Inefficiency in the Shadow of Unobservable Reservation Payoffs

Madhav S. Aney*

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Abstract

This paper considers the problem of allocating an object between two players in an environment with one sided asymmetric information when their reservation payoffs depend on the type of the informed player, causing the reservation payoff of the uninformed player to be unobservable to her. Inefficiency arises naturally in this setting and can be characterized by a simple condition on the reservation payoffs that is necessary and sufficient. I derive the necessary and sufficient condition for the existence of an implementable allocation that at least weakly dominates the reservation payoffs. Under a mild assumption on the distribution of types, I characterize the surplus maximizing mechanism in the second best setting. I argue that the model applies to an environment where property rights over the object are not well defined and are subject to costly enforcement. In such cases, type dependent reservation payoffs arise naturally as the uninformed player's expectation from the enforcement process. The model can explain why the best ways of avoiding costly dispute resolution, such as arbitration as a way of avoiding litigation, typically involve a degree of inefficiency.

JEL Codes: D82, D74, D61

1 Introduction

This paper considers the problem of efficiently allocating an object between two players in an environment where it is clear which of the two values the object more. I present a model where the valuation of one of the two players is observable and known to be higher than that

*School of Economics, Singapore Management University, 90 Stamford Road, Singapore 178903. Email: madhavsa@smu.edu.sg I am particularly grateful to Shurojit Chatterji for guidance at every stage of this project. I'm also particularly grateful to Giovanni Ko and Claudio Mezzetti for detailed comments. I also thank Pallavi Gopinath Aney, Michael Brooks, Jiling Cao, Rahul Deb, Hugo Hopenhayn, Debasis Mishra, Janak Nabar, Clemens Puppe, Ludovic Renou, Hamid Sabourian, and the seminar participants at the SOE Internal Workshop, the Meeting of the Society for Social Choice and Welfare, and the Australasian Public Choice Conference, Asian Meeting of the Econometric Society, Centre for Mathematical Social Science, and three anonymous referees and an associate editor for their insightful suggestions.

of the other. When the reservation payoff of one of the two players is zero, like in the buyer-seller case (Myerson and Satterthwaite 1983), this problem is trivial as the object can be allocated to the player with the higher valuation, in exchange for a transfer that satisfies the IR constraint of the other player. However, this first best solution may not be implementable under budget balance even with one-sided private information when the reservation payoffs of the two players depend on the type of the player who observes his type privately. Such type dependence in the reservation payoffs arises naturally in settings with incomplete property rights where partial claims over an object lead to the type of both players influencing what each receives in the event the inefficient reservation payoffs are triggered.[1] I show that the first best solution is implementable under budget balance if and only if the reservations payoffs satisfy a simple condition (Lemma 1).

This paper introduces a new and potentially interesting mechanism design problem and its solution. Taking the dependence of reservation payoffs on the type of the informed player as exogenous, the main results of this paper derive the surplus maximizing mechanism when the first best is not implementable (Proposition 2) and the necessary and sufficient condition for its existence (Lemma 2). The technical difficulty in characterizing the surplus maximizing mechanism comes from the fact that the reservation payoffs depend on the type that is not publicly observed. Consequently, in contrast to the standard mechanism design problem where the IR constraint typically binds for only one type, in this case both the incentive and IR constraints may bind at several sub intervals of the type space.

It is possible to cast the inefficiency result of this model in the Myerson and Satterthwaite (1983) framework. I depart from the standard Myerson and Satterthwaite (1983) framework in the following ways: First, unlike, Myerson and Satterthwaite (1983), there is only one-sided private information. Second, there is no uncertainty about which of the two players values the object more. Third, the distribution of valuation of the player with private valuation is left unspecified and need not be continuous. However in a stronger assumption relative to Myerson and Satterthwaite (1983), I assume that the reservation payoffs depend on the privately observed type. In the Myerson and Satterthwaite (1983) world since property rights are well defined, the seller walks away with an undisputed ownership over the object in case there is no trade. Consequently the reservation payoff of the seller is simply her valuation and the reservation payoff of the buyer is zero. This is in contrast to the application in section 4 where the reservation payoffs for both players depend on their opponent's type through the choice of equilibrium effort in a game that determines their property rights over the object.[2]

Following this interpretation, the inefficiency result presented here indicates that when reservation payoffs represent the payoffs from conflict, mechanisms that help players avoid conflict may not be fully efficient. An example of this is arbitration as a mechanism to avoid litigation. Although the costs of arbitration are significantly lower than litigation, parties

[1] An application of this model on these lines is presented in section 4.

[2] For a somewhat different treatment of incomplete property rights see Schmitz (2001) where the lack of property rights worsens the reservation payoffs leading to full efficiency in the Myerson and Satterthwaite (1983) framework.

to arbitration typically hire lawyers to argue their case, and this is costly for both parties. The results rationalize the phenomenon of dispute resolution mechanisms such as arbitration involving a smaller but positive degree of inefficiency.

The inefficiency that arises from games such as a contest under complete information, disappears as soon as parties are allowed to meet each other costlessly at a stage prior to the contest as this leads to efficient bargaining that avoids the costs of the contest.[3] On the other hand inefficiencies arising in the problem of allocating an object disappear when it is clear who values the object more, even when the exact valuations are unobservable (Myerson and Satterthwaite 1983). This paper attempts to incorporate both these elements into one model. I consider the case when the reservation payoffs arise in a way that allows the informed player's type to enter the reservation payoff of the uninformed player. This induces a change in the character of the informational asymmetry from private values to an environment similar to interdependent preferences. Consequently the inefficiency that arises here is neither subsumed by our usual understanding of the inefficiencies arising from informational asymmetry with private values nor from surplus losses associated with inefficient games such as a contest.

This paper is related to the growing literature on mechanism design when the reservation payoff of a player is type dependent. Jehiel, Moldovanu, and Stacchetti (1996) analyze a mechanism design problem when the final payoffs of the players are not solely determined by whether or not they are allocated the good. Figueroa and Skreta (2009) study this problem further in the context of a revenue maximizing auction. In these settings the optimal mechanism must take into account the externalities arising from any allocation. In contrast to these papers, in my setting the type dependence of the reservation payoffs does not arise as a result of externalities of the allocation, and indeed there is common knowledge about what the first best is – It is always optimal to allocate the object to the uninformed player regardless of the type of the informed player. The inefficiency in this setting arises from the inability of the uninformed player to accurately know her own reservation payoff due to the unobservability of the informed player's type. This leads to the surplus maximizing mechanism allocating the object to the lower valuation player with positive probability.

Another related paper is Jullien (2000) where the contracting problem between a principal and agent is analyzed in a setting where the reservation payoff of the agent is dependent on his type, which is his private information. The focus there is on characterizing the profit maximizing allocation rule for the principal. In contrast, I will solely focus on inability of efficient mechanisms to deliver ex-post efficiency. Finally this paper is also related to Aney (2012), which shows the conditions under which inefficient reservation payoffs are always strictly preferred over any allocation that can be implemented using a mechanism. The key difference is that Aney (2012) assumes that parties cannot commit to a mechanism, and this implies that they must take into account how truth telling at the mechanism stage modifies their reservation payoff in the event one of the players vetoes the mechanism ex-post. In

[3] On the other hand if bargaining prior to the contest involves positive costs, it is possible to construct an equilibrium where players will forgo bargaining even when the contest is costlier. See Anderlini and Felli (2001).

contrast, this paper shows that when parties can commit to the outcome of the mechanism, although they can improve over their reservation payoffs, they may not be able to attain the first best.

The next section presents the model and shows how inefficiency arises naturally in this setting. A simple condition turns out to be both necessary and sufficient for the first best to be implementable (Lemma 1). Section 3 analyses the case when the first best is not implementable. For the existence of a surplus maximizing mechanism we need to derive the set of functions that are 'proximate' to the reservation payoffs that can be implemented, and we need to ensure this set is compact. Lemma 2 derives the necessary and sufficient condition required on this set for the existence of an allocation that is preferred to the reservation payoffs. Proposition 1 shows that, under mild condition on the reservation payoffs, we can construct such a set by following the procedure described in the appendix, and ensure its compactness. The main result of the paper is Proposition 2, which characterizes the surplus maximizing mechanism. This result is derived under a mild condition on the distribution of types.

Section 4 shows an application of the model where the dependence in the reservation payoffs on the privately observed type arises endogenously when the reservation payoffs arise from a contest. Since the reservation payoffs arise from the equilibrium of a Bayesian game (a Tullock contest), the overall game becomes a multi-stage game that must be solved using an appropriate equilibrium concept namely that of the perfect Bayesian equilibrium. I show that the solution to the mechanism design problem in section 3 is preserved under a restriction on the off-equilibrium beliefs. I discuss the ramifications of the assumption that the reservation payoffs arise as equilibrium payoffs from a Bayesian game in section 4.5. Finally section 5 concludes.

2 Model

There are two players with preferences that can be represented by the usual quasi linear utility functions. The players wish to allocate an object that may be divisible among themselves. The object is valued at θ_1 by player 1 (female) and θ_1 is publicly observed. Player 2 (male) privately observes his valuation θ_2. Since player 1 is uninformed about player 2's type, she treats it as a random variable Θ_2 that takes values between $[\underline{\theta}_2, \overline{\theta}_2]$ with a cdf $F(\theta_2)$. For now we need not impose any restriction on this distribution which may be continuous, discrete, or mixed. I assume that

$$\theta_1 > \overline{\theta}_2 > \underline{\theta}_2 \geq 0 \tag{1}$$

Unless both players agree to allocate the object using a mechanism, they both end up with their reservation payoffs. If each player chooses to participate in the mechanism, both forgo their reservation payoffs. Consequently the expected payoff from the mechanism must be weakly greater than the reservation payoff for each player (the IR constraint).

The reservation payoffs, which are assumed to be dependent on player 2's type, are denoted by

$$\mathbb{E}(v_1(\Theta_2)) \geq 0 \qquad \text{and} \qquad v_2(\theta_2) \geq 0 \tag{2}$$

for player 1 and 2 respectively. Note that $\mathbb{E}(v_1(\Theta_2))$, the reservation payoff for player 1, is an expectation since it depends on player 2's type, which she does not observe. On the other hand player 2 does observe his own type and his reservation payoff is simply $v_2(\theta_2)$. This formulation captures the idea that both players' reservation payoffs depend on the state of the world θ_2, which happens to be the willingness to pay of player 2. Nature chooses θ_2 according to an unspecified probability distribution. Player 1 does not learn θ_2 but player 2 does. Therefore, player 2 knows his willingness to pay and his reservation payoff but player 1 does not. Player 1's willingness to pay on the other hand is θ_1 and this is common knowledge. The reservation payoffs of the two players may also depend on θ_1. However since θ_1 is common knowledge, the presence or absence of dependence of the reservation payoffs on θ_1 will not play any role in the results.

Before going further it is important to restrict our attention to the case where it is inefficient for players to receive their reservation payoffs. If on the other hand the reservation payoffs are large enough, the first best would involve the players simply accepting their reservation payoffs, rather than attempting to forgo them by agreeing to an allocation of the object and transfers. I assume

$$\forall \theta_2 \qquad v_1(\theta_2) + v_2(\theta_2) < \theta_1. \tag{3}$$

This states that the sum of the ex-post reservation payoffs generates lower surplus than allocating the object to player 1 for all realization of θ_2. To focus on the interesting case where the reservation payoffs feature this inefficiency, and it is indeed efficient for players to participate in the mechanism and forgo their reservation payoffs, I will assume inequality (3) holds throughout the paper.

Even when the reservation payoffs satisfy inequality (3), it is possible that

$$\exists \theta_2 \qquad \text{such that} \qquad \mathbb{E}(v_1(\Theta_2)) + v_2(\theta_2) > \theta_1. \tag{4}$$

This is because the informational asymmetry constrains player 1's expected payoff to be $\mathbb{E}(v_1(\Theta_2))$ without regard to the actual realizations of θ_2. This will be the key to the inefficiency showcased in this model. In section 4 I construct an example where type dependent reservation payoffs arise as the equilibrium payoffs from a lottery contest. Although a contest is clearly inefficient, and consequently the equilibrium payoffs from a contest will endogenously satisfy the inequality in (3), we will find that the inequality in (4) may still hold.

2.1 First Best

In this model inefficiency arises in a natural and simple way. I characterize this in Lemma 1 and illustrate it in section 4 with the help of an example. The main result of the paper, namely the mechanism design problem, will be introduced and solved in section 3.

Observation 1. *Ex-post efficiency is attained only if the object is allocated to player 1.*

As argued earlier, when inequality (3) is satisfied, there is an inefficiency when the players end up with their reservation payoffs. Since player 1 always values the object more, it is more efficient to allocate it to her in exchange for a transfer to player 2. This allocation problem can be tackled using a mechanism design approach. Let the expected payoffs from the mechanism be μ_1 and $\mu_2(\theta_2)$ for players 1 and 2 where

$$\mu_1 = \beta_1\theta_1 + t_1 \qquad \text{and} \qquad \mu_2(\theta_2) = \beta_2(\theta_2)\theta_2 + t_2(\theta_2). \tag{5}$$

β_1 and $\beta_2(\theta_2)$ are the probabilities with which the object is allocated to player 1 and 2 respectively and t_1 and $t_2(\theta_2)$ are the corresponding transfers. As the object could be divisible, β_1 and $\beta_2(\theta_2)$ may also be interpreted as the share of allocated to player 1 and 2. These payoffs are the interim expected payoffs from the mechanism. The ex-post payoffs would typically differ from these. For instance, the actual transfer that player 1 makes may depend on the declaration of player 2. However since there is full commitment, once the players agree on a mechanism to allocate the surplus they must accept the ex-post allocation.[4] The allocations will need to satisfy the IC constraint for player 2 and the interim IR constraints for the two players. If player 1 unilaterally refuses to participate in the mechanism she expects to receive $\mathbb{E}(v_1(\Theta_2))$. Hence the IR constraint for player 1 is $\mu_1 \geq \mathbb{E}(v_1(\Theta_2))$. Similarly if player 2 unilaterally refuses to participate in the mechanism he expects to receive $v_2(\theta_2)$ and hence his IR constraint is $\mu_2(\theta_2) \geq v_2(\theta_2)$.

Since player 1's valuation of the object is always greater than that of player 2, for full efficiency $\beta_2(\theta_2) = 0, \forall\theta_2$ or conversely $\beta_1 = 1$ is necessary.[5] More generally we will assume ex-post surplus feasibility and ex-post budget feasibility throughout the paper.

Definition 1. *Allocation is defined to be ex-post surplus feasible when*

$$\beta_1(\theta_2) + \beta_2(\theta_2) \leq 1 \qquad \forall\theta_2 \tag{6}$$

and ex-post budget feasible when

$$t_1(\theta_2) + t_2(\theta_2) \leq 0 \qquad \forall\theta_2 \tag{7}$$

[4]Without full commitment, player 2 must consider the impact his declaration has on the ex-post reservation payoffs. This model is analyzed in Aney (2012).

[5]The object being allocated to player 1 is necessary but not sufficient for full efficiency since part of the transfers made by the players may be burnt.

Ex-post surplus feasibility implies that the probability or share of the object cannot add to greater than one across the two players for any realization of θ_2. Second, the sum of transfers made to the two players can never be positive since this implies the presence of a third party subsidy. Note that budget feasibility subsumes budget balance but allows some transfers to be burnt. However we will see in the surplus maximizing mechanism in section 3 that this is never optimal.

Lemma 1. *First best under ex-post surplus and budget feasibility is implementable if and only if*

$$\mathbb{E}(v_1(\Theta_2)) + v_2(\theta_2) \leq \theta_1 \qquad \forall \theta_2 \in [\overline{\theta}_2, \underline{\theta}_2]. \tag{8}$$

Proof. To start with note that since we require $\beta_2(\theta_2) = 0$ for all θ_2 in the first best, $t_2(\theta_2)$ must some constant t_2 to ensure incentive compatibility. If not, player 2 will make the declaration that yields the highest transfer. Hence we have $t_2(\theta_2) = t_2$ for all θ_2.

I will first prove sufficiency. Set $t_2 = \max\{v_2(\theta_2)\}$. This satisfies IR constraint of player 2 for any type θ_2. Since $\mathbb{E}(v_1(\Theta_2)) \leq \theta_1 - \max\{v_2(\theta_2)\}$ the IR constraint of player 1 is also satisfied. This shows that the first best is implementable.

I will now prove necessity. Consider the case when there exists a θ_2 such that $\mathbb{E}(v_1(\Theta_2)) + v_2(\theta_2) > \theta_1$. To ensure the IR constraint is satisfied for player 2 of any type we need $t_2 \geq \max\{v_2(\theta_2)\}$. This however violates the IR constraint for player 1. To see this note that ex-ante budget feasibility implies $t_2 \leq -t_1$. Hence we must at least have $-t_1 \geq \max\{v_2(\theta_2)\}$. Substituting this into the IR constraint of player 1 we find that $\mathbb{E}(v_1(\Theta_2)) > \theta_1 - \max\{v_2(\theta_2)\}$. This shows that the first best is not implementable. □

This observation shows that if the overestimation of her reservation payoff by player 1 is large enough, it is impossible to allocate the object to her while satisfying the IR constraint of player 2. It is worth noting that the condition under which the first best is possible does not rely directly on the distribution of player 2's type, which could be discrete or continuous.[6]

There is a connection between this inefficiency result and the one presented in the literature on interdependent valuations. Define net valuations as the difference between valuations and the reservation payoff. Then the net valuations of player 1 and 2 are

$$\theta_1 - \mathbb{E}(v_1(\Theta_2)) \qquad \text{and} \qquad \theta_2 - v_2(\theta_2) \tag{9}$$

and these depend on the type of the player 2. Jehiel and Moldovanu (2001) show in a related environment that efficiency is hard to get with interdependent valuations. Mezzetti (2004) critiques their paper and shows that efficiency is always possible if transfers can be conditioned on players' observation of their payoffs after an outcome is decided. However there are important differences between these results and the result in Lemma 1. First, the interdependence here is at the level of net valuations, not valuations. In particular, efficiency

[6] Condition in (8) is reminiscent of the condition in Makowsky and Mezzetti (1994) who show that this condition is necessary and sufficient for ex-post efficiency in a private values setting with several players. However for this the distribution of Θ_2 needs to satisfy a richness property that is not required here.

here does not mean assigning the object to the player with the highest net valuation, but the one with the highest valuation. As a result the inefficiency result here does not follow from Jehiel and Moldovanu (2001). Second, efficiency requires that the object be given to player 1 and the reservation payoffs are never realized. Consequently player 1 does not discover anything about player 2's type. Hence the innovation of the two stage mechanism used in Mezzetti (2004) would not work here.[7]

To see the intuition for this result note that the object must always be allocated to player 1 in exchange for a transfer to player 2 to attain the first best. Consequently the only incentive compatible transfer schedule is one that is flat in the declaration of player 2. However if $v_2(\theta_2)$ is large enough for some θ_2, it will not be possible to transfer enough to player 2 while satisfying the IR constraint of player 1. As a consequence of this whenever there exists a θ_2 such that $\mathbb{E}(v_1(\Theta_2)) + v_2(\theta_2) > \theta_1$, we will find in section 3 that there exists some inefficiency as we will need $\beta_2(\theta_2) > 0$ for some θ_2.

Corollary 1. *First best is implementable whenever ex-post budget feasibility is relaxed.*

Proof. Since player 1 values the object more than player 2, allocating the object to player 1 along with a transfer to player to of $t_2(\theta_2) = \max\{v_2(\theta_2)\}, \forall \theta_2$ will ensure the first best. These transfers are feasible if budget feasibility is relaxed since the constraint $t_2(\theta_2) \leq \theta_1 - \mathbb{E}(v_1(\Theta_2))$ no longer applies. □

This corollary follows from the results in Groves (1973), Arrow (1979) and d'Aspremont and Gerard-Varet (1979) that prove the feasibility of the first best in this environment whenever budget feasibility is relaxed.

In this section I have constructed an example where type dependent reservation payoffs arise as the equilibrium payoffs from a default game that is clearly inefficient but still satisfies (4). This clarifies the point that the reservation payoffs could arise from a class of inefficient games, that yield payoffs that satisfies inequality (3) and consequently players prefer to avoid. However since the payoffs also satisfy inequality (4), the first best allocation is not attainable. Moreover, this section shows how this model applies to a situation with incomplete property rights.

3 Second best

In this section I will derive the main results of the paper. Since we are concerned with the second best, we can restrict our attention to the case where there exists a θ_2 such that

$$\mathbb{E}(v_1(\Theta_2)) + v_2(\theta_2) - \theta_1 > 0. \tag{10}$$

When this condition is satisfied, we are in the second best where player 2 must be allocated the object with positive probability.

[7] I thank Claudio Mezzetti for pointing this out.

In this section I first derive the necessary and sufficient condition for the existence of an implementable allocation in Lemma 2. Even though this guarantees that players can do better than their reservation payoffs, it falls short of providing us the surplus maximizing mechanism. This question is tackled in Propositions 1 and 2. For characterizing the surplus maximizing mechanism we need to characterize the set of implementable functions that are 'proximate' to the reservation payoff of player 2. More importantly we need to ensure that this set is compact since this set is the domain over which the optimization yielding the surplus maximizing mechanism is conducted. In Proposition 1 I show that using a procedure described in the appendix, it is possible to recover such a set as long as the reservation payoff of player 2 satisfies some mild conditions. Moreover, this result also proves that the resulting set is compact. In Proposition 2, the surplus maximizing mechanism is derived using the set of functions derived in Proposition 1. Finally in section 4 present an application of the model, show the impossibility of reaching the first best, and solve for the second best mechanism.

To proceed further it is necessary to specify the distribution of θ_2. For now I assume that θ_2 is drawn from a continuous distribution with a probability density function $f(\theta_2)$ on the interval $[\underline{\theta}_2, \overline{\theta}_2]$. As a result of the revelation principle we can restrict our attention to a direct mechanism where player 2 makes a declaration $\tilde{\theta}_2$ and gets a payoff

$$\mu_2(\tilde{\theta}_2) = \theta_2 \beta_2(\tilde{\theta}_2) + t_2(\tilde{\theta}_2) \tag{11}$$

where $\beta_2(\tilde{\theta}_2) \in [0, 1]$. Since quasi linear payoffs of this form satisfy the single crossing property,

$$\theta_2 \beta_2'(\theta_2) + t_2'(\theta_2) = 0 \qquad \text{and} \qquad \beta_2'(\theta_2) \geq 0 \qquad \forall \theta_2, \tag{12}$$

are each necessary, and together sufficient, to ensure that the incentive compatibility constraint for player 2 is satisfied.[8] Using the well known procedure first introduced in Mirrlees (1971) we know that

$$\frac{\partial \mu_2(\theta_2)}{\partial \theta_2} = \beta_2(\theta_2) + \theta_2 \beta_2'(\theta_2) + t_2'(\theta_2) = \beta_2(\theta_2) \tag{13}$$

implying that under incentive compatibility the expected payoff from negotiations for player 2 of type θ_2 is

$$\mu_2(\theta_2) = \mu_2(\underline{\theta}_2) + \int_{\underline{\theta}_2}^{\theta_2} \beta_2(w) dw. \tag{14}$$

From (14) we have $\mu_2'(\theta_2) = \beta_2(\theta_2) \in [0, 1]$. Using this and (14) we can rewrite the constraints in (12) as constraints on $\mu_2(\theta_2)$ as

$$\mu_2'(\theta_2) \in [0, 1] \qquad \text{and} \qquad \mu_2''(\theta_2) \geq 0. \tag{15}$$

The constraints in (15) are merely a restatement of constraints that are known to be necessary

[8] See chapter 2.3.3.1 in Bolton and Dewatripont (2005).

and sufficient for incentive compatibility. In addition to the IC constraints in (15), player 2's payoff must also satisfy the IR constraint, which is

$$\mu_2(\theta_2) \geq v_2(\theta_2). \tag{16}$$

Since player 1's type is publicly observed we only need to satisfy her IR constraint, which is $\mu_1 \geq \mathbb{E}(v_1(\Theta_2))$. Once we find a $\mu_2(\theta_2)$ satisfying the constraints in (15) we can recover

$$\beta_2(\theta_2) = \mu_2'(\theta_2) \qquad \text{and} \qquad t_2(\theta_2) = \mu_2(\theta_2) - \theta_2\mu_2'(\theta_2). \tag{17}$$

The key problem of deriving the surplus maximizing mechanism that is unique to this setting is the following. Since we have not imposed any restriction on $v_2(\theta_2)$, it may be the case that $v_2(\theta_2)$ does not satisfy the conditions in (15), that is, we may not have $v_2'(\theta_2) \in [0,1]$ and $v_2''(\theta_2) \geq 0$. As a result of this we need to find the function that is 'closest' to $v_2(\theta_2)$ but does satisfy (15), so that we may implement it while minimizing the inefficiency arising from allocating the object to player 2.

This problem may be solved in the following three steps. First we find a function or functions that satisfy the IC constraints in (15) and the IR constraint in (16) for player 2. Second, we restrict our attention to those functions, which if implemented, would also satisfy the IR constraint of player 1. Finally, if there are two or more functions that satisfy these constraints we need to identify the one that minimizes the inefficiency that arises in the second best. Let us consider the first step and construct a set Ψ composed of all functions that satisfy the IR and IC constraints of player 2. We call such a function a proximate implementable function.

Definition 2. *A proximate implementable function $\eta(\theta_2)$ for $v_2(\theta_2)$ is defined on θ_2 in the interval $[\underline{\theta}_2, \overline{\theta}_2]$. It is differentiable except at finitely many points, continuous, and convex, with*

$$\eta'(\theta_2) \in [0,1], \qquad \textit{and} \qquad \eta(\theta_2) \geq v_2(\theta_2) \tag{18}$$

and there does not exist another function $\tilde{\eta}(\theta_2)$ satisfying the same constraints such that $\eta(\theta_2) \geq \tilde{\eta}(\theta_2)$ for all θ_2 and $\eta(\theta_2) > \tilde{\eta}(\theta_2)$ for some θ_2. Let Ψ be the set of all proximate implementable functions for $v_2(\theta_2)$.

Since we allow proximate functions to be non-differentiable at finitely many points, we need to appropriately define $\eta'(\theta_2)$ to ensure its existence for all θ_2. For each $\eta(\theta_2)$, let S be the set of points where $\eta(\theta_2)$ is non differentiable. Since the derivative of $\eta(\theta_2)$ is not defined at points $s \in S$, define

$$\eta'(\theta_2) = \begin{cases} \lim_{\theta_2 \to s-} \frac{\eta(\theta_2)-\eta(s)}{\theta_2 - s} & \text{if } \theta_2 = s \in S \\ \frac{\partial \eta(\theta_2)}{\partial \theta_2} & \text{otherwise.} \end{cases} \tag{19}$$

This definition merely ensures that $\eta'(\theta_2)$ is well defined at all points in the interval $[\underline{\theta}_2, \overline{\theta}_2]$.

We can now proceed to the second step of restricting our attention to those elements of Ψ that also satisfy the IR constraint for player 1 when implemented.

Lemma 2. *For any $v_2(\theta_2)$ defined on the interval $[\underline{\theta}_2, \overline{\theta}_2]$, an allocation satisfying ex-post budget and surplus feasibility, IC and IR constraints exists if and only if there exists an $\eta(\theta_2) \in \Psi$ such that*

$$\mathbb{E}(\eta(\Theta_2)) - \int_{\underline{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2)\eta'(\theta_2) f(\theta_2) d\theta_2 \leq \theta_1 - \mathbb{E}(v_1(\Theta_2)). \tag{20}$$

Proof. By definition 2 we know that Ψ is the collection of functions $\eta(\theta_2)$ that satisfy the IC and IR constraint of player 2. Moreover, these function cannot be improved upon since we rule out $\eta(\theta_2)$ if there exists a $\tilde{\eta}(\theta_2)$ that satisfies the constraints in (18) and $\eta(\theta_2) \geq \tilde{\eta}(\theta_2)$ for all θ_2 and $\eta(\theta_2) > \tilde{\eta}(\theta_2)$ for some θ_2.

Using (17), we can construct the corresponding allocation for a given $\eta(\theta_2)$, and we have $\beta_2(\theta_2) = \eta'(\theta_2)$ and $t_2(\theta_2) = \eta(\theta_2) - \theta_2\eta'(\theta_2)$ for player 2. For player 1 we can at most have $\beta_1 = 1 - \mathbb{E}(\beta_2(\Theta_2))$ and $t_1 = -\mathbb{E}(t_2(\Theta_2))$. Hence, when the payoff of player 2 is $\eta(\theta_2)$, the payoff of player 1 is at most $\beta_1\theta_1 + t_1$, which equals

$$\theta_1 - \mathbb{E}(\eta(\Theta_2)) - \int_{\underline{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2)\eta'(\theta_2) f(\theta_2) d\theta_2. \tag{21}$$

This satisfies the IR constraint if and only if (20) is satisfied.

□

This result shows the necessary and sufficient condition for the existence of an incentive compatible and ex-post budget feasible allocation that makes both players, at least weakly better off. This is what the inequality in (20) ensures. In general the function $\eta(\theta_2)$ need not be unique. Although all functions $\eta(\theta_2)$ that satisfy inequality in (20), make the two players better off, typically there will be one element that dominates the others – namely the surplus maximizing mechanism. This will be characterized in Proposition 2. Before turning to this problem, we need to know how the set Ψ is derived, and ensure its compactness. This is what Proposition 1 does.

Proposition 1. *When $v_2(\theta_2)$ is twice differentiable and either concave or convex, the set Ψ may be recovered using the procedure described in steps 1 – 8 in the appendix. Furthermore, under these conditions, the set Ψ is compact.*

Proof. Proof in the appendix. □

Proposition 1 shows that following the procedure described in the appendix we can recover the set Ψ which includes all proximate implementable functions for a given $v_2(\theta_2)$ under the assumption that $v_2(\theta_2)$ is twice differentiable and either concave or convex. In particular when $v_2(\theta_2)$ is convex, Ψ has only one element. This is established in step 5 of the procedure in the appendix. If $v_2(\theta_2)$ is concave, there are infinitely many proximate implementable functions that are potential candidates to be used for the construction of the surplus maximizing mechanism. The characterization of the surplus maximizing mechanism that follows in Proposition 2 requires optimization over the set Ψ. For this optimization exercise to yield a solution, we need Ψ to be compact. From Proposition 1 we know that this is guaranteed as long as $v_2(\theta_2)$ is either concave or convex.

Proposition 2. *Assume $v_2(\theta_2)$ is twice differentiable and either concave or convex, $(\theta_1 - \theta_2)f(\theta_2)$ is non-increasing in θ_2, and there exists an $\eta(\theta_2)$ satisfying (20). The surplus maximizing mechanism must take the form*

$$\begin{array}{llll} \beta_2(\theta_2) = \mu_2'(\theta_2) & \textit{and} & t_2(\theta_2) = \mu_2(\theta_2) - \theta_2\mu_2'(\theta_2) & \textit{if } \overline{\theta}_2 \geq \theta_2 > \hat{\theta}_2, \\ \beta_2(\theta_2) = 0 & \textit{and} & t_2(\theta_2) = \mu_2(\hat{\theta}_2) & \textit{if } \hat{\theta}_2 \geq \theta_2 \geq \underline{\theta}_2, \\ \beta_1 = 1 - \mathbb{E}(\beta_2(\Theta_2)) & \textit{and} & t_1 = -\mathbb{E}(t_2(\Theta_2)). & \end{array} \tag{22}$$

where

$$\mu_2(\theta_2) = \underset{\eta(\theta_2)\in\Psi}{\operatorname{argmax}} \left(\int_{\hat{\theta}_2}^{\overline{\theta}_2} (\theta_2 - \theta_1)\eta'(\theta_2) f(\theta_2) d\theta_2 \right) \tag{23}$$

and $\hat{\theta}_2$ is the highest value of θ_2 that satisfies

$$\mathbb{E}(v_1(\Theta_2)) = \theta_1 - F(\hat{\theta}_2)\eta(\hat{\theta}_2) - \int_{\hat{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2)\eta'(\theta_2) f(\theta_2) d\theta_2. \tag{24}$$

Proof. The social planner's problem is to maximize $\mu_1 + \mathbb{E}(\mu_2(\Theta_2))$ subject to the IR constraints for the two players and the IC constraint for player 2. To begin with note that in the surplus maximizing mechanism we must have

$$\beta_1 = 1 - \mathbb{E}(\beta_2(\Theta_2)) \qquad \text{and} \qquad t_1 = -\mathbb{E}(t_2(\Theta_2)). \tag{25}$$

This is because

$$\beta_1 > 1 - \mathbb{E}(\beta_2(\Theta_2)) \qquad \text{or} \qquad t_1 > -\mathbb{E}(t_2(\Theta_2)) \tag{26}$$

will violate budget or surplus feasibility and

$$\beta_1 < 1 - \mathbb{E}(\beta_2(\Theta_2)) \qquad \text{or} \qquad t_1 < -\mathbb{E}(t_2(\Theta_2)) \tag{27}$$

can always be improved upon by allocating the unused expected surplus or expected transfers to player 1 without violating any constraints, thereby increasing total surplus. Using these two constraints, the social planner's problem modifies to

$$\min_{\beta_2(\theta_2)} \int_{\underline{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2)\beta_2(\theta_2) f(\theta_2) d\theta_2, \tag{28}$$

subject to the IR constraints of the two players, and the IC constraint of player 2.

Ignoring for now the problem of identifying the optimal $\eta(\theta_2) \in \Psi$, and assuming that the optimal $\eta(\theta_2)$ is known, we must have $\mu_2(\theta_2) \geq \eta(\theta_2)$ for all θ_2 since $\mu_2(\theta_2) < \eta(\theta_2)$ will violate either the participation or the IC constraints of player 2. Given (14), for each θ_2 we need

$$\int_{\underline{\theta}_2}^{\theta_2} \beta_2(x)dx \geq \eta(\theta_2) - \mu_2(\underline{\theta}_2). \tag{29}$$

We will see that the solution to (28), will correspond to the solution of minimizing

$$\int_{\underline{\theta}_2}^{\overline{\theta}_2} \beta_2(x)dx \tag{30}$$

with respect to $\beta_2(x)$ subject to (29). To minimize (30) while ensuring that (29) is satisfied, we can see that $\mu_2(\underline{\theta}_2)$ must be set as high as possible as this allows us to lower $\int_{\underline{\theta}_2}^{\overline{\theta}_2} \beta_2(x)dx$. Let $\hat{\theta}_2$ be the value of $\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]$ such that $\eta(\hat{\theta}_2) = \mu_2(\underline{\theta}_2)$ is the highest value of $\mu_2(\underline{\theta}_2)$ feasible due to budget feasibility. For a player 2 with a type $\theta_2 > \hat{\theta}_2$ the constraint (29) modifies to

$$\int_{\hat{\theta}_2}^{\theta_2} \beta_2(x)dx \geq \eta(\theta_2) - \eta(\hat{\theta}_2). \tag{31}$$

The lowest possible $\beta_2(\theta_2)$ that satisfies this is

$$\beta_2(\theta_2) = \eta'(\theta_2) \qquad \text{for } \theta_2 > \hat{\theta}_2. \tag{32}$$

By construction this satisfies the IR constraint and the IC constraint for player 2 with type $\theta_2 > \hat{\theta}_2$. For $\theta_2 \leq \hat{\theta}_2$ we can simply set $\beta_2(\theta_2) = 0$ and $\mu_2(\theta_2) = t_2(\theta_2) = \eta(\hat{\theta}_2)$. This satisfies the IR and IC constraints for a player 2 with type $\theta_2 \leq \hat{\theta}_2$. Since $\int_{\hat{\theta}_2}^{\overline{\theta}_2} \beta_2(\theta_2)d\theta_2$ is decreasing in $\hat{\theta}_2$ we can solve for the highest possible $\hat{\theta}_2$ that satisfies equation (24), which is the IR constraint for player 1. This gives us the allocation in (22).

We will now see that the solution we have derived corresponds to the solution for (28). Consider a $b(\theta_2) \neq \beta_2(\theta_2)$ such that

$$\int_{\underline{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2)b(\theta_2)f(\theta_2)d\theta_2 < \int_{\underline{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2)\beta_2(\theta_2)f(\theta_2)d\theta_2. \tag{33}$$

I will show by contradiction that an implementable $b(\theta_2)$ cannot exist since $(\theta_1 - \theta_2)f(\theta_2)$ is non-increasing in θ_2. First note that $\beta_2(\theta_2)$ minimizes $\int_{\underline{\theta}_2}^{\overline{\theta}_2} \beta_2(x)dx$ subject to the constraint in (29). Hence we must have

$$\int_{\underline{\theta}_2}^{\overline{\theta}_2} b(x)dx > \int_{\underline{\theta}_2}^{\overline{\theta}_2} \beta_2(x)dx. \tag{34}$$

Second note that we must have $b(\theta_2) > 0$ for some interval in $[\underline{\theta}_2, \hat{\theta}_2]$. If not, following from

(33), we must have

$$\int_{\hat{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2) b(\theta_2) f(\theta_2) d\theta_2 < \int_{\hat{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2) \beta_2(\theta_2) f(\theta_2) d\theta_2. \tag{35}$$

But since $\beta_2(\theta_2)$ is determined when the constraint in (29) binds for $\theta_2 > \hat{\theta}_2$, $b(\theta_2)$ must violate this constraint for the inequality in (35) to be satisfied, and consequently such a $b(\theta_2)$ is not implementable. Hence we must have an interval in $[\underline{\theta}_2, \hat{\theta}_2]$ where $b(\theta_2) > 0$. However, recall that $(\theta_1 - \theta_2) f(\theta_2)$ is non-increasing in θ_2. This implies that (33) cannot be true. To see this construct $c(\theta_2)$ such that

$$\int_{\underline{\theta}_2}^{\overline{\theta}_2} b(x) dx = \int_{\underline{\theta}_2}^{\overline{\theta}_2} c(x) dx, \tag{36}$$

and $c(\theta_2) = 0$ for $\theta_2 \leq \hat{\theta}_2$. The total inefficiency with $c(\theta_2)$ must satisfy

$$\int_{\hat{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2) c(\theta_2) f(\theta_2) d\theta_2 < \int_{\underline{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2) b(\theta_2) f(\theta_2) d\theta_2, \tag{37}$$

since $(\theta_1 - \theta_2) f(\theta_2)$ is non-increasing in θ_2. However since $\int_{\underline{\theta}_2}^{\overline{\theta}_2} c(x) dx = \int_{\underline{\theta}_2}^{\overline{\theta}_2} b(x) dx > \int_{\underline{\theta}_2}^{\overline{\theta}_2} \beta_2(x) dx$, and $c(\theta_2) = \beta_2(\theta_2) = 0$ for $\theta_2 \leq \hat{\theta}_2$, we must have

$$\int_{\underline{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2) c(\theta_2) f(\theta_2) d\theta_2 > \int_{\underline{\theta}_2}^{\overline{\theta}_2} (\theta_1 - \theta_2) \beta_2(\theta_2) f(\theta_2) d\theta_2. \tag{38}$$

This contradicts (33), proving that $\beta_2(\theta_2)$ is optimal given the optimal $\eta(\theta_2)$.

Finally the optimal $\eta(\theta_2) \in \Psi$ is found by choosing $\eta(\theta_2)$ that minimizes the inefficiency resulting from the second best allocation, which is equivalent to the solution of (23). A solution to (23) is guaranteed to exist by the Weierstrass theorem since the integral is a continuous function over the domain Ψ, which is guaranteed to be compact by Proposition 1 when $v_2(\theta_2)$ is either concave or convex.

□

Proposition 2 characterizes the surplus maximizing mechanism. We see that in the surplus maximizing mechanism, a player 2 with type θ_2 less than a threshold $\hat{\theta}_2$ will be allocated the object with zero probability. The threshold $\hat{\theta}_2$ is endogenously determined, and is set as high as possible at the point where the IR constraint of player 1 binds. This result relies on $(\theta_1 - \theta_2) f(\theta_2)$ being non-increasing in θ_2. If this assumption is violated, it is not possible to impose a structure on the surplus maximizing mechanism and the shape of $\beta_2(\theta_2)$ will vary with the distribution of θ_2. In general, it may then become optimal to allocate surplus to player 2 with type lower than $\hat{\theta}_2$ with positive probability. However, although it is not possible to characterize the surplus maximizing mechanism in this case, Lemma 2 shows that as long as there exists at least one $\eta(\theta_2)$ in the set Ψ that satisfies (20), it is possible to implement an allocation that at least weakly dominates the reservation payoffs for both players.

The result in Proposition 2 also relies on $v_2(\theta_2)$ being twice differentiable and either concave or convex. This is a simplifying assumption that is needed in Proposition 1 to ensure that Ψ is compact. Compactness is sufficient (although not necessary) for the optimization exercise in Proposition 2 to yield the surplus maximizing mechanism.[9] In the absence of compactness we are left only with the result in Lemma 2 that guarantees at least one implementable allocation that dominates the reservation payoffs, but without the certainty about the existence the surplus maximizing mechanism. In particular in the the absence of compactness of Ψ it may be possible that for any $\eta(\theta_2) \in \Psi$ that satisfies the IR constraint of player 1, there may exist another element in Ψ that generates more total surplus.

4 An Application

Consider an intellectual property dispute between two firms. The two firms are using a technology that each firm claims to have patented. Since the scope of the patent held by each firm is somewhat broad, both firms have a plausible claim over the technology they are using. Firm 1 is a large publicly listed firm with expected profits that are publicly observable. Consequently it is known that if firm 1 operates the technology exclusively, it will lead to expected profits of θ_1. Firm 2 is a small new firm and certain aspects of its production process are not publicly observable. However it is known that since it is a smaller firm, it does not enjoy economies of scale, and is consequently less efficient than firm 1. If firm 2 has the exclusive right to operate the technology, its profits are either $\overline{\theta}_2$ with probability $q \in (0,1)$ or $\underline{\theta}_2 = 0$ with probability $1-q$. If either of the two firms chooses not to negotiate, the dispute will be resolved by the court. Assume for now that the refusal to negotiate reveals nothing about the type of player 2 and that the posterior beliefs of firm 1 are the same as its prior. We will see in section 4.3 that refusal to negotiate is off-equilibrium and we are free to specify off-equilibrium beliefs in this way. To simplify things further, assume that the market value of the technology is zero and this implies that neither firm wants to acquire exclusive right to it for its resale value. We can treat this dispute as a multi-stage game where the stages are specified as follows:

Timeline:

1. The two firms independently choose whether to litigate or negotiate.
2. If both firms have chosen to negotiate, their payoffs are determined by the surplus maximizing mechanism in Proposition 3 and the game ends.
3. If either of the two firms has chosen not to negotiate, litigation is triggered. In this case firm 1 updates its belief about firm 2 and the two firms simultaneously choose litigation effort. The court determines the allocation and the game ends.

[9] For a formal statement of the Weierstrass Theorem see Ok (2007).

We can solve this game backwards by starting with the litigation sub-game. The payoff from litigation will become the reservation payoffs that define the IR constraints of the firms in stage 1. In the following section I model litigation as a contest. In section 4.4 the results are extended to the case where litigation is an unspecified Bayesian game and the distribution of types for firm 2 is continuous.

4.1 Litigation Game

Following a large literature[10] that models litigation as a contest, I assume that the two firms face the following objective functions in court.

$$\theta_1 \mathbb{P}(\tilde{x}_1, \tilde{x}_2) - \tilde{x}_1 \qquad \text{and} \qquad \theta_2(1 - \mathbb{P}(\tilde{x}_1, \tilde{x}_2)) - \tilde{x}_2$$

where

$$\mathbb{P}(\tilde{x}_1, \tilde{x}_2) = \begin{cases} 1 & \text{if } \tilde{x}_1 = \tilde{x}_2 = 0 \\ \frac{\tilde{x}_1^\lambda}{\tilde{x}_1^\lambda + \tilde{x}_2^\lambda} & \text{otherwise, where } \lambda \in (0,1). \end{cases} \tag{39}$$

The Tullock contest[11] has been used extensively for modeling litigation.[12] Firm 1 and 2 non-cooperatively choose $\tilde{x}_1$ and $\tilde{x}_2$ respectively. These are the amounts the two firms spend on litigation, for example the costs of legal counsel. Since the patent is indivisible, ex-post it must be allocated to one of the two firms. $\mathbb{P}(\tilde{x}_1, \tilde{x}_2)$ is the probability with which the court decides in favor of firm 1. We see that the probability with which the court allocates the patent to a firm is increasing in its effort and decreasing in the effort of its opponent.[13] This could reflect the fact that lawyers who are more persuasive in court, are also more expensive as they help their client win with a higher probability.[14]

We can solve for the equilibrium litigation payoffs for the firms. These will become $\mathbb{E}(v_1(\Theta_2))$ and $v_2(\theta_2)$, the reservation payoffs of the firms, and any expected allocation at the mechanism design stage must be greater than these to satisfy the IR constraints of the two firms. To derive the equilibrium payoffs we must first compute the Bayesian Nash equilibrium effort levels x_1 and $x_2(\theta_2)$.

[10]See Cooter and Rubinfeld (1989) and Hay and Spier (1998) for a review.

[11]This is a close variant of the Tullock "lottery" contest function that has been studied in the contest literature. See Skaperdas (1996) for its axiomatic foundations. The slight variation comes from the fact that $\mathbb{P}(\tilde{x}_1, \tilde{x}_2) = 1$ rather than $1/2$ when $\tilde{x}_1 = \tilde{x}_2 = 0$. This helps in avoiding issues of existence of equilibrium but is otherwise innocuous. $\lambda \in (0,1)$ guarantees that the objective function is concave which ensures that we can rely on the first order conditions to characterize the optimal efforts of the firms.

[12]Hirshleifer and Osborne (2001), uses the same function as the one used here. Farmer and Pecorino (1999), and Katz (1988) also use the Tullock form while allowing for the court to treat the two parties asymmetrically. It is easy to generalize the results presented here to the case when one of the two firms has a stronger claim to the technology and consequently the court is more likely to rule in its favor. Since this doesn't add anything substantial to this example, I opt for the simpler formulation where the two sides are symmetric before the court. Finally Garfinkel and Skaperdas (2007), and Robson and Skaperdas (2008), also use a generalization of the Tullock contest function in the context of litigation.

[13]Since the firms bear their own costs, in this example the court follows the US fee shifting rule rather than English one where it is not uncommon for the loser to be made to pay part of the winner's costs.

[14]See Skaperdas and Vaidya (2009) for the axiomatic foundation for how persuasion may be modeled as a contest.

First note that $x_2(0) = 0$ as firm 2 is strictly worse off by exerting any positive effort when its valuation is zero. This implies that the equilibrium payoff $v_2(0) = 0$ irrespective of the effort of firm 1. Hence the optimal effort levels for firm 1 and high value firm 2 are

$$x_1 = \underset{\tilde{x}_1}{\text{argmax}} \left(\theta_1 q \frac{\tilde{x}_1^\lambda}{\tilde{x}_1^\lambda + x_2(\overline{\theta}_2)^\lambda} + (1-q)\theta_1 - \tilde{x}_1 \right) \tag{40}$$

and

$$x_2(\overline{\theta}_2) = \underset{\tilde{x}_2}{\text{argmax}} \left(\overline{\theta}_2 \frac{\tilde{x}_2^\lambda}{{x_1}^\lambda + \tilde{x}_2^\lambda} - \tilde{x}_2 \right). \tag{41}$$

The first order conditions of the two firms give us

$$\frac{q\theta_1}{\overline{\theta}_2} = \frac{x_1}{x_2(\overline{\theta}_2)}. \tag{42}$$

Substituting this into the objective functions, and setting $\frac{\overline{\theta}_2^\lambda}{(q\theta_1)^\lambda + \overline{\theta}_2^\lambda} =: \gamma$, we have

$$\mathbb{E}(v_1(\Theta_2)) = \theta_1 q(1-\gamma)(1-\lambda\gamma) + \theta_1(1-q), \qquad v_2(0) = 0, \qquad \text{and} \qquad v_2(\overline{\theta}_2) = \overline{\theta}_2\gamma(1-\lambda(1-\gamma)). \tag{43}$$

Since γ is a function of q, we observe that $v_2(\overline{\theta}_2)$ depends on the belief of firm 1.

4.2 Negotiations

In this section we will see that the first best outcome, that of allocating the surplus to firm 1 in exchange for a transfer to firm 2, is not implementable under budget balance. I will also derive the surplus maximizing mechanism.

Before we do this we need to be mindful of the following issue when defining the IR constraints of the two firms. In Bayesian games the equilibrium payoffs of the players are a function of the beliefs of the other players. With the Tullock contest in (39), we see from the expressions in (43) that the equilibrium payoffs for both firms are a function of q, the prior with which firm 1 believes firm 2 to be a high type. This is not a problem when firm 1 contemplates a refusal to negotiate – in case of refusal to negotiate firm 1 learns nothing about firm 2's type and the reservation payoff of firm 1 is correctly computed to be $\mathbb{E}(v_1(\Theta_2))$. However it is possible that refusal to negotiate by firm 2 conveys some information about the type of firm 2. Since the Tullock contest equilibrium effort levels x_1 and $x_2(\overline{\theta}_2)$ depend on this belief, this modifies $v_2(\theta_2)$. In other words, to compute the correct reservation payoff that defines its IR constraint, firm 2 needs to consider what its reservation payoff would be under the belief of firm 1 that is induced by firm 2's refusal to negotiate. This issue does not arise in section 2 and 3 since there the reservation payoff of player 2 does not depend on player 1's beliefs.

To deal with this issue I use perfect Bayesian equilibrium as the solution concept to solve this multi-stage game. In short this problem is resolved by adopting the following off-equilibrium belief for firm 1– whenever firm 2 refuses to negotiate, firm 1's posterior is equal to its prior. Since in equilibrium firm 2 always negotiates, this restriction on off-equilibrium beliefs will imply that the beliefs of firm 1 remain the same regardless of whether firm 2 negotiates, and this allows us to pin down the IR constraint of firm 2 to $v_2(\theta_2)$ in (43), allowing us to solve for the surplus maximizing mechanism.

4.2.1 First Best

Before we verify that the first best is not implementable, note that condition (3) is satisfied here and it is efficient to avoid litigation. To see this note

$$v_1(\overline{\theta}_2) + v_2(\overline{\theta}_2) < \theta_1 \qquad \text{and} \qquad v_1(0) + v_2(0) < \theta_1 \tag{44}$$

where

$$v_1(\overline{\theta}_2) = \theta_1 \mathbb{P}(x_1, x_2(\overline{\theta}_2)) - x_1 \qquad \text{and} \qquad v_1(0) = \theta_1 \mathbb{P}(x_1, x_2(0)) - x_1. \tag{45}$$

Noting that $x_2(\overline{\theta}_2)$ and x_1 are strictly positive it is easy to see that the inequalities in (44) are satisfied. This is because both firms burn resources during litigation and this is a standard inefficiency associated with contests. As a result it is more efficient to avoid litigation by allocating the object to firm 1 in exchange for a transfer to firm 2. To see that this first best allocation is not implementable we check that inequality in (4) may also hold in this example. For this it is sufficient to show that

$$\mathbb{E}(v_1(\Theta_2)) + v_2(\overline{\theta}_2) > \theta_1 \tag{46}$$

$$\Leftrightarrow \theta_1 q(1-\gamma)(1-\lambda\gamma) + \theta_1(1-q) + \overline{\theta}_2\gamma(1-\lambda(1-\gamma)) > \theta_1 \tag{47}$$

$$\Leftrightarrow \theta_1 q(1-\gamma)(1-\lambda\gamma) + \overline{\theta}_2\gamma(1-\lambda(1-\gamma)) > q\theta_1 \tag{48}$$

$$\Leftrightarrow \overline{\theta}_2(1-\lambda(1-\gamma)) > q\theta_1(2-\lambda\gamma). \tag{49}$$

Since γ is decreasing in q, the left hand side of (49) also decreases in q, while the right hand side is increasing in q. Moreover $\gamma \to 1$ as $q \to 0$ and consequently as $q \to 0$, the left hand side goes to $\overline{\theta}_2$ and the right hand side goes to zero. The continuity and monotonicity of the two sides in q implies that there must exist a threshold $\hat{q} \in (0,1)$ such that inequality (4) is satisfied for all $q < \hat{q}$. Using Lemma 1 we can state the following.

Observation 2. *It is not possible to allocate the patent to firm 1 in exchange for a transfer to firm 2 when $q < \hat{q}$.*

To summarize, in this example we find that if the probability with which firm 2 is a high type is low enough, it will not be possible for the two firms to reach the first best by allocating the right to use the technology to firm 1 with probability one, in exchange for a transfer to firm 2.

4.2.2 Second Best

We can solve for the surplus maximizing mechanism in this application for the case when the first best is not implementable. The reservation payoffs that define the firm's IR constraint are $\mathbb{E}(v_1(\Theta_2))$ and $v_2(\theta_2)$ from (43). Since the distribution of types here is discrete, the result in this section is not subsumed in the result in Proposition 2 which was derived under the assumption that the distribution of θ_2 is continuous.

Proposition 3. *The surplus maximizing mechanism always exists and comprises of*

$$\begin{array}{lll} \beta_2(\overline{\theta}_2) = \frac{\mathbb{E}(v_1(\Theta_2))+v_2(\overline{\theta}_2)-\theta_1}{\overline{\theta}_2-q\theta_1} & \textit{and} & t_2(\overline{\theta}_2) = v_2(\overline{\theta}_2) - \beta_2(\overline{\theta}_2)\overline{\theta}_2 \\ \beta_2(0) = 0 & \textit{and} & t_2(0) = v_2(\overline{\theta}_2) - \beta_2(\overline{\theta}_2)\overline{\theta}_2 \\ \beta_1 = 1 - q\beta_2(\overline{\theta}_2) & \textit{and} & t_1 = \beta_2(\overline{\theta}_2)\overline{\theta}_2 - v_2(\overline{\theta}_2). \end{array} \tag{50}$$

Proof. The IR constraints for firm 1 and 2 are

$$\beta_1\theta_1 + t_1 \geq \mathbb{E}(v_1(\Theta_2)), \qquad \beta_2(\overline{\theta}_2)\overline{\theta}_2 + t_2(\overline{\theta}_2) \geq v_2(\overline{\theta}_2), \qquad \text{and} \qquad t_2(0) \geq 0 \tag{51}$$

The IC constraints are

$$(\beta_2(\overline{\theta}_2) - \beta_2(0))\overline{\theta}_2 \geq t_2(0) - t_2(\overline{\theta}_2) \geq (\beta_2(\overline{\theta}_2) - \beta_2(0)) \cdot 0. \tag{52}$$

To minimize the inefficiency of the second best allocation we need to minimize $\beta_2(\theta_2)$. The inequalities in (52) indicate that we can simultaneously reduce $\beta_2(\overline{\theta}_2)$ and $\beta_2(0)$ while keeping IC constraints intact. Hence we can set $\beta_2(0) = 0$. The IC constraints simplify to $t_2(0) = t_2(\overline{\theta}_2) = t_2$.

In the surplus maximizing mechanism we must have

$$\beta_1 = 1 - \mathbb{E}(\beta_2(\Theta_2)) \qquad \text{and} \qquad t_1 = -\mathbb{E}(t_2(\Theta_2)) \tag{53}$$

since

$$\beta_1 > 1 - \mathbb{E}(\beta_2(\Theta_2)) \qquad \text{or} \qquad t_1 > -\mathbb{E}(t_2(\Theta_2)) \tag{54}$$

are ruled out by surplus and budget feasibility and

$$\beta_1 < 1 - \mathbb{E}(\beta_2(\Theta_2)) \qquad \text{or} \qquad t_1 < -\mathbb{E}(t_2(\Theta_2)) \tag{55}$$

can be improved by allocating the excess surplus to firm 1 without violating any constraint. Substituting $\beta_1 = 1 - \mathbb{E}(\beta_2(\Theta_2)) = 1 - q\beta_2(\overline{\theta}_2)$ and $t_1 = -\mathbb{E}(t_2(\Theta_2)) = -t_2$ into the IR constraints of two firms in (51) we have

$$(1 - q\beta_2(\overline{\theta}_2))\theta_1 - t_2 \geq \mathbb{E}(v_1(\Theta_2)), \qquad \beta_2(\overline{\theta}_2)\overline{\theta}_2 + t_2 \geq v_2(\overline{\theta}_2), \qquad \text{and} \qquad t_2 \geq 0. \tag{56}$$

The IR constraint of firm 1 and high type firm 2 indicate that we can keep both intact if we simultaneously decrease $\beta_2(\overline{\theta}_2)$ and increase t_2 till both constraints hold with an equality. Once we do this we can solve for $\beta_2(\overline{\theta}_2)$ and find that

$$\beta_2(\overline{\theta}_2) = \frac{\mathbb{E}(v_1(\Theta_2)) + v_2(\overline{\theta}_2) - \theta_1}{\overline{\theta}_2 - q\theta_1}. \tag{57}$$

To check that $\beta_2(\overline{\theta}_2) \in (0,1)$ note first that inequality (4) simplifies to inequality in (49) when the reservation payoffs arise as the equilibrium payoffs from the contest in this example. Since $2 - \lambda\gamma > 1 - 1\lambda(1-\gamma)$, inequality (49) implies that $\overline{\theta}_2 > q\theta_1$ must hold when the first best is not attainable. This implies that the denominator in (57) is positive. For $\beta_2(\overline{\theta}_2) < 1$ we need $\overline{\theta}_2 + (1-q)\theta_1 > v_1 + v_2(\overline{\theta}_2)$. To see this always holds, use expressions in (43) for $\mathbb{E}(v_1(\Theta_2))$ and $v_2(\overline{\theta}_2)$, and note that

$$\overline{\theta}_2 + (1-q)\theta_1 > \mathbb{E}(v_1(\Theta_2)) + v_2(\overline{\theta}_2) \tag{58}$$

$$\Leftrightarrow \overline{\theta}_2 + (1-q)\theta_1 > q\theta_1(1-\gamma)(1-\lambda\gamma) + (1-q)\theta_1 + \overline{\theta}_2\gamma(1-\lambda(1-\gamma)) \tag{59}$$

$$\Leftrightarrow \overline{\theta}_2(1+\lambda\gamma) > q\theta_1(1-\lambda\gamma), \tag{60}$$

which always holds since we have just shown that inequality (49) implies $\overline{\theta}_2 > q\theta_1$. This shows that $\beta_2(\overline{\theta}_2)$ always exists. We can derive β_1, t_1, and $t_2 = t_2(0) = t_2(\overline{\theta}_2)$ as

$$t_2 = v_2(\overline{\theta}_2) - \beta_2(\overline{\theta}_2)\overline{\theta}_2 \qquad t_1 = -t_2 \qquad \text{and} \qquad \beta_1 = 1 - q\beta_2(\overline{\theta}_2)$$

□

Proposition 3 shows the best the two firms can do in this setting. Although this mechanism involves some inefficiency since $\beta_2(\overline{\theta}_2) > 0$, it still pareto dominates the reservation payoffs. At the optimal allocation, the IR constraint of firm 1 and a high type firm 2 bind and consequently they must receive $\mathbb{E}(v_1(\Theta_2))$ and $v_2(\overline{\theta}_2)$ respectively. However a low type firm 2 receives $t_2 > 0$ and this is strictly greater that its payoff under the contest which is zero. This illustrates the point that in this setting it is the low valuation firm 2 that receives an informational rent. This appears to be a robust feature in this model as we also see it in Proposition 2.

4.3 Equilibrium

Finally, using the allocations from the surplus maximizing mechanism derived in Proposition 3 we can derive the equilibrium of this game. I use perfect Bayesian equilibrium as the solution concept. As is well known, such an equilibrium is defined by two elements – first, the strategy of each firm must be sequentially rational, and second, the beliefs of firm 1 must be updated using the Bayes rule wherever possible. The strategy for each firm comprises of a unilateral decision of whether or not to negotiate and a litigation effort if litigation is triggered. In addition to this we need to fully specify the beliefs of firm 1 about firm 2's type given the actions of firm 2 at each node.

Observation 3. *The following is a perfect Bayesian equilibrium of the game.*

1. *Firm 1 chooses to negotiate. If firm 2 chooses litigation (off-equilibrium) firm 1 chooses effort x_1 derived in equation* (40)*.*
2. *Firm 2 chooses to negotiate. If firm 1 chooses litigation (off-equilibrium) it chooses effort 0 if $\theta_2 = 0$ and the effort derived in equation* (41) *if $\theta_2 = \overline{\theta}_2$.*
3. *Firm 1's posterior is equal to its prior when firm 2 chooses to negotiate or litigate.*

We can see that negotiation is an equilibrium by noting that given firm 1 chooses to negotiate, a low type firm 2 is strictly better off by choosing to negotiate, and a high type firm 2 is indifferent between negotiation and litigation. Hence negotiation is a best response for firm 2. Similarly if firm 2 chooses to negotiate, it is a best response for firm 1 to negotiate since it does not gain by triggering litigation. This is because we know from Proposition 3 that firm 1 and high type firm 2 are pushed to their reservation payoffs in the surplus maximizing mechanism whereas a low type firm 2 receives rents. Since firm 2 always negotiates, in equilibrium the posterior belief of firm 1 must remain the same as its prior by Bayes rule. The equilibrium satisfies this as the prior of firm 1 is used as the distribution to derive the surplus maximizing mechanism in Proposition 3.

Note that it is also an equilibrium for both firms to choose litigation since conditional on one firm triggering litigation, it is a best response for the other firm to also choose litigation. However this equilibrium is less plausible since it involves the firms playing weakly dominated strategies.

4.4 Extension

In this section I allow the litigation sub-game to be an unspecified Bayesian game. Moreover instead of a discrete distribution, now firm 2's type is drawn from a continuous distribution with a probability density function $f(\theta_2)$ on the interval $[\underline{\theta}_2, \overline{\theta}_2]$ where $(\theta_1 - \theta_2)f(\theta_2)$ is non-increasing in θ_2. The timeline of the game is the same as the one we used at the start of section 4. We will see that the surplus maximizing mechanism exists in this setting and is characterized by the one in Proposition 2.

Let the equilibrium payoffs of the two firms in the litigation sub-game, under the prior belief of firm 1, be $\mathbb{E}(v_1(\Theta_2))$ and $v_2(\theta_2)$. Assume that the Bayesian game is ex-post inefficient in that these payoffs satisfy the inefficiency condition in (3). Assume for now that the posterior belief of firm 1 about firm 2's type remains the same as the prior if firm 2 chooses to litigate. Just as before, litigation will be off-equilibirum here and we will be free to specify the belief this way in the perfect Bayesian equilibrium that follows. If $\mathbb{E}(v_1(\Theta_2))$ and $v_2(\theta_2)$ satisfy the condition in (4), then the first best will not be implementable since Lemma 1 applies. Hence the patent cannot always be allocated to firm 1. The following Proposition shows that in this setting the surplus maximizing mechanism from Proposition 2 applies.

Proposition 4. *Assume that $(\theta_1 - \theta_2)f(\theta_2)$ is non-increasing in θ_2, and the reservation payoffs arise from an inefficient Bayesian game where $v_2(\theta_2)$ is twice differentiable. The surplus maximizing mechanism is characterized by the one in Proposition 2.*

Proof. To prove this I will first show that $\eta(\theta_2) = v_2(\theta_2)$ is unique. Next we will see that $v_2(\theta_2)$ must satisfy the inequality in (20) since it represents the equilibrium payoff from a Bayesian game that is inefficient. Finally the assumption that $(\theta_1 - \theta_2)f(\theta_2)$ is non-increasing in θ_2 ensures that all conditions in Proposition 2 are satisfied and it follows that the surplus maximizing mechanism is the one characterized there.

First, we will see that when $v_2(\theta_2)$ arises from a Bayesian game and the preferences of the firms are quasi-linear, we must have $v_2'(\theta_2) \in [0, 1]$ and $v_2''(\theta_2) \geq 0$. By the revelation principle, for any equilibrium of a Bayesian game there exists an equivalent direct mechanism where firm 2 declares its true type. The equilibrium payoff $v_2(\theta_2)$ can be represented as

$$v_2(\theta_2) = \theta_2\alpha(\theta_2) + x(\theta_2), \tag{61}$$

where $\alpha(\theta_2)$ is equilibrium probability with which the patent is allocated to firm 2 of type θ_2 and $x(\theta_2)$ is the corresponding equilibrium transfer in the direct mechanism. Let firm 2 declare a type $\tilde{\theta}_2$ in the direct mechanism. Hence while playing the direct mechanism firm 2 maximizes

$$\max_{\tilde{\theta}_2 \in [\underline{\theta}_2, \bar{\theta}_2]} (\theta_2\alpha(\tilde{\theta}_2) + x(\tilde{\theta}_2)) \tag{62}$$

For $\tilde{\theta}_2 = \theta_2$ to be the equilibrium, the first and second order conditions simplify to

$$\alpha(\theta_2) \geq 0 \qquad \text{and} \qquad \alpha'(\theta_2) \geq 0. \tag{63}$$

Since $\alpha(\theta_2)$ must take a value in the interval $[0, 1]$, and $v_2'(\theta_2) = \alpha(\theta_2)$, we find that $v_2'(\theta_2) \in [0, 1]$. Similarly since $v_2''(\theta_2) = \alpha'(\theta_2)$, we must have $v_2''(\theta_2) \geq 0$. This implies that $v_2(\theta_2) = \eta(\theta_2)$ is unique.

Second, we have assumed that the Bayesian game is inefficient in that condition (3) is satisfied. This implies that we must have

$$\mathbb{E}(v_1(\Theta_2)) + \mathbb{E}(v_2(\Theta_2)) < \theta_1 \tag{64}$$

which is sufficient for the inequality in (20) to be satisfied since $\eta(\theta_2) = v_2(\theta_2)$. Finally, we have assumed that $(\theta_1 - \theta_2)f(\theta_2)$ is non-increasing in θ_2. This ensures all three conditions required for the surplus maximizing mechanism in Proposition 2 are satisfied. □

Proposition 4 shows that we can use the surplus maximizing mechanism from Proposition 2 when reservation payoffs arise from an inefficient Bayesian game. Using this we can construct a perfect Bayesian equilibrium such that both firms prefer to negotiate and receive these allocations rather than choosing to litigate.

To make this point clearly define σ_1 and $\sigma_2(\theta_2)$ as the equilibrium strategies in the litigation sub-game when it is played under the prior beliefs of firm 1. These are the strategies that lead to the equilibrium payoffs $\mathbb{E}(v_1(\Theta_2))$ and $v_2(\theta_2)$. We must remain agnostic about what these strategies are since we have not specified the litigation sub-game. In particular they may represent pure or mixed strategies. Moreover the litigation sub-game could be a simultaneous or a sequential game played over one or many periods with moves and countermoves. In this case σ_1 and $\sigma_2(\theta_2)$ would represent the entire equilibrium profile for each firm that specify the strategy and beliefs conditional on each node of the litigation sub-game being reached.

Observation 4. *The following is a perfect Bayesian equilibrium of the game.*

1. *Firm 1 chooses to negotiate. If firm 2 chooses litigation (off-equilibrium) firm 1 chooses action σ_1.*
2. *Firm 2 chooses to negotiate. If firm 1 chooses litigation (off-equilibrium) firm 2 chooses action $\sigma_2(\theta_2)$.*
3. *Firm 1's posterior is equal to its prior when firm 2 chooses to negotiate or litigate.*

This observation shows that given the existence of a surplus maximizing mechanism we can construct an equilibrium where the firms choose to negotiate.[15] As before firm 1's posterior being equal to its prior is consistent in equilibrium since firm 2 always negotiates.

4.5 Discussion

The negotiation equilibrium derived here relies on the assumption that the posterior belief of firm 1 is equal to its prior in the event firm 2 chooses the off-equilibrium action of triggering litigation. It is possible that the surplus maximizing mechanism changes when this assumption is changed and that allocations that were previously not implementable become implementable and vice versa. This assumption is made to micro-found the idea that the reservation payoffs can arise as equilibrium payoffs from a Bayesian game. However, deriving the surplus maximizing mechanism while allowing the posterior of firm 1 to be affected by firm 2's choice of triggering litigation in a more general way is an interesting avenue for future work.

A firm chooses to negotiate only when it expects to receive an allocation that weakly dominates its payoff from litigation. This is true as long as the firms expect negotiations to yield a pareto dominating outcome relative to their reservation payoffs. However in the real world there are situations where this may not be true. If for example, the negotiations are constrained by a particular bargaining protocol, the expected allocation from negotiation will be generally inferior to the one in the surplus maximizing mechanism and may even be lower than the litigation payoff for at least one of the two firms.

[15] To be precise the existence of a surplus maximizing mechanism is sufficient but not necessary for constructing such an equilibrium. We merely need that the reservation payoffs satisfy the condition in (20) which guarantees the existence of at least one allocation that weakly dominates the reservation payoffs.

It is also possible that ex-post, once firm 2 declares its type, firm 1's modified reservation payoff (now foregone) is actually greater than its ex-post allocation under the mechanism. Since the firms commit to forgoing the default game once they agree to negotiate, the possibility of a change in the reservation payoffs induced by firms 2's declaration does not create any problems. However, if firms lacked the ability to commit, this imposes additional constraints on the surplus maximizing mechanism possibly leading it its non-existence.[16] Hence there is an important implicit assumption in this framework – that negotiations between the firms as captured by the mechanism are only constrained by IR constraints of the two firms and the IC of firm 2 and moreover when a surplus maximizing allocation exists, then the firms expect it to be picked when they choose to negotiate.

5 Conclusion

This paper embeds the Myerson and Satterthwaite (1983) model in a setting where property rights are insecure. This insecurity allows for ex-post inefficiency under much weaker conditions, in particular even when there is no uncertainty about which player values the object more. This inefficiency arises when the reservation payoffs of the players depend on the type of the informed player. This causes the reservation payoff of the uninformed player to become unobservable to her. Consequently there are states where the uninformed player overestimates her reservation payoff. I have shown that even when the reservation payoffs involve some inefficiency, if this overestimation is large enough, it is impossible to implement the first best. Taking type dependent reservation payoffs as given, the paper characterizes the surplus maximizing mechanism and the necessary and sufficient condition required for its existence. Finally I show an application of the model to a case of an intellectual property dispute between two firms where type dependent reservation payoffs arise endogenously as a consequence of property rights not being well defined.

This paper can be seen as an attempt to characterize a failure of the Coase theorem in the environment where parties attempt to resolve their disputes efficiently when their reservation payoffs are determined by conflict. The paper shows that even the best alternatives to conflict may not deliver ex-post efficiency.

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Appendix

I now describe a procedure that will allow us to construct the set Ψ for a given $v_2(\theta_2)$, under the assumption that $v_2(\theta_2)$ is twice differentiable, and either concave or convex on the interval $[\underline{\theta}_2, \overline{\theta}_2]$.

1. Check if there exists a θ_2 such that $v_2''(\theta_2) < 0$. If yes, proceed to step 6. If no, then proceed to step 2.

2. Construct a function $\eta_0(\theta_2) = v_2(\theta_2)$ for all θ_2. If $\eta_0(\theta_2)$ is non-decreasing, relable $\eta_0(\theta_2)$ as $\eta_3(\theta_2)$ and proceed to step 5. If not, proceed to step 3.

3. Since $\eta_0(\theta_2)$ is convex, the lowest point in $[\underline{\theta}_2, \overline{\theta}_2]$ where $\eta_0(\theta_2)$ is decreasing must be $\underline{\theta}_2$. Set
$$\eta_1(\theta_2) = \eta_0(\underline{\theta}_2) \qquad \forall \theta_2. \tag{65}$$
Check if $\eta_1(\theta_2) \geq \eta_0(\theta_2)$. If yes, then relabel $\eta_1(\theta_2)$ as $\eta_3(\theta_2)$ and proceed to step 5. If no, then proceed to step 4.

4. Find the point $a \in [\underline{\theta}_2, \overline{\theta}_2]$ such that $\eta_0(a) = \eta_1(a)$ and $\eta_1(\theta_2) < \eta_0(\theta_2)$ for all $\theta_2 \in (a, \overline{\theta}_2]$ and set
$$\begin{aligned} \eta_2(\theta_2) &= \eta_1(\theta_2) \quad \theta_2 \leq a \\ \eta_2(\theta_2) &= \eta_0(\theta_2) \quad \theta_2 > a \end{aligned} \tag{66}$$
Since $\eta_0(\theta_2)$ is convex, $\eta_2(\theta_2)$ must be convex and we must have $\eta_2(\theta_2) \geq v_2(\theta_2)$. Proceed to step 5.

5. Using (19) check if $\eta_2'(\overline{\theta}_2) \leq 1$. If yes, then relabel $\eta_2(\theta_2)$ as $\eta(\theta_2)$ and the procedure finishes. If no, then identify the highest point a such that $a + \eta_2(\overline{\theta}_2) - \overline{\theta}_2 = \eta_2(a)$. If no $a \in [\underline{\theta}_2, \overline{\theta}_2]$ exists then construct the function $\eta(\theta_2)$ such that
$$\eta(\theta_2) = \theta_2 + \eta_2(\overline{\theta}_2) - \overline{\theta}_2 \quad \theta_2 \in [\underline{\theta}_2, \overline{\theta}_2] \tag{67}$$
and the procedure ends. If $a \in [\underline{\theta}_2, \overline{\theta}_2]$ does exist then set
$$\begin{aligned} \eta(\theta_2) &= \theta_2 + \eta_2(\overline{\theta}_2) - \overline{\theta}_2 \quad && \theta_2 \in [a, \overline{\theta}_2] \\ \eta(\theta_2) &= \eta_2(\theta_2) && \theta_2 < a. \end{aligned} \tag{68}$$

Now we must have $\eta'(\theta_2) \leq 1$ for all θ_2 and the procedure ends.

6. Construct a function $\eta_0(\theta_2) = v_2(\theta_2)$ for all θ_2. If $\eta_0(\theta_2)$ is non-decreasing, relable $\eta_0(\theta_2)$ as $\eta_1(\theta_2)$ and proceed to step 7. If $\eta_0(\theta_2)$ is non-increasing, set $\eta(\theta_2) = \eta_0(\underline{\theta}_2)$ for all θ_2 and the procedure ends. If $\eta_0(\theta_2)$ increases and then decreases, then find $a \in [\underline{\theta}_2, \overline{\theta}_2]$ where $\eta_0(\theta_2)$ attains its maximum and set

$$\begin{aligned} \eta_1(\theta_2) &= \eta_0(\theta_2) \quad \theta_2 < a \\ \eta_1(\theta_2) &= \eta_0(a) \quad\ \ \theta_2 \geq a. \end{aligned} \tag{69}$$

   and proceed to step 7.

7. Check if $\eta_1'(\theta_2) \leq 1$. If yes, relabel $\eta_1(\theta_2)$ as $\eta_2(\theta_2)$ and proceed to step 8. If not, identify the point $a \in [\underline{\theta}_2, \overline{\theta}_2]$ such that $\eta_1'(a) = 1$. Due to concavity of $\eta_1(\theta_2)$ we must have $\eta_1'(\theta_2) > 1$ for $\theta_2 < a$. Construct

$$\begin{aligned} \eta_2(\theta_2) &= \theta_2 + \eta_1(a) - a \quad \theta_2 < a \\ \eta_2(\theta_2) &= \eta_1(\theta_2) \quad\quad\quad\quad\ \theta_2 \geq a. \end{aligned} \tag{70}$$

   Proceed to step 8.

8. For each point $x \in [\underline{\theta}_2, \overline{\theta}_2]$ construct a function

$$\eta(\theta_2) = \theta_2 \eta_2'(x) + \eta_2(x) - x\eta_2'(x) \tag{71}$$

   and assign each $\eta(\theta_2)$ to the set Ψ, and this procedure ends.

Proof of Proposition 1. When $v_2(\theta_2)$ is convex, steps 3 and 4 ensure that $\eta(\theta_2)$ is increasing, and step 5 ensures that $1 \geq \eta'(\theta_2) \geq 0$. Note that in step 5 $\eta'(\theta_2) < 1$ for $\theta_2 > a$ is not possible since $\eta(\theta_2)$ must be continuous and $\eta'(\theta_2) < 1$ for some $\theta_2 > a$ implies that we must also have $\eta'(\theta_2) > 1$ for some $\theta_2 > a$ to satisfy the IR constraint $\eta(\theta_2) \geq \eta_2(\theta_2)$. Hence the lowest possible $\eta(\theta_2)$ is defined by (67) and (68). We see from step 5 that $\eta(\theta_2)$ is unique when $v_2(\theta_2)$ is convex. When $v_2(\theta_2)$ is concave, step 6 ensures that $\eta(\theta_2)$ is increasing and step 7, like step 5 in case of convex $v_2(\theta_2)$, ensures that $1 \geq \eta'(\theta_2) \geq 0$. In this case we have a set of functions Ψ with elements $\eta(\theta_2)$ which may be denoted as $\eta(\theta_2; x)$ that correspond to each point $x \in [\underline{\theta}_2, \overline{\theta}_2]$.

I will now prove that Ψ is compact. When $v_2(\theta_2)$ is convex, $\eta(\theta_2)$ is unique, and the set Ψ is trivially compact. Consider the case when $v_2(\theta_2)$ is concave. In this case Ψ may be populated with a continuum of functions. Let $\psi : [\underline{\theta}_2, \overline{\theta}_2] \to \Psi$, be defined by $\psi(x)(\theta_2) = \eta(\theta_2; x)$ for all $x \in [\underline{\theta}_2, \overline{\theta}_2]$. This function maps $[\underline{\theta}_2, \overline{\theta}_2]$ onto Ψ since for all $\eta \in \Psi$, there exists an $x \in [\underline{\theta}_2, \overline{\theta}_2]$ such that $\psi(x) = \eta(\cdot; x)$. To see this note from step 8 that for each $x \in [\underline{\theta}_2, \overline{\theta}_2]$, there exists an $\eta(\theta_2; x)$ that is constructed as a tangent to $\eta_2(\theta_2)$ at x.

Hence as $[\underline{\theta}_2, \overline{\theta}_2]$ is compact, if $\psi(x)$ is continuous for all $x \in [\underline{\theta}_2, \overline{\theta}_2]$, then Ψ which is the image of ψ is compact. By definition, $\psi(x)$ is continuous at $x \in [\underline{\theta}_2, \overline{\theta}_2]$ if for any $\epsilon > 0$ there exists a δ such that $\|\psi(x) - \psi(y)\| < \varepsilon$ for all $y \in [\underline{\theta}_2, \overline{\theta}_2]$ such that $|x - y| < \delta$, where $\|\cdot\|$

is the uniform norm. Therefore, by the definitions of ψ and η, we need to show that for any $\varepsilon > 0$, there exists a δ such that for all $y \in [\underline{\theta}_2, \overline{\theta}_2]$ such that $|x - y| < \delta$,

$$\|\psi(x) - \psi(y)\| = \sup_{\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]} |\eta(\theta_2; x) - \eta(\theta_2; y)| \tag{72}$$

$$= \sup_{\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]} |\theta_2 \eta_2'(x) + \eta_2(x) - x\eta_2'(x) - \theta_2 \eta_2'(y) - \eta_2(y) + y\eta_2'(y)| \tag{73}$$

$$\leq |\eta_2(x) - \eta_2(y)| + \overline{\theta}_2 |\eta_2'(x) - \eta_2'(y)| + |x\eta_2'(x) - y\eta_2'(y)| < \varepsilon\,. \tag{74}$$

From step 8, we know that $\eta_2(\theta_2)$, $\eta_2'(\theta_2)$, and hence $\theta_2 \eta_2'(\theta_2)$, are continuous when $v_2(\theta_2)$ is twice differentiable and concave for all $\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]$. Hence, for any $x \in [\underline{\theta}_2, \overline{\theta}_2]$ and any $\varepsilon/3 > 0$ we can find $\delta_1, \delta_2, \delta_3$ such that $|\eta_2(x) - \eta_2(y)| < \varepsilon/3$ for all $y \in [\underline{\theta}_2, \overline{\theta}_2]$ with $|x - y| < \delta_1$, $\overline{\theta}_2 |\eta_2'(x) - \eta_2'(y)| < \varepsilon/3$ for all $y \in [\underline{\theta}_2, \overline{\theta}_2]$ with $|x - y| < \delta_2$, and $|x\eta_2'(x) - y\eta_2'(y)| < \varepsilon/3$ for all $y \in [\underline{\theta}_2, \overline{\theta}_2]$ with $|x - y| < \delta_3$. Let $\delta := \min(\delta_1, \delta_2, \delta_3)$. Then for any $y \in [\underline{\theta}_2, \overline{\theta}_2]$ such that $|x - y| < \delta$, the inequality in (74) is satisfied, since each of the three terms in the sum is less than $\varepsilon/3$, showing that $\psi(x)$ is indeed continuous at any $x \in [\underline{\theta}_2, \overline{\theta}_2]$.

□