i) of Definition 2 holds for a set of second period prices Π which contains $B(\overline{p}_2(\overline{x}), \varepsilon)$. Then, the Condition (ii) of Definition 2 is satisfied whenever $\phi^i(\overline{p}_1(\overline{x}))(B(\overline{p}_2(\overline{x}), \varepsilon))$ is large enough $\forall i \in I^-(\overline{x})$. One has therefore that the strongly attainable allocation $\overline{x}$ arises as a perfectly contracted allocation whenever at the period one price $\overline{p}_1(\overline{x})$, each $i \in I^-(\overline{x})$ attaches enough probability to a neighborhood of the period two price $\overline{p}_2(\overline{x})$. Notice that there is *no* restriction on the probability measures forecasted by agents in $I \backslash I^-(\overline{x})$ at the period one price $\overline{p}_1(\overline{x})$.

Therefore, there exists a neighborhood Π of second period prices containing $\overline{p}_2(\overline{x})$ such that $\overline{x}$ is a perfectly contracted allocation whenever for each $i \in I^{-}(\overline{x})$, $\phi^i(\overline{p}_1(\overline{x}))(\Pi)$ is large enough.

Remark **3** It is straightforward to note that a similar result applies starting from an efficient allocation corresponding to a contractable price pair i.e., there exists $\varepsilon > 0$ such that if the contract (allocation) $(x_1, x_2) \gg 0$, is efficient at a contractable price pair p_1, p_2, it is possible to construct a contract $s_\Pi = (x_1, z_2(\cdot))$ which is efficient at each price pair $(p_1, \tilde{p}_2)$, $\tilde{p}_2 \in B(p_2, \varepsilon)$ and satisfies $z_2(p_2) = x_2$ (we show this explicitly in the proof of the above Lemma). At (p_1, p_2), by construction, the individual rationality constraint (Definition 2 (i)) holds for each $i \in I$ (with strict inequality for some i). As utility functions are monotone, without loss of generality, we may require that at (p_1, p_2), the individual rationality constraints hold as a strict inequality holds for all agents. As $x_2 \gg 0$ and $(\xi_1^i(p_1, p_1 \cdot x_1^i), \xi_2^i(p_2, p_2 \cdot x_2^i))_{i \in I} \gg 0$, by continuity of the marginal utilities and the continuity of demand functions, if $\|p_2 - p_2'\| < \varepsilon$, $p_2, p_2' \in \Pi$ for $\varepsilon > 0$ but small enough, there exists a redistribution of x_2 contingent on $p_2' \in \Pi$, denoted by $x_2(p_2')$ with $x_2(p_2) = x_2$ such that for each pair of agents $i, j \in I$, we have $MRS_{qr}^i(\xi^i(p', x^i(p_2'))) = MRS_{qr}^j(\xi^j(p', x^j(p_2')))$, for some $q \in M, r \in N$ (as long as N has at least two elements) where $p' = (p_1, p_2')$ and $\xi^i(p', x^i(p_2')) = (\xi_1^i(p_1, p_1 \cdot x_1^i(p_2')), \xi_2^i(p_2', p_2' \cdot x_2^i(p_2')))$ (See proof of Lemma). Moreover, as the individual rationality constraints holds as a strict inequality for each $i \in I$ at $x_2(p_2) = x_2$, as long as $\varepsilon > 0$ but small enough, the individual rationality constraints (Definition 2 (i)) continue to be satisfied at each $p' = (p_1, p_2')$. It follows that if the contract (allocation) $(x_1, x_2) \gg 0$, is efficient and individually rational at a contractable price pair p_1, p_2, there exists a neighborhood Π of second period prices containing p_2 such that the contract can be extended to Π while preserving efficiency and individual rationality whenever for each $i \in I^{-}(x)$, $\phi^i(p_1)(\Pi)$ is large enough.

Remark **4** In constructing contracts that lead to a strongly attainable allocation $\overline{x}$, we have started with the contractable pair $\overline{p}_1(\overline{x})$, $\overline{p}_2(\overline{x})$ and then extended the contract to a set Π containing $\overline{p}_2(\overline{x})$. In view of Remark 3, we could alternatively start with a contractable pair $\overline{p}_1(\overline{x})$ and $\tilde{p}_2$ where $\tilde{p}_2$ is "close" to $\overline{p}_2(\overline{x})$ and then extend the contract to a set Π containing $\overline{p}_2(\overline{x})$.

5 Concluding remarks and future research

We have presented a model of trade where agents need not be subjectively certain of period two spot prices and where intertemporal transfers of wealth are conducted via contracts that are price contingent. We propose an equilibrium for this model and characterize the set of interior allocations that can result under this equilibrium. These are a subset of the set of Pareto optimal allocations and include the interior Arrow-Debreu allocations for the economy.

We have attempted to formulate a model where in principle cooperative solutions (the contracts may be the result of the application of a cooperative solution concept or some non-cooperative procedure), expectations, and the Walrasian adjustment procedure may interact and, in principle, generate Pareto optimal allocations in the absence of informational asymmetries. The model shows that for Pareto optimal allocations to result, one does not need perfect foresight; what is needed is that agents be able to write contracts that exhaust gains to trade conditional on particular second period prices arising in the future. Viewed thus, the fact that agents cannot predict future prices in itself need not lead to inefficient resource allocations provided there is a frictionless contracting process in price contingent contracts available in the current period that allows agents to exhaust potential gains from intertemporal trade.

Our requirement of efficient contracting requires an economy wide coordination which is considerably more complex than the decentralized trade postulated in a competitive asset market as in the Radner model with perfect foresight. The paper indicates that financial institutions that facilitate the formation of these contracts, may have relevance in our formulation. Though this paper has not fully explored this issue, there is presumably a role for such institutions in the model we have proposed. For instance, it may be of interest to model the financial system using the language of networks and examine conditions under which stable networks that deliver efficient contracts emerge endogenously. It would also be useful to have an existence result which has as its focus the interaction between a fixed contracting procedure and the Walrasian adjustment process.

The presence of informational asymmetries will only exacerbate the problem of writing contracts that lead to Pareto optimality. One should not in general expect Pareto optimality with additional informational complexities. However, our model may be useful in formulating a theory of second best dynamic resource allocation in a more general set up with asymmetric information; one that does not rely too heavily on agents being sufficiently clairvoyant so as to be able to divine future prices.

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Appendix

A Proof of the Theorem

Let $x^* \gg 0$ be the perfectly contracted allocation associated with the pair (p^*, s_Π). We first show that x^* is Pareto optimal. Our assumptions on the utility functions of the agents and the requirement $x^* \gg 0$ ensure that the marginal rates of substitution of every pair of agents for every pair of commodities are well defined. For each $i \in I$, $x_1^{*i} = \xi_1^i(p_1^*, s_\Pi)$ is utility maximizing with respect to the price vector p_1^* and accordingly one has $MRS^i_{q,\overline{q}}(x^{*i}) = MRS^j_{q,\overline{q}}(x^{*j})$, $\forall i, j \in I$ and $\forall q, \overline{q} \in M$. Analogously, for each $i \in I$, $x_2^{*i} = \xi_2^i(p_2^*, s_\Pi)$ is utility maximizing with respect to the price vector p_2^* and here too one has $MRS^i_{\overline{r},r}(x^{*i}) = MRS^j_{\overline{r},r}(x^{*j})$, $\forall i, j \in I$ and $\forall \overline{r}, r \in N$.

To establish the Pareto optimality of x^*, it suffices to show that for any pair of agents i, j and any pair of commodities $q \in M, r \in N$, we have $MRS^i_{q,r}(x^{*i}) = MRS^j_{q,r}(x^{*j})$. Since s_Π is an efficient contract at p_1^*, it follows that there exist $\overline{q} \in M, \overline{r} \in N$, such that $MRS^i_{\overline{q},\overline{r}}(x^{*i}) = MRS^j_{\overline{q},\overline{r}}(x^{*j})$. The conclusion follows by observing that $MRS^k_{q,r}(x^{*k}) = MRS^k_{q,\overline{q}}(x^{*k}) MRS^k_{\overline{q},\overline{r}}(x^{*k}) MRS^k_{\overline{r},r}(x^{*k})$ for $k = i, j$.

Next, we show that (x_1^*, x_2^*) is an attainable Pareto optimal allocation. Clearly, (x_1^*, x_2^*) is supported by the price vector (p_1^*, p_2^*) such that $p_2^* \in \Pi$. By the requirement that s_Π satisfies individual rationality at p_1^* and that $p_2^* \in \Pi$, one has

$$\begin{aligned} U^i(x^{*i}) = u_1^i(x_1^{*i}) + u_2^i(x_2^{*i}) &= u_1^i(\xi_1^i(p_1^*, s_\Pi)) + u_2^i(\xi_2^i(p_2^*, s_\Pi)) \\ &\geq u_1^i(\overline{\xi}_1^i(p_1^*, p_1^* \cdot \omega_1^i)) + u_2^i(\overline{\xi}_2^i(p_2^*, p_2^* \cdot \omega_2^i)). \end{aligned}$$

Since x_1^* and x_2^* are utility maximizing at p_1^* and p_2^* respectively, p_1^* and p_2^* must respectively coincide with the price subvectors $\overline{p}_1$ and $\overline{p}_2$ that support the subvectors x_1^* and x_2^*. Therefore

$$u_1^i(\overline{\xi}_1^i(p_1^*, p_1^* \cdot \omega_1^i)) + u_2^i(\overline{\xi}_2^i(p_2^*, p_2^* \cdot \omega_2^i)) = u_1^i(\overline{\xi}_1^i(\overline{p}_1, \overline{p}_1 \cdot \omega_1^i)) + u_2^i(\overline{\xi}_2^i(\overline{p}_2, \overline{p}_2 \cdot \omega_2^i)) = \mathcal{U}^i_{x^*}$$

and one has accordingly $U^i(x^{*i}) \geq \mathcal{U}^i_{x^*}$ as required for attainability. This completes the proof of the first statement of the theorem.

Let $\overline{x} \gg 0$ be an attainable Pareto optimal allocation. We show that it is a perfectly contracted equilibrium allocation for some specification of $(\phi^i)_{i \in I}$. Recall that the price subvectors $\overline{p}_1(\overline{x})$ and $\overline{p}_2(\overline{x})$ support respectively the subvectors $\overline{x}_1$ and $\overline{x}_2$. Define for $\forall i \in I$, $\phi^i(\overline{p}_1)$ as the probability distribution that gives probability one to $\overline{p}_2(\overline{x})$, set $\Pi = \{\overline{p}_2(\overline{x})\}$, and consider the contract $s_\Pi = (\overline{x}_1, \overline{x}_2)$. For each $i \in I$, the price contingent vector $\xi^i(\overline{p}(\overline{x}), s_\Pi) = (\xi_1^i(\overline{p}_1(\overline{x}), s_\Pi), \xi_2^i(\overline{p}_2(\overline{x}), s_\Pi))$ associated with this contract for $\overline{p}_2(\overline{x})$ is indeed $(\overline{x}_1, \overline{x}_2)$. The triple $(\overline{p}(\overline{x}), \overline{x}, s_\Pi)$ satisfy Conditions (i)-(iii) of a perfectly contracted equilibrium (Definition 4). It remains to verify at s_Π is efficient and individually rational at $\overline{p}_1$. By the Pareto optimality of $(\overline{x}_1, \overline{x}_2)$ we have that s_Π is efficient at $\overline{p}_1(\overline{x})$. Since $(\overline{x}_1, \overline{x}_2)$ is attainable, it follows that s_Π satisfies Condition (i) of Definition 2. Since $\phi^i(\overline{p}_1(\overline{x})), \forall i \in I$ is the probability distribution that gives probability one to $\overline{p}_2(\overline{x})$, and we have set $\Pi = \{\overline{p}_2(\overline{x})\}$, by Observation 1, part (i) of Definition 2 implies part (ii) of Definition 2, and consequently s_Π is individually rational at $\overline{p}_1(\overline{x})$. This completes the proof of the second statement of the theorem.

Let (p', x') be an Arrow-Debreu equilibrium such that $x' \gg 0$ and $p' \in \Re_+^{M+N}$. Interpreting p'_1, p'_2 as spot prices, observe that since $\overline{\xi}_t^i(p'_t, p'_t \cdot \omega_t^i)$ solves Max $u_t^i(x_t^i)$ subject to $p'_t \cdot x_t^i \leq p'_t \cdot \omega_t^i$, for $t = 1, 2$, the vector $(\overline{\xi}_1^i(p'_1, p'_1 \cdot \omega_1^i), \overline{\xi}_2^i(p'_2, p'_2 \cdot \omega_2^i))$ belongs to the Arrow-Debreu budget set $p'_1 \cdot x_1^i + p'_2 \cdot x_2^i \leq p'_1 \cdot \omega_1^i + p'_2 \cdot \omega_2^i$, $\forall i \in I$. Therefore, $U^i(x_1'^i, x_2'^i) \geq u_1^i(\overline{\xi}_1^i(p'_1, p'_1 \cdot \omega_1^i)) + u_2^i(\overline{\xi}_2^i(p'_2, p'_2 \cdot \omega_2^i))$, $\forall i \in I$. Since x' is Pareto optimal, the subvectors x'_1 and x'_2 have supporting price subvectors which are denoted $\overline{p}_t \in \mathcal{P}_t, t = 1, 2$ respectively. By utility maximization, it must be the case that p'_t is proportional to the supporting price subvector $\overline{p}_t \in \mathcal{P}_t, t = 1, 2$, that is, $p'_t = \lambda_t \overline{p}_t, t = 1, 2$. Consequently, for each $i \in I$, $(\overline{\xi}_1^i(p'_1, p'_1 \cdot \omega_1^i), \overline{\xi}_2^i(p'_2, p'_2 \cdot \omega_2^i)) = (\overline{\xi}_1^i(\overline{p}_1, \overline{p}_1 \cdot \omega_1^i), \overline{\xi}_2^i(\overline{p}_2, \overline{p}_2 \cdot \omega_2^i))$ and we have $U^i(x'^i) \geq u_1^i(\overline{\xi}_1^i(p'_1, p'_1 \cdot \omega_1^i)) + u_2^i(\overline{\xi}_2^i(p'_2, p'_2 \cdot \omega_2^i)) = u_1^i(\overline{\xi}_1^i(\overline{p}_1, \overline{p}_1 \cdot \omega_1^i)) + u_2^i(\overline{\xi}_2^i(\overline{p}_2, \overline{p}_2 \cdot \omega_2^i)) = \mathcal{U}_{x'}^i$ as required for the attainability of x'. ∎

B Proof Proposition 1

Given a strongly attainable allocation $\overline{x} = (\overline{x}_1, \overline{x}_2) \gg 0$, consider the supporting price subvectors $\overline{p}_1(\overline{x})$ and $\overline{p}_2(\overline{x})$ interpreted as spot prices. Since A1 holds and $\overline{p}_1(\overline{x}), \overline{p}_2(\overline{x}) \gg 0$, we know that for all $i \in I$ and $\tau_1^i, \tau_2^i > 0$, $\xi_1^i(\overline{p}_1(\overline{x}), \tau_1^i)$ and $\xi_2^i(\overline{p}_2(\overline{x}), \tau_2^i)$ are interior solutions of the following maximization problem: *Max* $u_1^i(x_1^i)$ subject to $\overline{p}_1(\overline{x}) \cdot x_1^i \leq \tau_1^i$ and *Max* $u_2^i(x_2^i)$ subject to $\overline{p}_2(\overline{x}) \cdot x_2^i \leq \tau_2^i$ respectively. Furthermore, two restrictions are binding under $(\xi_1^i(\overline{p}_1(\overline{x}), \tau_1^i), \xi_2^i(\overline{p}_2(\overline{x}), \tau_2^i))$. Therefore, we have that for all $i \in I$; $q, q' \in M$ and $r, r' \in N$,

$$\overline{p}_1(\overline{x}) \cdot \frac{\partial \xi_1^i(\overline{p}_1(\overline{x}), \tau_1^i)}{\partial \tau_1^i} = 1, \qquad \overline{p}_2(\overline{x}) \cdot \frac{\partial \xi_2^i(\overline{p}_2(\overline{x}), \tau_2^i)}{\partial \tau_2^i} = 1,$$

$$\frac{\partial u_1^i(\xi_1^i(\overline{p}_1(\overline{x}), \tau_1^i))}{\partial x_{1,q}} = \overline{\lambda}^i \overline{p}_{1,q}(\overline{x}), \qquad \frac{\partial u_2^i(\xi_2^i(\overline{p}_2(\overline{x}), \tau_2^i))}{\partial x_{2,r}} = \overline{\lambda}^i \overline{p}_{2,r}(\overline{x}),$$

$$\frac{u_1^i(\xi_1^i(\overline{p}_1(\overline{x}), \tau_1^i))}{\partial x_{1,q}} \Big/ \frac{\partial u_1^i(\xi_1^i(\overline{p}_1(\overline{x}), \tau_1^i))}{\partial x_{1,q'}} = \frac{\overline{p}_{1,q}(\overline{x})}{\overline{p}_{1,q'}(\overline{x})} \quad \text{and} \quad \frac{u_2^i(\xi_2^i(\overline{p}_2(\overline{x}), \tau_2^i))}{\partial x_{2,r}} \Big/ \frac{u_2^i(\xi_2^i(\overline{p}_2(\overline{x}), \tau_2^i))}{\partial x_{2,r'}} = \frac{\overline{p}_{2,r}(\overline{x})}{\overline{p}_{2,r'}(\overline{x})}.$$

It follows that for each $i \in I$,

$$\begin{aligned}
\sum_{q' \in M} \frac{1}{\overline{\lambda}^i} \frac{\partial u_1^i(\xi_1^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i))}{\partial x_{1,q'}} \frac{\partial \xi_{1,q'}^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i)}{\partial \tau_1^i} &= 1, \\
\sum_{r' \in N} \frac{1}{\overline{\lambda}^i} \frac{\partial u_2^i(\xi_2^i(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{x}_2^i))}{\partial x_{2,r'}} \frac{\partial \xi_{2,r'}^i(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{x}_2^i)}{\partial \tau_2^i} &= 1
\end{aligned}$$

Further, for all $i \in I$, $q \in M$ and $r \in N$,

$$MRS^i_{q,r}(\xi^i(\overline{p}(\overline{x}), \overline{p}(\overline{x}) \cdot \overline{x}^i)) \equiv \frac{\partial u^i_1(\xi^i_1(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}^i_1))}{\partial x_{1,q}} \Big/ \frac{\partial u^i_2(\xi^i_2(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{x}^i_2))}{\partial x_{2,r}} = \frac{\overline{p}_{1,q}(\overline{x})}{\overline{p}_{2,r}(\overline{x})},$$

where $\xi^i(\overline{p}(\overline{x}), \overline{p}(\overline{x}) \cdot \overline{x}^i) = \left(\xi^i_1(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}^i_1), \xi^i_2(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{x}^i_2)\right)$.

Let $V^i(\overline{p}_1(\overline{x}), \overline{p}_2(\overline{x}), \tau^i_1, \tau^i_2) = u^i_1(\xi^i_1(\overline{p}_1(\overline{x}), \tau^i_1)) + u^i_2(\xi^i_2(\overline{p}_2(\overline{x}), \tau^i_2))$, $\forall i \in I$. Note that for each $i \in I$, $u^i_t(\xi^i_t(\overline{p}_t(\overline{x}), \tau^i_t))$ is the value function associated with maximization problem $Max\ u^i_t(x^i_t)$ subject to $\overline{p}_t(\overline{x}) \cdot x^i_t \leq \tau^i_t$, $t = 1, 2$, and as $u^i_t(\cdot)$ is strictly concave, $t = 1, 2$, $V^i(\overline{p}_1(\overline{x}), \overline{p}_2(\overline{x}), \tau^i_1, \tau^i_2)$ is strictly concave in τ^i_1, τ^i_2 [12] and therefore, $\mathcal{CB}$ is a well-defined concave maximization problem. To check that the Slater condition is satisfied in $\mathcal{CB}$, observe that by setting $\tau^i_t = \overline{p}_t(\overline{x}) \cdot \overline{x}^i_t > 0$, $t = 1, 2$, the individual rationality constraint is satisfied as a strict inequality for each $i \in I$ (as $\overline{x}$ is a strongly attainable allocation) so that by continuity of $\overline{\xi}^i_t(\overline{p}_t(\overline{x}), \tau^i_t)$ and $u^i_t(\cdot)$, $\forall i \in I$ and $t = 1, 2$, there exists strictly positive β strictly less than, but close to one for which all the three constriants in $\mathcal{CB}$ are satisfied as strict inequalities when $\tau^{i,\beta}_t = \beta \tau^i_t$. Therefore, the Slater condition is satisfied and it is possible to use the Kuhn-Tucker conditions to characterize the solution to $\mathcal{CB}$. To this end, the Lagrangian of $\mathcal{CB}$ is specified as follows:

$$\begin{aligned} L \;=\; & \sum_{i \in I} \theta^i V^i(\overline{p}_1(\overline{x}), \overline{p}_2(\overline{x}), \tau^i_1, \tau^i_2) - \mu_1 \Big[\sum_{i \in I} \tau^i_1 - \sum_{i \in I} \overline{p}_1(\overline{x}) \cdot \omega^i_1 \Big] - \mu_2 \Big[\sum_{i \in I} \tau^i_2 - \sum_{i \in I} \overline{p}_2(\overline{x}) \cdot \omega^i_2 \Big] \\ & + \sum_{i \in I} \mu^i \Big[u^i_1(\xi^i_1(\overline{p}_1(\overline{x}), \tau^i_1)) + u^i_2(\xi^i_2(\overline{p}_2(\overline{x}), \tau^i_2)) - u^i_1(\overline{\xi}^i_1(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \omega^i_1)) - u^i_2(\overline{\xi}^i_2(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \omega^i_2)) \Big] \end{aligned}$$

where $\theta^i = \frac{1}{\lambda^i} > 0$, $\mu_1, \mu_2 \geq 0$ and $\mu^i \geq 0$, $\forall i \in I$. Now, consider $\tau^{*i}_t = \overline{p}_t(\overline{x}) \cdot \overline{x}^i_t > 0$, $t = 1, 2$. We check that the FOCs characterizing the Kuhn-Tucker are satisfied so that $\tau^{*i}_t = \overline{p}_t(\overline{x}) \cdot \overline{x}^i_t$

[12] This statement can be derived explicitly as follows. Let $\tau = \lambda \tau' + (1 - \lambda)\tau''$ and $x_1 = \lambda \overline{\xi}^i_1(\overline{p}_1(\overline{x}), \tau') + (1 - \lambda) \overline{\xi}^i_1(\overline{p}_1(\overline{x}), \tau'')$. Then, clearly

$$\overline{p}_1(\overline{x}) \cdot x_1 = \lambda \overline{p}_1(\overline{x}) \cdot \overline{\xi}^i_1(\overline{p}_1(\overline{x}), \tau') + (1 - \lambda) \overline{p}_1(\overline{x}) \cdot \overline{\xi}^i_1(\overline{p}_1(\overline{x}), \tau'') = \lambda \tau' + (1 - \lambda)\tau'' = \tau$$

so that

$$\begin{aligned} u^i_1(\overline{\xi}^i_1(\overline{p}_1(\overline{x}), \tau)) &\geq u^i_1(x_1) \\ &= u^i_1\left(\lambda \overline{\xi}^i_1(\overline{p}_1(\overline{x}), \tau') + (1 - \lambda) \overline{\xi}^i_1(\overline{p}_1(\overline{x}), \tau'')\right) \\ &> \lambda u^i_1(\overline{\xi}^i_1(\overline{p}_1(\overline{x}), \tau')) + (1 - \lambda) u^i_1(\overline{\xi}^i_1(\overline{p}_1(\overline{x}), \tau'')) \end{aligned}$$

(where the final inequality follows from the strict concavity of $u^i_1(\cdot)$) thus establishing the strict concavity of $u^i_1(\overline{\xi}^i_1(\overline{p}_1(\overline{x}), \tau))$ in τ. A symmetric argument establishes that $u^i_2(\overline{\xi}^i_2(\overline{p}_1(\overline{x}), \tau))$ is strictly concave in τ. The strict concavity of $V^i(\overline{p}_1(\overline{x}), \overline{p}_2(\overline{x}), \tau^i_1, \tau^i_2)$ in τ^i_1, τ^i_2 for each i follows.

is the maximum. First, note that as $\overline{x}$ is a strongly attainable allocation, $\mu^i = 0$, $\forall i \in I$. At *any* solution to the maximization problem $(\tau_1^{*i}, \tau_2^{*i})_{i\in I} \gg 0$, we have that

$$\theta^i \sum_{q'\in M} \frac{\partial u_1^i(\xi_1^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i))}{\partial x_{1,q'}} \frac{\partial \xi_{1,q'}^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i)}{\partial \tau_1^i} = \mu_1.$$

Now,

$$\theta^i \sum_{q'\in M} \overline{\lambda}^i \overline{p}_{1,q'}(\overline{x}) \frac{\partial \xi_{1,q'}^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i)}{\partial \tau_1^i} = \theta^i \overline{\lambda}^i \sum_{q'\in M} \overline{p}_{1,q'}(\overline{x}) \frac{\partial \xi_{1,q'}^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i)}{\partial \tau_1^i} = 1 = \mu_1.$$

By a symmetric argument, $\mu_2 = 1$. Since $\xi_1^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i)$ is an interior solution, we know that $\frac{\partial u_1^i(\xi_1^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i))}{\partial x_{1,q}} > 0$. Further, we can re-write the FOCs as for each $i \in I$,

$$\begin{aligned}
\sum_{q'\in M} \theta^i \frac{\partial u_1^i(\xi_1^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i))}{\partial x_{1,q'}} \frac{\partial \xi_{1,q'}^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i)}{\partial \tau_1^i} &= 1, \\
\sum_{r'\in M} \theta^i \frac{\partial u_2^i(\xi_2^i(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{x}_2^i))}{\partial x_{2,r'}} \frac{\partial \xi_{2,r'}^i(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{x}_2^i)}{\partial \tau_2^i} &= 1.
\end{aligned}$$

As by assumption, $\theta^i = \frac{1}{\overline{\lambda}^i}$, $\tau_t^{*i} = \overline{p}_t(\overline{x}) \cdot \overline{x}_t^i$ is a maximum; uniqueness follows as each $V^i(\overline{p}_1(\overline{x}), \overline{p}_2(\overline{x}), \tau_1^i, \tau_2^i)$ is strictly concave in (τ_1^i, τ_2^i). It follows that the allocation corresponding to $(\tau_1^{*i}, \tau_2^{*i})$ is $\left(\xi_1^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{x}_1^i), \xi_2^i(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{x}_2^i)\right)_{i\in I} = \overline{x}$. ■

C Proof of Proposition 2

Fix a contractable price pair (p_1, p_2) with $p_1 \in \mathcal{P}_1$ and $p_2 \in \Pi$ and the initial endowments ω. Consider the sequence of allocations $\left\{y_m = (y_m^1, ..., y_m^I)\right\}_{m=1}^{\infty}$ and the corresponding sequence of utilities $\left\{U_m = (U_m^1, ..., U_m^I)\right\}_{m=1}^{\infty}$ generated by an admissible contracting pattern with where $U_m^i = U^i(\xi_1^i(p_1, p_1 \cdot y_{1,m}^i), \xi_2^i(p_2, p_2 \cdot y_{2,m}^i))$, $\forall i \in I$ and $\forall m \geq 1$; $y_0 = (\omega^1, ..., \omega^I)$; $U_0^i = U^i(\overline{\xi}_1^i(p_1, p_1 \cdot \omega_1^i), \overline{\xi}_2^i(p_2, p_2 \cdot \omega_2^i))$, $\forall i \in I$ and $U_0 = \left(U_0^1, \ldots, U_0^I\right)$. By construction each element in the sequence of allocations $\{y_m\}_{m=1}^{\infty}$ is a feasible allocation. Moreover, as $\omega_t^i \gg 0$, $\forall i \in I$ and $t = 1, 2$ and the indifference curves of each individual through $\Re_{++}^M$ (respectively, $\Re_{++}^N$) do not intersect the boundaries of $\Re_+^M$ (respectively, $\Re_+^N$), it follows that $\{y_m\}_{m=1}^{\infty}$ is contained in a compact subset of $\Re_{++}^{M+N}$.

As $\xi_1^i(p_1, p_1 \cdot y_{1,m}^i)$ (respectively, $\xi_2^i(p_2, p_2 \cdot y_{2,m}^i)$) is the solution to Max $u_1^i(x_1^i)$ subject to $p_1 \cdot x_1^i \leq p_1 \cdot y_{1,m}^i$ (respectively, the solution to Max $u_2^i(x_2^i)$ subject to $p_2 \cdot x_2^i \leq p_2 \cdot y_{2,m}^i$), each $\{y_m\}_{m=1}^{\infty}$ is contained in a compact subset of $\Re_{++}^{M+N}$, and by assumption the indifference curves of each individual through $\Re_{++}^M$ (respectively, $\Re_{++}^N$) do not intersect the boundaries of $\Re_+^M$ (respectively, $\Re_+^N$), it follows that $\left(\xi_1^i(p_1, p_1 \cdot y_{1,m}^i), \xi_2^i(p_2, p_2 \cdot y_{2,m}^i)\right) \gg$

0 and that these are continuous functions of y_m, $\forall m \geq 1$; therefore, for each $i \in I$, $\big(\xi_1^i(p_1, p_1 \cdot y_{1,m}^i), \xi_2^i(p_2, p_2 \cdot y_{2,m}^i)\big)$ is contained in a compact subset of $\Re_{++}^{M+N}$.

Next, it follows that, by continuity of utility functions, the sequence $\{U_m\}_{m=1}^{\infty}$ is contained in a compact set. Moreover, by construction, $U_m \geq U_{m-1}$, $\forall m \geq 1$ so that $\{U_m\}_{m=1}^{\infty}$ is an increasing sequence and as every increasing sequence in a compact set converges to a supremum, it follows that $\{U_m\}_{m=1}^{\infty}$ converges to $\overline{U} = (\overline{U}^1, ..., \overline{U}^I)$, the component-wise supremum of $\{U_m\}_{m=1}^{\infty}$. Then, as the set of feasible allocations is compact, $\{y_m\}_{m=1}^{\infty}$ has a limit point $\overline{y}$. As $\{y_m\}_{m=1}^{\infty}$ is contained in a compact subset of $\Re_{++}^{M+N}$, it follows that $\overline{y} \in \Re_{++}^{M+N}$ and further, by continuity of $(\xi_1^i(p_1, \cdot), \xi_2^i(p_2, \cdot))_{i \in I}$ and utility functions $(U^i)_{i \in I}$,

$$\begin{aligned} U(\overline{y}) &= \big(U^1(\xi_1^1(p_1, p_1 \cdot \overline{y}_1^1), \xi_2^1(p_2, p_2 \cdot \overline{y}_2^1)), \ldots, U^I(\xi_1^I(p_1, p_1 \cdot \overline{y}_1^I), \xi_2^I(p_2, p_2 \cdot \overline{y}_2^I))\big) \\ &= \overline{U}. \end{aligned}$$

Define $\succ$ in the set of feasible allocations F as follows. Given any two feasible allocations x, y, $x \succ y$ if either (i) y is an immediate predecessor of x in a bilateral contracting pattern and x is a desirable move from y, or (ii) starting from y, x is the limit point of a sequence of bilateral moves contained in a bilateral contracting pattern with at least one such move being desirable. Clearly, $\succ$ is transitive on F: if x, y, z are all feasible allocations and $x \succ y$ and $y \succ z$, then by (ii), $x \succ z$. Therefore, $\succ$ is a partial order on F (Kelley [1955], page 13) and $\{y_m\}_{m=1}^{\infty}$ is a chain under $\succ$. Therefore, Zorn's lemma (Kelley [1955], page 33), guarantees that F has a maximal element. A maximal element in F under $\succ$ cannot be preceded by any other element of F under $\succ$. Therefore, if $\overline{y}$ is an upper bound of the chain $\{y_m\}_{m=1}^{\infty}$, $\overline{y}$ is bilaterally unimprovable.

As $\overline{y} \gg 0$ and by assumption that the indifference curves of each individual through $\Re_{++}^M$ (respectively, $\Re_{++}^N$) do not intersect the boundaries of $\Re_+^M$ (respectively, $\Re_+^N$), by continuity of $(\xi_1^i(p_1, \cdot), \xi_2^i(p_2, \cdot))_{i \in I}$, $\big(\xi_1^i(p_1, p_1 \cdot \overline{y}_1^i), \xi_2^i(p_2, p_2 \cdot \overline{y}_2^i)\big)_{i \in I} \gg 0$ and hence, it is efficient i.e., for each pair of agents $i, j \in I$, we have $MRS_{q,r}^i(\xi^i(p, p \cdot \overline{y}^i)) = MRS_{q,r}^j(\xi^j(p, p \cdot \overline{y}^j))$, $\forall q \in M$ and $\forall r \in N$. ■

D Proof of Proposition 3

We take as given a strongly attainable allocation $\overline{x} \gg 0$; its supporting prices $\overline{p}_1(\overline{x}), \overline{p}_2(\overline{x})$ are a contractable price pair. Assume that at the price pair $\overline{p}_1(\overline{x}), \overline{p}_2(\overline{x})$ and allocation (y_1, y_2), whenever a desirable bilateral move exists for a pair of agents $\{i, j\}$, the desirable move that is implemented is made as follows.

Firstly, starting from any allocation (y_1, y_2), a contract is specified by choosing a wealth vector $(\tau_1^{*k}, \tau_2^{*k})_{k \in \{i,j\}}$ which solves the following maximization problem $(\mathcal{K})$:

$$\begin{aligned}
\max_{(\tau_1^k,\tau_2^k)_{k\in\{i,j\}}} \quad & \sum_{k\in\{i,j\}} \theta^k[u_1^k(\xi_1^k(\overline{p}_1(\overline{x}),\tau_1^k)) + u_2^k(\xi_2^k(\overline{p}_2(\overline{x}),\tau_2^k))] \\
s.t. \quad & \sum_{k\in\{i,j\}} \tau_1^k \le \sum_{k\in\{i,j\}} \overline{p}_1(\overline{x})\cdot y_1^k \\
& \sum_{k\in\{i,j\}} \tau_2^k \le \sum_{k\in\{i,j\}} \overline{p}_2(\overline{x})\cdot y_2^k \\
& \sum_{t=1}^{2} u_t^k(\overline{\xi}_t^k(\overline{p}_t(\overline{x}),\overline{p}_t(\overline{x})\cdot y_t^k)) \le \sum_{t=1}^{2} u_t^k(\xi_t^k(\overline{p}_t(\overline{x}),\tau_t^k)),\ \forall k\in\{i,j\}
\end{aligned}$$

where $\theta^i = \frac{1}{\overline{\lambda}^i}$, $\forall i\in I$.

This is a well-defined concave maximization problem (see the proof of Proposition 1). Let (y_1', y_2'), where $\sum_{k\in\{i,j\}}(y_1'^k, y_2'^k) = \sum_{k\in\{i,j\}}(y_1^k, y_2^k)$ be such that $\left(\tau_1^{*k} = \overline{p}_1(\overline{x})\cdot y_1'^k, \tau_2^{*k} = \overline{p}_2(\overline{x})\cdot y_2'^k\right)_{k\in\{i,j\}}$. At the price pair $\overline{p}_1(\overline{x}), \overline{p}_2(\overline{x})$, the allocation (y_1', y_2') results from a *bilateral move* from allocation (y_1, y_2).

Consider the sequence of allocations $\{y_m\}_{m=1}^{\infty}$ and the corresponding sequence of utilities $\{U_m\}_{m=1}^{\infty}$ generated by bilateral contracting where a bilateral move is required to be consistent with the maximization problem $(\mathcal{K})$ and $y_0 = (\omega^i)_{i\in I}$ and $U_0 = (U^i(\overline{\xi}_1^i(\overline{p}_1(\overline{x}),\overline{p}_1(\overline{x})\cdot \omega_1^i), \overline{\xi}_2^i(\overline{p}_2(\overline{x}),\overline{p}_2(\overline{x})\cdot\omega_2^i)))_{i\in I}$. Following the steps in the proof of Proposition 2, we conclude that $\{U_m\}_{m=1}^{\infty}$ converges to $\overline{U} = (\overline{U}^1, ..., \overline{U}^I)$, the component-wise supremum of $\{U_m\}_{m=1}^{\infty}$. Moreover, each $\overline{y}$ which is an upper bound of the chain $\{y_m\}_{m=1}^{\infty}$ with $\overline{U} = U(\overline{y})$ is also bilaterally unimprovable and as $\overline{y} \gg 0$,

$$MRS_{q,r}^i(\xi^i(\overline{p}(\overline{x}),\overline{p}(\overline{x})\cdot\overline{y}^i)) = MRS_{q,r}^j(\xi^j(\overline{p}(\overline{x}),\overline{p}(\overline{x})\cdot\overline{y}^j)) = \frac{\overline{p}_{1,q}(\overline{x})}{\overline{p}_{2,r}(\overline{x})},\ \forall q\in M \text{ and } \forall r\in N.$$

Moreover, at $\overline{y}$, $\overline{p}_t(\overline{x})\cdot\overline{y}_t^i > 0$, $\forall i\in I$ and $t=1,2$. Therefore, for each $i\in I$, $\xi^i(\overline{p}(\overline{x}),\overline{p}(\overline{x})\cdot\overline{y}^i)$ must satisfy the condition that for some $\lambda^i > 0$,

$$\begin{aligned}
\frac{\partial u_1^i(\xi_1^i(\overline{p}_1(\overline{x}),\overline{p}_1(\overline{x})\cdot\overline{y}_1^i))}{\partial x_{1,q}} - \lambda^i\overline{p}_{1,q}(\overline{x}) &= 0,\ \forall q\in M,\\
\frac{\partial u_2^i(\xi_2^i(\overline{p}_2(\overline{x}),\overline{p}_2(\overline{x})\cdot\overline{y}_2^i))}{\partial x_{2,r}} - \lambda^i\overline{p}_{2,r}(\overline{x}) &= 0,\ \forall r\in N.
\end{aligned}$$

Furthermore, we must also have that

$$\begin{aligned}
\sum_{q'\in M}\frac{1}{\overline{\lambda}^i}\frac{\partial u_1^i(\xi_1^i(\overline{p}_1(\overline{x}),\overline{p}_1(\overline{x})\cdot\overline{y}_1^i))}{\partial x_{1,q'}}\frac{\xi_{1,q'}^i(\overline{p}_1(\overline{x}),\overline{p}_1(\overline{x})\cdot\overline{y}_1^i)}{\partial\tau_1^i} &= 1,\\
\sum_{r'\in N}\frac{1}{\overline{\lambda}^i}\frac{\partial u_2^i(\xi_2^i(\overline{p}_2(\overline{x}),\overline{p}_2(\overline{x})\cdot\overline{y}_2^i))}{\partial x_{2,r'}}\frac{\xi_{2,r'}^i(\overline{p}_2(\overline{x}),\overline{p}_2(\overline{x})\cdot\overline{y}_2^i)}{\partial\tau_1^i} &= 1.
\end{aligned}$$

Next, we show that $\overline{x}$ is an upper bound of the chain. To this end, we begin by setting $\lambda^i = \overline{\lambda}^i$ and show that there exists $\overline{y}$ such that $\overline{y} = \overline{x}$. Note that, by construction, each $\overline{y}$ solves $(\mathcal{K})$ for each pair of agents i, j and moreover, as the individual rationality constraint is always trivially satisfied at $\overline{y}$ (so that the associated Lagrangian multiplier can be set equal to zero for each pair of agents), by adapting the argument and the relevant expressions in Proposition 1, we must have that

$$\begin{aligned}
\sum_{q' \in M} \theta^k \frac{\partial u_1^k(\xi_1^k(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{y}_1^k))}{\partial x_{1,q'}} \frac{\partial \xi_{1,q'}^k(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{y}_1^k)}{\partial \tau_1^k} &= \mu_1^{ij}, \\
\sum_{r' \in N} \theta^k \frac{\partial u_2^k(\xi_2^k(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{y}_2^k))}{\partial x_{2,r'}} \frac{\partial \xi_{2,r'}^k(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{y}_2^k)}{\partial \tau_2^k} &= \mu_2^{ij},
\end{aligned}$$

(where $k \in \{i, j\}$ and μ_t^{ij}, $t = 1, 2$, is the Lagrangian multiplier associated with the constraint $\sum_{k \in \{i,j\}} \tau_t^k \leq \sum_{k \in \{i,j\}} \overline{p}_t(\overline{x}) \cdot \overline{y}_t^k$). Furthermore, fixing agent $i \in I$, note that at each $t = 1, 2$, as $(\mathcal{K})$ must have the same solution for any other agent j she is matched with, we have that $\mu_t^{ij} = \theta^i \overline{\lambda}^i = \mu_t = 1$. Moreover, $\xi_t^i(\overline{p}_t(\overline{x}), \overline{p}_t(\overline{x}) \cdot \overline{y}_t^i)$ must satisfy both the two preceeding equations and the following two equations

$$\begin{aligned}
\sum_{q' \in M} \frac{1}{\overline{\lambda}^i} \frac{\partial u_1^i(\xi_1^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{y}_1^i))}{\partial x_{1,q'}} \frac{\partial \xi_{1,q'}^i(\overline{p}_1(\overline{x}), \overline{p}_1(\overline{x}) \cdot \overline{y}_1^i)}{\partial \tau_1^i} &= 1, \\
\sum_{r' \in N} \frac{1}{\overline{\lambda}^i} \frac{\partial u_2^i(\xi_2^i(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{y}_2^i))}{\partial x_{2,r'}} \frac{\partial \xi_{2,r'}^i(\overline{p}_2(\overline{x}), \overline{p}_2(\overline{x}) \cdot \overline{y}_2^i)}{\partial \tau_2^i} &= 1
\end{aligned}$$

as well. It follows that $\overline{p}_t(\overline{x}) \cdot \overline{y}_t^i = \overline{p}_t(\overline{x}) \cdot \overline{x}_t^i$ and therefore, $\xi_t^i(\overline{p}_t(\overline{x}), \overline{p}_t(\overline{x}) \cdot \overline{y}_t^i) = \overline{x}_t^i$ so that $\overline{y}_t^i = \overline{x}_t^i$, $\forall i \in I$ and $t = 1, 2$ (by strict concavity of the utility function of each individual the maximization problem $Max\ u_t^i(x_t^i)$ subject to $\overline{p}_t(\overline{x}) \cdot x_t^i \leq \overline{p}_t(\overline{x}) \cdot \overline{x}_t^i$, has a unique solution at each $t = 1, 2$), as required. Further, $\overline{U} = U(\overline{x})$ and for any other upper bound $\overline{y}$ of the chain $\{y_m\}_{m=1}^{\infty}$, (i) $U(\overline{x}) = U(\overline{y})$ so that $\overline{x}^i$ and $\overline{y}^i$ lie on the same indifference curve for each individual $i \in I$, and (ii) further, for each individual $i \in I$,

$$MRS_{q,r}^i(\xi^i(\overline{p}(\overline{x}), \overline{p}(\overline{x}) \cdot \overline{x}^i)) = MRS_{q,r}^i(\xi^i(\overline{p}(\overline{x}), \overline{p}(\overline{x}) \cdot \overline{y}^i)) = \frac{\overline{p}_{1,q}(\overline{x})}{\overline{p}_{2,r}(\overline{x})},\ \forall q \in M \text{ and } \forall r \in N,$$

so that as $U^i(\cdot)$ is strictly concave, $\xi^i(\overline{p}(\overline{x}), \overline{p}(\overline{x}) \cdot \overline{x}^i) = \xi^i(\overline{p}(\overline{x}), \overline{p}(\overline{x}) \cdot \overline{y}^i) = \overline{x}^i$ for each $i \in I$. ∎

E Proof of the Lemma

Under A1 and A2, as $\omega^i \gg 0$, $\forall i \in I$ and the indifference curves of each individual through $\Re_{++}^{M+N}$ do not intersect the boundaries of $\Re_{+}^{M+N}$, both $\overline{x}_2 \gg 0$ and

$$\left(\xi_1^i(p_1, p_1 \cdot \overline{x}_1^i) = \overline{x}_1^i, \xi_2^i(p_2, p_2 \cdot \overline{x}_2^i) = \overline{x}_2^i\right)_{i \in I} \gg 0.$$

Pick a pair of commodities m in period one and n in period two. At the attainable allocation, we have that for all $i, j \in I$, $\frac{\partial u_1^i(\overline{x}_1^i)}{\partial x_{1,m}} / \frac{\partial u_2^i(\overline{x}_2^i)}{x_{2,n}} = \frac{\partial u_1^j(\overline{x}_1^j)}{\partial x_{1,m}} / \frac{\partial u_2^j(\overline{x}_2^j)}{x_{2,n}} \Leftrightarrow \frac{\partial u_1^i(\overline{x}_1^i)}{\partial x_{1,m}} / \frac{\partial u_1^j(\overline{x}_1^j)}{x_{1,m}} = \frac{\partial u_2^i(\overline{x}_2^i)}{\partial x_{2,n}} / \frac{\partial u_2^j(\overline{x}_2^j)}{x_{2,n}}$. Let $m(\overline{x}) = \frac{\partial u_1^i(\overline{x}_1^i)}{\partial x_{1,m}} / \frac{\partial u_2^j(\overline{x}_1^j)}{x_{1,m}}$. The function defined by

$$\varphi(\overline{x}, \omega, p_2, x_2, x_2', \lambda) = \begin{cases} \frac{\partial u_2^i(\overline{x}_2^i)}{\partial x_2} - \lambda^i p_2, & i = 1, 2, ..., I \\ p_2 \cdot (x_2^i - x_2'^i), & i = 1, 2, ..., I \\ \frac{\partial u_2^1(\overline{x}_2^1)}{\partial x_{2,n}} m(\overline{x}) - \frac{\partial u_2^j(\overline{x}_2^j)}{\partial x_{2,n}}, & j = 2, ..., I \\ \sum_{i \in I}(x_2'^i - \omega_2^i) \end{cases}$$

characterizes the conditions for solution that have to met at an efficient contract at each $p_2 \in B(\overline{p}_2(\overline{x}), \varepsilon) = \{p_2 \in \mathcal{P}_2 | \|p_2 - \overline{p}_2(\overline{x})\| < \varepsilon\}$: at $p_2 \in B(\overline{p}_2(\overline{x}), \varepsilon)$ and the attainable allocation $\overline{x}$, it is necessary and sufficient for an efficient contract that $\varphi(\overline{x}, \omega, p_2, x_2, x_2', \lambda) = 0$ for some Lagrangian multipliers $\lambda = (\lambda^i)_{i \in I} \gg 0$. The dimension of the domain of the function φ is greater than the dimension of its range provided $N \geq 2$. Moreover, as long as the utility function of each individual satisfies the assumption that the utility functions within each period has a non-zero Gaussian curvature, zero is a regular point for the function $\varphi(\overline{x}, \omega, p_2, x_2, \overline{x}_2, \overline{\lambda})$ for some Lagrangian multipliers $\overline{\lambda} = (\overline{\lambda}^i)_{i \in I} \gg 0$. Therefore, if $p_2 \in B(\overline{p}_2(\overline{x}), \varepsilon)$ for some $\varepsilon > 0$ but small enough, there exists a redistributions of commodities in period two, $x_2'^i - \omega_2^i$, such that $\varphi(\overline{x}, \omega, p_2, x_2, x_2', \lambda) = 0$ for some Lagrangian multipliers $\lambda = (\lambda^i)_{i \in I} \gg 0$.

Finally note that if the starting point is an efficient allocation y that is not attainable, then the above argument continues to apply if we replace $\overline{x}$ by y in $\varphi(\cdot, \omega, p_2, x_2, x_2', \lambda)$. ■

F Computational details of Subsection 3.1.2

Case A:
$\mathcal{C} \equiv (p, \Pi, (\phi^i)_{i \in I}) = (9, \{4\}, (\phi^i(9)(\{4\}) = 1)_{i \in I})$.

The Walrasian demand with a contract $(\varepsilon, \delta^\varepsilon(4))$ is

$$\xi^1 = ((\tfrac{9\varepsilon+1}{10}, \tfrac{9\varepsilon+1}{10}), (\tfrac{4[1-\delta^\varepsilon(4)]}{5}, \tfrac{4[1-\delta^\varepsilon(4)]}{5})) \text{ and } \xi^2 = ((\tfrac{9(1-\varepsilon)}{10}, \tfrac{9(1-\varepsilon)}{10}), (\tfrac{4\delta^\varepsilon(4)+1}{5}, \tfrac{4\delta^\varepsilon(4)+1}{5})).$$

Next, the Walrasian demand without a contract is

$$\overline{\xi}^1 = ((\tfrac{1}{10}, \tfrac{1}{10}), (\tfrac{4}{5}, \tfrac{4}{5})) \text{ and } \overline{\xi}^2 = ((\tfrac{9}{10}, \tfrac{9}{10}), (\tfrac{1}{5}, \tfrac{1}{5})).$$

Under a contract, $MRS^1_{x_{11}, x_{21}}(\xi^1) = \frac{9[1-\delta^\varepsilon(4)]}{9\varepsilon+1}$ and $MRS^2_{x_{11}, x_{21}}(\xi^2) = \frac{1+4\delta^\varepsilon(4)}{4(1-\varepsilon)}$. Hence, $\delta^\varepsilon(4) = \frac{7-9\varepsilon}{8}$. Substituting into $\xi^i, i = 1, 2$, we get

$$\xi^1 = ((\tfrac{1+9\varepsilon}{10}, \tfrac{1+9\varepsilon}{10}), (\tfrac{1+9\varepsilon}{10}, \tfrac{1+9\varepsilon}{10})) \text{ and } \xi^2 = ((\tfrac{9(1-\varepsilon)}{10}, \tfrac{9(1-\varepsilon)}{10}), (\tfrac{9(1-\varepsilon)}{10}, \tfrac{9(1-\varepsilon)}{10})).$$

The indirect utility with a contract is

$$\mathcal{U}^1(\varepsilon) \equiv \mathcal{U}^1(\xi^1) = 2\ln\tfrac{9\varepsilon+1}{10} \text{ and } \mathcal{U}^2(\varepsilon) \equiv \mathcal{U}^2(\xi^2) = 2\ln\tfrac{9(1-\varepsilon)}{10}.$$

The indirect utility without a contract (Threat point) is

$$\bar{\mathcal{U}}^1 \equiv \mathcal{U}^1(\bar{\xi}^1) = \ln\tfrac{2}{25} \text{ and } \bar{\mathcal{U}}^2 \equiv \mathcal{U}^2(\bar{\xi}^2) = \ln\tfrac{9}{50}.$$

Therefore, by individual rationality,

$$\begin{aligned}
\mathcal{F}(\mathcal{C}) &= \{(\varepsilon, \delta^\varepsilon(4)) \,|\, 0 \le \varepsilon \le 1; (1+9\varepsilon)^2 \ge 8; (1-\varepsilon)^2 \ge \frac{2}{9}; \delta^\varepsilon(4) = \frac{7-9\varepsilon}{8}\} \\
&= \{(\varepsilon, \delta^\varepsilon(4)) \,|\, \frac{2\sqrt{2}-1}{9} \le \varepsilon \le \frac{3-\sqrt{2}}{3}; \delta^\varepsilon(4) = \frac{7-9\varepsilon}{8}\}
\end{aligned}$$

Using the indirect utility with a contract $(\mathcal{U}^1(\varepsilon),\mathcal{U}^2(\varepsilon))$, we get the utility possiblity set: $\mathcal{U} \equiv \{(\mathcal{U}^1,\mathcal{U}^2) \in \Re^2 \,|\, e^{\frac{\mathcal{U}^1}{2}} + e^{\frac{\mathcal{U}^2}{2}} \le 1\}$ which is convex.

The Nash Bargaining solution is the following:

$$\begin{aligned}
\varepsilon^{NB} &= arg \max_{\varepsilon \in \mathcal{F}(\mathcal{C})} [\mathcal{U}^1(\varepsilon) - \bar{\mathcal{U}}^1] \cdot [\mathcal{U}^2(\varepsilon) - \bar{\mathcal{U}}^2] \\
&= arg \max_{\varepsilon \in \mathcal{F}(\mathcal{C})} [2\ln\frac{9\varepsilon+1}{10} - \ln\frac{2}{25}] \cdot [2\ln\frac{9(1-\varepsilon)}{10} - \ln\frac{9}{50}]
\end{aligned}$$

The f.o.c. is $\frac{9}{1+9\varepsilon}[2\ln(1-\varepsilon)+\ln\frac{9}{2}] - \frac{1}{1-\varepsilon}[2\ln(1+9\varepsilon)+\ln\frac{1}{8}] = 0$. Then, we get $\varepsilon^{NB} \approx 0.361828$. Accordingly, the perfectly contracted allocation is $\xi^1 = ((0.425645, 0.425645), (0.425645, 0.425645))$; $\xi^2 = ((0.574355, 0.574355), (0.574355, 0.574355))$.

Case B:

$\mathcal{C} \equiv (p, \Pi, (\phi^i)_{i\in I}) = (9, [4-\sigma, 4+\sigma], (\phi^i(9)(q) = \frac{q+\sigma-4}{2\sigma})_{i\in I})$, $0 < \sigma < 4$.

As in Case A, we first compute

$$\begin{aligned}
\xi^1(\varepsilon, q) &= ((\frac{1+9\varepsilon}{10}, \frac{1+9\varepsilon}{10}), (\frac{4[1-\delta^\varepsilon(q)]}{5}, \frac{q[1-\delta^\varepsilon(q)]}{5})), \\
\xi^2(\varepsilon, q) &= ((\frac{9(1-\varepsilon)}{10}, \frac{9(1-\varepsilon)}{10}), (\frac{4[1+q\delta^\varepsilon(q)]}{5q}, \frac{[1+q\delta^\varepsilon(q)]}{5})); \\
\bar{\xi}^1(q) &= ((\frac{1}{10}, \frac{1}{10}), (\frac{4}{5}, \frac{q}{5})) \text{ and } \bar{\xi}^2(q) = ((\frac{9}{10}, \frac{9}{10}), (\frac{4}{5q}, \frac{1}{5})).
\end{aligned}$$

Under a contract, $MRS^1_{x_{11},x_{21}}(\xi^1(\varepsilon, q)) = \frac{9[1-\delta^\varepsilon(q)]}{9\varepsilon+1}$ and $MRS^2_{x_{11},x_{21}}(\xi^2(\varepsilon, q)) = \frac{1+q\delta^\varepsilon(q)}{q(1-\varepsilon)}$, $\forall q \in \Pi$. Hence, $\delta^\varepsilon(q) = \frac{9-9\varepsilon(1+\frac{1}{q})-\frac{1}{q}}{10}$, for $\forall q \in \Pi$. Substituting into $\xi^i, i = 1, 2$, we get

$$\begin{aligned}
\xi^1(\varepsilon, q) &= ((\frac{1+9\varepsilon}{10}, \frac{1+9\varepsilon}{10}), (\frac{1+9\varepsilon}{10}\frac{4(1+q)}{5q}, \frac{1+9\varepsilon}{10}\frac{1+q}{5})) \quad \text{and} \\
\xi^2(\varepsilon, q) &= ((\frac{9(1-\varepsilon)}{10}, \frac{9(1-\varepsilon)}{10}), (\frac{9(1-\varepsilon)}{10}\frac{4(1+q)}{5q}, \frac{9(1-\varepsilon)}{10}\frac{1+q}{5})).
\end{aligned}$$

Given $\forall q \in \Pi$, the indirect utility with and without contract are:

$$\begin{aligned}
\mathcal{U}^1(\varepsilon,q) &\equiv \mathcal{U}^1(\xi^1(\varepsilon,q)) = 2\ln\frac{1+9\varepsilon}{10} + \ln(1+\frac{1}{q}) + \frac{4}{5}\ln\frac{4}{5} + \frac{1}{5}\ln\frac{q}{5}\\
\mathcal{U}^2(\varepsilon,q) &\equiv \mathcal{U}^1(\xi^2(\varepsilon,q)) = 2\ln\frac{9(1-\varepsilon)}{10} + \ln(1+\frac{1}{q}) + \frac{4}{5}\ln\frac{4}{5} + \frac{1}{5}\ln\frac{q}{5}\\
\bar{\mathcal{U}}^1(q) &\equiv \mathcal{U}^1(\bar{\xi}^1(q)) = \ln\frac{1}{10} + \frac{4}{5}\ln\frac{4}{5} + \frac{1}{5}\ln\frac{q}{5}\\
\bar{\mathcal{U}}^2(q) &\equiv \mathcal{U}^2(\bar{\xi}^2(q)) = \ln\frac{9}{10} + \frac{4}{5}\ln\frac{4}{5q} + \frac{1}{5}\ln\frac{1}{5}
\end{aligned}$$

Hence, $\mathcal{U}^1(\varepsilon,q) - \bar{\mathcal{U}}^1(q) = 2\ln(1+9\varepsilon) + \ln(1+\frac{1}{q}) + \ln\frac{1}{10}$ and $\mathcal{U}^2(\varepsilon,q) - \bar{\mathcal{U}}^2(q) = 2\ln(1-\varepsilon) + \ln(1+q) + \ln\frac{9}{10}$.

Since $\frac{\partial[\mathcal{U}^1(\varepsilon,q)-\bar{\mathcal{U}}^1(q)]}{\partial q} = -\frac{1}{q^2+q} < 0$ and $\frac{\partial[\mathcal{U}^2(\varepsilon,q)-\bar{\mathcal{U}}^2(q)]}{\partial q} = \frac{1}{1+q} > 0$, $\forall q \in \Pi$, in view of individual rationality, we only need to consider $\mathcal{U}^1(\varepsilon, 4+\sigma) - \bar{\mathcal{U}}^1(4+\sigma)$ and $\mathcal{U}^2(\varepsilon, 4-\sigma) - \bar{\mathcal{U}}^2(4-\sigma)$, which gives

$$\mathcal{F}(\mathcal{C}) = \Big\{(\varepsilon, \delta^\varepsilon(q)), q\in\Pi | 0\le\varepsilon\le 1; (1+9\varepsilon)^2 \ge \frac{10(4+\sigma)}{5+\sigma}; (1-\varepsilon)^2 \ge \frac{10}{9(5-\sigma)}; \delta^\varepsilon(q) = \frac{9-9\varepsilon(1+\frac{1}{q})-\frac{1}{q}}{10}\Big\}.$$

The Nash Bargaining solution is based on the expected utilities of the agents, which are

$$\begin{aligned}
\mathcal{U}^1(\varepsilon) \equiv E_q[\mathcal{U}^1(\varepsilon,q)] &= \int_{4-\sigma}^{4+\sigma} \mathcal{U}^1(\varepsilon,q)\frac{1}{2\sigma}dq\\
&= 2\ln\frac{1+9\varepsilon}{10} + \frac{4}{5}\ln\frac{4}{5} + \frac{1}{5}\ln\frac{1}{5} - \frac{1}{5} - \frac{2}{5\sigma}[(4+\sigma)\ln(4+\sigma) - (4-\sigma)\ln(4-\sigma)]\\
&\quad + \frac{1}{2\sigma}[(5+\sigma)\ln(5+\sigma) - (5-\sigma)\ln(5-\sigma)]
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}^2(\varepsilon) \equiv E_q[\mathcal{U}^2(\varepsilon,q)] &= \int_{4-\sigma}^{4+\sigma} \mathcal{U}^2(\varepsilon,q)\frac{1}{2\sigma}dq\\
&= 2\ln\frac{9(1-\varepsilon)}{10} + \frac{4}{5}\ln\frac{4}{5} + \frac{1}{5}\ln\frac{1}{5} - \frac{1}{5} - \frac{2}{5\sigma}[(4+\sigma)\ln(4+\sigma) - (4-\sigma)\ln(4-\sigma)]\\
&\quad + \frac{1}{2\sigma}[(5+\sigma)\ln(5+\sigma) - (5-\sigma)\ln(5-\sigma)]
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{U}}^1 \equiv E_q[\bar{\mathcal{U}}^1(q)] &= \int_{4-\sigma}^{4+\sigma} \bar{\mathcal{U}}^1(q)\frac{1}{2\sigma}dq\\
&= \ln\frac{1}{10} + \frac{4}{5}\ln\frac{4}{5} + \frac{1}{5}\ln\frac{1}{5} - \frac{1}{5} + \frac{1}{10\sigma}[(4+\sigma)\ln(4+\sigma) - (4-\sigma)\ln(4-\sigma)]
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{U}}^2 \equiv E_q[\bar{\mathcal{U}}^2(q)] &= \int_{4-\sigma}^{4+\sigma} \bar{\mathcal{U}}^2(q)\frac{1}{2\sigma}dq\\
&= \ln\frac{9}{10} + \frac{4}{5}\ln\frac{4}{5} + \frac{1}{5}\ln\frac{1}{5} + \frac{4}{5} - \frac{2}{5\sigma}[(4+\sigma)\ln(4+\sigma) - (4-\sigma)\ln(4-\sigma)]
\end{aligned}$$

According to the indirect utility with contract $(\mathcal{U}^1(\varepsilon),\mathcal{U}^2(\varepsilon))$, we get the utility possible set: $\mathcal{U} \equiv \{(\mathcal{U}^1,\mathcal{U}^2) \in \Re^2 \,|\, e^{\frac{\mathcal{U}^1-f(\sigma)}{2}} + e^{\frac{\mathcal{U}^2-f(\sigma)}{2}} \leq 1\}$, where $f(\sigma) = \frac{1}{2\sigma}[(5+\sigma)\ln(5+\sigma) - (5-\sigma)\ln(5-\sigma)] - \frac{2}{5\sigma}[(4+\sigma)\ln(4+\sigma) - (4-\sigma)\ln(4-\sigma)] + \frac{4}{5}\ln\frac{4}{5} + \frac{1}{5}\ln\frac{1}{5} - \frac{1}{5}$. Similar to Case A, the utility possible set is convex.

The Nash Bargaining solution is the following:

$$\begin{aligned}
\varepsilon^{NB} &= arg \max_{\varepsilon \in \mathcal{F}(\mathcal{C})} [\mathcal{U}^1(\varepsilon) - \bar{\mathcal{U}}^1] \cdot [\mathcal{U}^2(\varepsilon) - \bar{\mathcal{U}}^2] \\
&= arg \max_{\varepsilon \in \mathcal{F}(\mathcal{C})} [2\ln(1+9\varepsilon) + \ln\frac{1}{10} + c_1(\sigma) - c_2(\sigma)] \cdot [\,2\ln(1-\varepsilon) + \ln\frac{9}{10} - 1 + c_1(\sigma)]
\end{aligned}$$

where $c_1(\sigma) = \frac{1}{2\sigma}[(5+\sigma)\ln(5+\sigma) - (5-\sigma)\ln(5-\sigma)]$ and $c_2(\sigma) = \frac{1}{2\sigma}[(4+\sigma)\ln(4+\sigma) - (4-\sigma)\ln(4-\sigma)]$.

Setting $\sigma = 2.25$, we get $\varepsilon^{NB} \approx 0.3556$. The perfectly contracted allocation is $\xi^1 = ((0.42004, 0.42004), (0.42004, 0.42004))$; $\xi^2 = ((0.57996, 0.57996), (0.57996, 0.57996))$. ■