Supplement to
"Semi-parametric GMM Estimation of Spatial Autoregressive Models"

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THIS APPENDIX PROVIDES PROOFS FOR TECHNICAL LEMMAS IN THE ABOVE PAPER.

Let C signify a generic constant whose exact value may vary from case to case. Frequently we will use two evident facts (see, e.g., Kelejian and Prucha, 1999; Lee, 2002):

Fact 1: If the row and column sums of the $n \times n$ matrices B_{1n} and B_{2n} are uniformly bounded in absolute value, then the row and column sums of $B_{1n}B_{2n}$ are also uniformly bounded in absolute value.

Fact 2: If the row (resp. column) sums of B_{1n} are uniformly bounded in absolute value and B_{2n} is a conformable matrix whose elements are uniformly $O(o_n)$, then so are the elements of $B_{1n}B_{2n}$ (resp. $B_{2n}B_{1n}$).

For example, the row and column sums of $G_{1n} = W_{1n}\left(I_n - \rho_n^0 W_{1n}\right)^{-1}$ are uniformly bounded by Assumption 1 and Fact 1. Noting that the elements $\mathbf{s}_{h\lambda,ij}$ of $\mathbf{S}_{h\lambda}$ are uniformly $O(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1})$, so are elements of $G_{1n}\mathbf{S}_{h\lambda}$ or $\mathbf{S}_{h\lambda}G_{1n}$ by Assumption 1 and Facts 1-2.

A Proof of Lemma A.2

Recall that $\mathbf{S}_{h\lambda} = (\mathbf{s}_{h\lambda}(x_{n,1}), ..., \mathbf{s}_{h\lambda}(x_{n,n}))'$. Denote the (i,j)th element of $\mathbf{S}_{h\lambda}$ as $\mathbf{s}_{h\lambda,ij}$. To study the properties of $\mathbf{s}_{h\lambda,ij}$, we need to distinguish whether $x_{n,j}^c \equiv (x_{n,j1}^c, \cdots x_{n,jp_c}^c)'$ is a boundary point in the compact support $\mathcal{X}^c$ of $f\left(x^c, x^d\right)$. Without loss of generality, we assume $\mathcal{X}^c = \Pi_{s=1}^{p_c} S_s$, where $S_s \equiv [\underline{x}_s, \overline{x}_s]$. A point $x_{n,j}^c$ is said to be a boundary point in $\mathcal{X}^c$ if there exists $s \in \{1, 2, \cdots, p_c\}$ such that $x_{n,js}^c = \underline{x}_s + b_s h_s$ or $x_{n,js}^c = \overline{x}_s - c_s h_s$ for some finite positive numbers b_s and c_s. Otherwise, we say that $x_{n,j}^c$ is not a boundary point. In the following, when $x_{n,j}^c$ is a boundary point, we assume that it is a pure "lower" boundary point such that we can write $x_{n,j}^c = \underline{x} + b \odot h_s$, where $\underline{x} = (\underline{x}_1, \cdots \underline{x}_{p_c})'$ and $b = (b_1, \cdots b_{p_c})'$. Other cases of boundary points can be analogously analyzed.

Define

$$\mathbf{A}_b(x) = \begin{pmatrix} \kappa_{b,0}\varphi(x) & \varphi(x)\kappa_{b,1}' \\ \varphi(x) \otimes \kappa_{b,1} & \varphi(x) \otimes \kappa_{b,2} \end{pmatrix}$$

where $\kappa_{b,0}=\int_{-b}^{\infty}\Pi_{s=1}^{p_c}q\left(u_s\right)du$, $\kappa_{b,1}=\int_{-b}^{\infty}u\Pi_{s=1}^{p_c}q\left(u_s\right)du$, and $\kappa_{b,2}=\int_{-b}^{\infty}uu'\Pi_{s=1}^{p_c}q\left(u_s\right)du$. Note that when $b=\infty$, $\Pi_b=\Pi_{\infty}=\Pi$, where Π is defined in Section 3.3.

Lemma A.2 *(a)* $\sum_{j=1}^{n}s_{h\lambda,ij}=1$ *and* $\sum_{j=1}^{n}s_{h\lambda,ij}\left(x_{n,j}^{c}-x_{n,i}^{c}\right)=0_{p_c\times 1}$ *for each* i;

(b) $A_{n,h\lambda}\left(x_{n,j}\right)=f\left(x_{n,j}\right)\overline{\mathbf{A}}\left(x_{n,j}\right)+o\left(1\right)$ *for each* j;

(c) the row and column sums of $S_{h\lambda}=\left(s_{h\lambda,ij}\right)$ *are uniformly bounded in absolute value for sufficiently large* n,

where $\overline{\mathbf{A}}=A$ *if* $x_{n,j}$ *is not a boundary point, and* $\overline{\mathbf{A}}=A_b$ *if* $x_{n,j}^{c}=\underline{x}+b\odot h$. *For other cases of boundary points,* $\overline{\mathbf{A}}$ *can be similarly defined.*

Proof. To show (a), write

$$\begin{aligned}\mathbf{s}_{h\lambda}\left(x_{n,i}\right)' &\equiv e_1'\left(\mathbf{A}_{n,h\lambda}\left(x_{n,i}\right)'\mathbf{A}_{n,h\lambda}\left(x_{n,i}\right)\right)^{-1}\mathbf{A}_{n,h\lambda}\left(x_{n,i}\right)'\mathbf{Z}_{n,h}\left(x_{n,i}\right)'\operatorname{diag}\left(\mathbf{k}_{h\lambda}\left(x_{n,i}\right)\right)\\ &= \left(\mathbf{s}_{h\lambda,i1},...,\mathbf{s}_{h\lambda,in}\right). \qquad\qquad \text{(A.1)}\end{aligned}$$

Noting that $\left(\mathbf{A}_{n,h\lambda}\left(x_{n,i}\right)'\mathbf{A}_{n,h\lambda}\left(x_{n,i}\right)\right)^{-1}\mathbf{A}_{n,h\lambda}\left(x_{n,i}\right)'\mathbf{A}_{n,h\lambda}\left(x_{n,i}\right)=I_{p_c+1}$, the results then follow from the definitions of $\mathbf{A}_{n,h\lambda}$, $\mathbf{Z}_{n,h}$ and $\mathbf{k}_{h\lambda}$. For (b), we have

$$\mathbf{A}_{n,h\lambda}\left(x_{n,j}\right)=n^{-1}\sum_{i=1}^{n}\mathbf{z}_{h,ij}\boldsymbol{\tau}_{h,ij}'K_{h\lambda,ij}=\begin{pmatrix}\Gamma_{nj,11} & \Gamma_{nj,12}\\ \Gamma_{nj,21} & \Gamma_{nj,22}\end{pmatrix}$$

with

$$\begin{aligned}\Gamma_{nj,11} &= n^{-1}\sum_{i=1}^{n}K_{h\lambda,ij}z_{n,i}^{(1)},\ \Gamma_{nj,12}=n^{-1}\sum_{i=1}^{n}K_{h\lambda,ij}z_{n,i}^{(1)}(x_{n,i}^{c}-x_{n,j}^{c})'/h,\\ \Gamma_{nj,21} &= n^{-1}\sum_{i=1}^{n}K_{h\lambda,ij}z_{n,i}^{(1)}\otimes\left((x_{n,i}^{c}-x_{n,j}^{c})/h\right),\text{ and}\\ \Gamma_{nj,22} &= n^{-1}\sum_{i=1}^{n}K_{h\lambda,ij}z_{n,i}^{(1)}\otimes\left((x_{n,i}^{c}-x_{n,j}^{c})/h\right)\left((x_{n,i}^{c}-x_{n,j}^{c})/h\right)'.\end{aligned}$$

We first focus on the case where $x_{n,j}^{c}$ is not a boundary point. By Assumptions 3-4,

$$\begin{aligned}\Gamma_{nj,11} &= n^{-1}\sum_{i=1}^{n}z_{n,i}^{(1)}\left[\prod_{t=1}^{p_c}h_t^{-1}q\left(\frac{x_{n,it}^{c}-x_{n,jt}^{c}}{h_t}\right)\right]\left[\prod_{s=1}^{p_1}\lambda_s^{\left|\overrightarrow{x}_{n,is}^{d}-\overrightarrow{x}_{n,js}^{d}\right|}\right]\left[\prod_{s=1}^{p_2}\lambda_{s+p_1}^{1-\mathbf{1}(\widetilde{x}_{n,is}^{d}=\widetilde{x}_{n,js}^{d})}\right]\\ &= n^{-1}\sum_{i=1}^{n}z_{n,i}^{(1)}\left[\prod_{t=1}^{p_c}h_t^{-1}q\left(\frac{x_{n,it}^{c}-x_{n,jt}^{c}}{h_t}\right)\right]\left\{\mathbf{1}(x_{n,i}^{d}=x_{n,j}^{d})+O\left(\|\lambda\|\right)\mathbf{1}(x_{n,i}^{d}\neq x_{n,j}^{d})\right\}\\ &= \int_{\mathbb{R}^{p_c}}\prod_{t=1}^{p_c}q\left(u_t\right)\varphi_n\left(x_{n,j}^{c}+h\odot u,x_{n,j}^{d}\right)f_n\left(x_{n,j}^{c}+h\odot u,x_{n,j}^{d}\right)du+o\left(1\right)+O\left(\|\lambda\|\right)\\ &= \varphi\left(x_{n,j}\right)f\left(x_{n,j}\right)+o\left(1\right),\end{aligned}$$

and

$$\begin{aligned}\Gamma_{nj,12} &= n^{-1}\sum_{i=1}^{n}z_{n,i}^{(1)}\frac{\left(x_{n,i}^{c}-x_{n,j}^{c}\right)'}{h}\left[\prod_{t=1}^{p_c}h_t^{-1}q\left(\frac{x_{n,it}^{c}-x_{n,jt}^{c}}{h_t}\right)\right]\mathbf{1}(x_{n,i}^{d}=x_{n,j}^{d})+O\left(\|\lambda\|\right)\\ &= \int_{\mathbb{R}^{p_c}}\varphi_n\left(x_{n,j}^{c}+h\odot u,x_{n,j}^{d}\right)u'\prod_{t=1}^{p_c}q\left(u_t\right)f_n\left(x_{n,j}^{c}+h\odot u,x_{n,j}^{d}\right)du+o\left(1\right)\\ &= O(\|h\|^2)+o\left(1\right)=o\left(1\right).\end{aligned}$$

Similarly, we can show that $\digamma_{nj,21} = o(1)$, and $\digamma_{nj,22} = \kappa_{21} f(x_{n,j})(\varphi(x_{n,j}) \otimes I_{p_c}) + o(1)$. Consequently, $n^{-1}\mathbf{Z}'_{h,j}\text{diag}(\mathbf{k}_{h\lambda,\cdot j})\,\mathbf{Z}_{h,j} = f(x_{n,j}) \begin{pmatrix} \varphi(x_{n,j}) & 0 \\ 0 & \kappa_{21}\varphi(x_{n,j}) \otimes I_{p_c} \end{pmatrix} + o(1)$ when $x^c_{n,j}$ is not a boundary point.

If $x^c_{n,j}$ is a boundary point, we only need to change the upper or lower limit of integration in the above proof. For example, if $x^c_{n,j} = \underline{x} + b \odot h$, then

$$\begin{aligned} \digamma_{nj,11} &= \int_{-b}^{\infty} \prod_{t=1}^{p_c} q(u_t)\, \varphi\left(x^c_{n,j} + h \odot u, x^d_{n,j}\right) f\left(x^c_{n,j} + h \odot u, x^d_{n,j}\right) du + o(1) + O(\|\lambda\|) \\ &= \varphi(x_{n,j})\, f(x_{n,j})\, \kappa_{b,0} + o(1). \end{aligned}$$

This proves (b).

Let $e'_1 \left(\overline{\mathbf{A}}(x)' \overline{\mathbf{A}}(x)\right)^{-1} \overline{\mathbf{A}}(x)' = (a_1(x)', a_2(x)')$, where a'_1 and a'_2 are $1 \times q_1$ and $1 \times q_1 p_c$ vectors, respectively. By (b) and (A.1),

$$\mathbf{s}_{h\lambda,ij} = n^{-1}\left\{ f^{-1}(x_{n,i}) \left[a_1(x_{n,i})' z^{(1)}_{n,j} + a_2(x_{n,i})' \left(z^{(1)}_{n,j} \otimes ((x^c_{n,j} - x^c_{n,i})/h) \right) \right] + o(1) \right\} K_{h\lambda,ij}.$$

By Assumptions 3-4 and similar arguments to the proof of (b), we have as $n \to \infty$,

$$\begin{aligned} \sum_{i=1}^{n} |\mathbf{s}_{h\lambda,ij}| &\leq n^{-1} \sum_{i=1}^{n} f^{-1}(x_{n,i}) \left| a_1(x_{n,i})' z^{(1)}_{n,j} + a_2(x_{n,i})' \left(z^{(1)}_{n,j} \otimes ((x^c_{n,j} - x^c_{n,i})/h) \right) \right| K_{h\lambda,ij} \\ &\quad + o(1)\, n^{-1} \sum_{i=1}^{n} K_{h\lambda,ij} \\ &= \int \prod_{t=1}^{p_c} q(u_t)\left(\overline{a}_1 c_{z1} + \overline{a}_2 c_{z1} \|u\|\right) du + o(1) \int \prod_{t=1}^{p_c} q(u_t)\, f(x_{n,i})\, du + o(1) \\ &\leq c_{z1}\overline{a}_1 + c_{z1}\overline{a}_2\overline{c}_q + o(1), \end{aligned}$$

where for some large n_0, $\overline{a}_l = \sup_{1 \leq i \leq n, n \geq n_0} \|\pi_l(x_{n,i})\|$ for $l = 1, 2$, $c_{z1} = \sup_{1 \leq i \leq n, n \geq n_0} ||z^{(1)}_{n,j}||$, and the integration is taken over $\mathbb{R}^{p_c}$ or a subset of $\mathbb{R}^{p_c}$ depending on whether $x^c_{n,i}$ is a boundary point, and $\overline{c}_q = \int_{\mathbb{R}^{p_c}} \Pi_{t=1}^{p_c} q(u_t) \|u\|\, du$. This implies that $\sum_{i=1}^{n} |\mathbf{s}_{h\lambda,ij}|$ is bounded uniformly in j for sufficiently large n. Similarly, one can show that $\sum_{j=1}^{n} |\mathbf{s}_{h\lambda,ij}|$ is bounded uniformly in i when n is sufficiently large. This proves (c). ■

B Proof of Lemma C.1

Lemma C.1 *Let $\mathcal{A}_n$ be a real nonstochastic $n \times n$ matrix whose row and column sums are uniformly bounded in absolute value: $sup_{1 \leq j \leq n, n \geq 1} \sum_{i=1}^{n} |\alpha_{n,ij}| \leq c_\alpha$ and $sup_{1 \leq i \leq n, n \geq 1} \sum_{j=1}^{n} |\alpha_{n,ij}| \leq c_\alpha$ for some $c_\alpha < \infty$. Let $\vartheta_n = n^{-1} U'_n \mathcal{A}_n U_n$ and $\widetilde{\vartheta}_n = n^{-1} \widetilde{U}'_n \mathcal{A}_n \widetilde{U}_n$. Then*

(a) $E[\vartheta_n] = O(1)$, $Var(\vartheta_n) = O(n^{-1}) = o(1)$, and $\vartheta_n - E[\vartheta_n] = O_p\left(n^{-1/2}\right)$;

(b) $n^{-1/2}\widetilde{U}'_n \mathcal{A}_n \widetilde{U}_n = n^{-1/2} U'_n \mathcal{A}_n U_n - n^{-1/2} \delta_{\rho n} E[U'_n \overline{G}_{12n} \varepsilon_n] + o_p(1)$, where $\delta_{\rho n} \equiv \widetilde{\rho}_n - \rho^0_n$, $\overline{G}_{1n} \equiv G'_{1n}(\mathcal{A}_n + \mathcal{A}'_n)$ and $\overline{G}_{12n} \equiv \overline{G}_{1n}\left(I_n - \gamma^0_n W_{2n}\right)^{-1}$.

Proof. (a) Let $\mathcal{B}_n = (1/2)\left(I_n - \gamma_n^0 W_{2n}'\right)^{-1}(\mathcal{A}_n + \mathcal{A}_n')\left(I_n - \gamma_n^0 W_{2n}\right)^{-1}$. Then $\vartheta_n = n^{-1}\varepsilon_n'\mathcal{B}_n\varepsilon_n$. By the assumption on $\mathcal{A}_n$, Assumption 6 and Fact 1, the row and column sums of $\mathcal{B}_n$ are uniformly bounded in absolute value. By the Cauchy-Schwarz inequality, $E|\vartheta_n| = n^{-1}\sum_{i=1}^n\sum_{j=1}^n |\beta_{n,ij}| \times E|\varepsilon_{n,i}\varepsilon_{n,j}| \leq n^{-1}\sum_{i=1}^n\sum_{j=1}^n |\beta_{n,ij}|\sigma_{n,i}\sigma_{n,j} \leq \overline{\sigma}^2 C$, where $\beta_{n,ij}$ denotes the (i,j) element of $\mathcal{B}_n$. Similarly,

$$\begin{aligned}\mathrm{Var}(\vartheta_n) &= 2n^{-2}\mathrm{tr}(\mathcal{B}_n\Sigma_n\mathcal{B}_n\Sigma_n) + n^{-2}\sum_{i=1}^n \beta_{n,ii}^2\left[E\varepsilon_{n,i}^4 - 3\sigma_{n,i}^4\right] \\ &\leq 2n^{-1}c_\beta + n^{-1}c_\beta^2\overline{\overline{\mu}}_4 = O(n^{-1})\end{aligned}$$

where $\overline{\overline{\mu}}_4 \equiv \sup_{1\leq i\leq n, n\geq 1}\left[E\varepsilon_{n,i}^4 - 3\sigma_{n,i}^4\right]$, c_β is a common bound for the row and column sums of the absolute elements of $\mathcal{B}_n$ and $\mathcal{B}_n\Sigma_n\mathcal{B}_n\Sigma_n$ and of their respective elements. By the Chebyshev inequality,

$$\vartheta_n - E\vartheta_n = O_p(n^{-1/2}). \tag{B.1}$$

(b) Note that $\widetilde{U}_n = Y_n - \widetilde{\mathbf{m}}(X_n) - \widetilde{\rho}_n W_{1n}Y_n = U_n - D_n - \delta_{\rho n}\overline{Y}_n$, where $D_n \equiv \widetilde{\mathbf{m}}(X_n) - \mathbf{m}(X_n)$. So

$$\begin{aligned}\widetilde{\vartheta}_n - \vartheta_n &= -n^{-1}\delta_{\rho n}\overline{Y}_n'(\mathcal{A}_n + \mathcal{A}_n')U_n + n^{-1}\delta_{\rho n}^2\overline{Y}_n'\mathcal{A}_n\overline{Y}_n - n^{-1}D_n'(\mathcal{A}_n + \mathcal{A}_n')U_n \\ &\quad + n^{-1}\delta_{\rho n}D_n'(\mathcal{A}_n + \mathcal{A}_n')\overline{Y}_n + n^{-1}D_n'\mathcal{A}_nD_n \\ &\equiv -T_{n3} + T_{n4} - T_{n5} + T_{n6} + T_{n7}.\end{aligned} \tag{B.2}$$

We will show that T_{n3} is a dominant term in the above expression and $T_{nj} = o_p\left(n^{-1/2}\right)$ for $j = 4, 5, 6, 8$.

First, recall $\overline{G}_{1n} = G_{1n}'(\mathcal{A}_n + \mathcal{A}_n')$ and $\overline{G}_{12n} = \overline{G}_{1n}\left(I_n - \gamma_n^0 W_{2n}\right)^{-1}$. Then by the assumption on $\mathcal{A}_n$, Assumptions 1 and 6 and Fact 1, the row and column sums of $\overline{G}_{1n}$ and $\overline{G}_{12n}$ are uniformly bounded in absolute value. Write

$$\begin{aligned}T_{n3} &= n^{-1}\delta_{\rho n}\overline{Y}_n'(\mathcal{A}_n + \mathcal{A}_n')U_n = \delta_{\rho n}[n^{-1}\mathbf{m}(X_n)'\overline{G}_{12n}\varepsilon_n + n^{-1}U_n'\overline{G}_{12n}\varepsilon_n] \\ &\equiv \delta_{\rho n}(T_{n3a} + T_{n3b}).\end{aligned}$$

Note that $E[T_{n3a}] = 0$, and

$$\begin{aligned}\mathrm{Var}(T_{n3a}) &= n^{-2}\mathbf{m}(X_n)'\overline{G}_{12n}\Sigma_n\overline{G}_{12n}'\mathbf{m}(X_n) = n^{-2}\sum_{i=1}^n\sum_{j=1}^n m(x_{n,i})\,m(x_{n,j})\,\overleftarrow{\sigma}_{n,ij} \\ &\leq C^2 n^{-2}\sum_{i=1}^n\sum_{j=1}^n |\overleftarrow{\sigma}_{n,ij}| = O\left(n^{-1}\right),\end{aligned}$$

where $\overleftarrow{\sigma}_{n,ij}$ is the (i,j)th element of $\overleftarrow{\Sigma}_n \equiv \overline{G}_{12n}\Sigma_n\overline{G}_{12n}'$, and the row and column sums of $\overleftarrow{\Sigma}_n$ are uniformly bounded in absolute value. It follows from the Chebyshev inequality that $T_{n3a} = O_p(n^{-1/2})$. Analogously to the proof of (B.1), we can show that $T_{n3b} = n^{-1}E[U_n'\overline{G}_{12n}\varepsilon_n] + O_p\left(n^{-1/2}\right)$, where $n^{-1}E[U_n'\overline{G}_{12n}\varepsilon_n] = n^{-1}E[\varepsilon_n'\overline{G}_{212n}\varepsilon_n] = n^{-1}\sum_{i=1}^n \sigma_{n,i}^2\overline{g}_{212n,ii} = O(1)$, and $\overline{g}_{212n,ij}$ is the (i,j)th element of $\overline{G}_{212n} \equiv \left(I_n - \gamma_n^0 W_{2n}'\right)^{-1}\overline{G}_{12n}$. Hence

$$T_{n3} = n^{-1}\delta_{\rho n}E[U_n'\overline{G}_{12n}\varepsilon_n] + o_p(n^{-1/2}). \tag{B.3}$$

Second, let $r = 2 + \eta_1/2$ and $s > 1$ such that $1/r + 1/s = 1$, where η_1 is a small positive number defined in Assumption 2. By the assumption on $\mathcal{A}_n$, Assumption 1 and Facts 1-2, the row and column sums and each element of $\overline{\mathcal{A}}_n \equiv W_{1n}' \mathcal{A}_n W_{1n}$ are uniformly bounded in absolute value by $c_{\overline{\alpha}}$, say, each element $\overline{\alpha}_{n,ij}$ of $\overline{\mathcal{A}}_n$ is also uniformly bounded by $c_{\overline{\alpha}}$, and $\sum_{j=1}^n |\overline{\alpha}_{n,ij}|^s \leq c_{\overline{\alpha}}^{s-1} \sum_{j=1}^n |\overline{\alpha}_{n,ij}| \leq c_{\overline{\alpha}}^s$. By (3.1) and the definitions of $\overline{Y}_n$ and G_{1n}, $\overline{Y}_n = G_{1n}(\mathbf{m}(X_n) + U_n) = G_{1n}\mathbf{m}(X_n) + G_{12n}\varepsilon_n$, where $G_{12n} \equiv G_{1n} \left(I_n - \gamma_n^0 W_{2n}\right)^{-1}$.

$$
\begin{aligned}
T_{n4} &= n^{-1}\delta_{\rho n}^2 (G_{1n}\mathbf{m}(X_n) + G_{12n}\varepsilon_n)' \mathcal{A}_n (G_{1n}\mathbf{m}(X_n) + G_{12n}\varepsilon_n) \\
&= \delta_{\rho n}^2 \{ n^{-1}\mathbf{m}(X_n)' G_{1n}' \mathcal{A}_n G_{1n} \mathbf{m}(X_n) + n^{-1}\mathbf{m}(X_n)' G_{1n}' \mathcal{A}_n G_{12n}\varepsilon_n \\
&\quad + n^{-1}\varepsilon_n' G_{12n}' \mathcal{A}_n G_{1n}\mathbf{m}(X_n) + n^{-1}\varepsilon_n' G_{12n}' \mathcal{A}_n G_{12n}\varepsilon_n \} \\
&\equiv \delta_{\rho n}^2 (T_{n41} + T_{n42} + T_{n43} + T_{n44}).
\end{aligned}
$$

Note that the row and column sums of $G_{1n}'\mathcal{A}_n G_{1n}$, $G_{1n}'\mathcal{A}_n G_{12n}$, $G_{12n}'\mathcal{A}_n G_{1n}$ and $G_{12n}'\mathcal{A}_n G_{12n}$ are all uniformly bounded in absolute value by Assumptions 1, 6, the assumption on $\mathcal{A}_n$, and Fact 1. Let $c_{g\alpha}$ to denote the common bound. Clearly, $T_{n41} = O(1)$. Let $C_{n1} \equiv G_{1n}'\mathcal{A}_n G_{12n}$ with typical element $c_{n1,ij}$. Noting that $E[T_{n42}] = 0$ and $\text{Var}(T_{n42}) = n^{-2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n m(x_{n,i})\, c_{n1,ij} m(x_{n,k})\, c_{n1,kj} \sigma_{n,j}^2$ $\leq Cn^{-2} \sum_{i=1}^n \sum_{j=1}^n |c_{n1,ij}| \sum_{k=1}^n |c_{n1,kj}| = O\left(n^{-1}\right)$,it follows from the Chebyshev inequality that $T_{n42} = O_p(n^{-1/2})$. Similarly, $T_{n43} = O_p(n^{-1/2})$. Now, let $C_{n2} \equiv G_{12n}'\mathcal{A}_n G_{12n}$ with typical element $c_{n2,ij}$. It is easy to show that $E[T_{n44}] = n^{-1}\sum_{i=1}^n c_{n2,ii}\sigma_{n,i}^2 = O(1)$, and $\text{Var}(T_{n44}) = O(1)$ and hence $T_{n44} = O_p(1)$. Consequently,

$$
T_{n4} = O_p(n^{-1}) = o_p(n^{-1/2}). \tag{B.4}
$$

Third, write $\widetilde{m}(x) - m(x) = [\mathbf{s}_{\widetilde{h}\widetilde{\lambda}}(x)' (\mathbf{m}(X_n) + U_n) + \mathbf{s}_{\widetilde{h}\widetilde{\lambda}}(x)' W_{1n} Y_n (\rho_n^0 - \widetilde{\rho}_n)] - m(x) = d_{n1}(x) + d_{n2}(x) + d_{n3}(x)$, where

$$
\begin{aligned}
d_{n1}(x) &\equiv \frac{1}{2}\sum_{i=1}^n \mathbf{s}_{\widetilde{h}\widetilde{\lambda}}(x_{n,i}, x) \left(x_{n,i}^c - x^c\right)' \ddot{m}(x) \left(x_{n,i}^c - x^c\right) \mathbf{1}\left(x_{n,i}^d = x^d\right) \\
&\quad + \sum_{i=1}^n \mathbf{s}_{\widetilde{h}\widetilde{\lambda}}(x_{n,i}, x) \left[m(x_{n,i}) - m(x)\right] \mathbf{1}\left(x_{n,i}^d \neq x^d\right), \\
d_{n2}(x) &\equiv \mathbf{s}_{\widetilde{h}\widetilde{\lambda}}(x)' U_n, \text{ and} \\
d_{n3}(x) &\equiv -\delta_{\rho n} \widetilde{\mathbf{s}}_{\widetilde{h}\widetilde{\lambda}}(x)' \overline{Y}_n.
\end{aligned}
$$

By assumption and the proof of Theorem 3.2, $d_{n1}(x) = O(||\widetilde{h}||^2 + ||\widetilde{\lambda}||) = o(n^{-1/4})$ uniformly in x. Let $d_{nj,i} = d_{nj}(x_{n,i})$ and $D_{nj} = (d_{nj,1}, d_{nj,2}, \cdots, d_{nj,n})'$ for $j = 1, 2, 3$. Then

$$
D_{n2} = \mathbf{S}_{\widetilde{h}\widetilde{\lambda}} U_n, \; D_{n3} = -\delta_{\rho n} \mathbf{S}_{\widetilde{h}\widetilde{\lambda}} \overline{Y}_n, \text{ and } D_n = \sum_{j=1}^3 D_{nj}. \tag{B.5}
$$

Let $\overrightarrow{\mathcal{A}}_n = (\mathcal{A}_n + \mathcal{A}_n')\left(I_n - \gamma_n^0 W_{2n}\right)^{-1}$. Then by the assumption on $\mathcal{A}_n$, Assumption 6 and Fact 1, the row and column sums of $\overrightarrow{\mathcal{A}}_n$ are uniformly bounded in absolute value. We can rewrite

$$
T_{n5} = n^{-1} D_n (\mathcal{A}_n + \mathcal{A}_n') U_n = \sum_{k=1}^3 n^{-1} D_{nk}' \overrightarrow{\mathcal{A}}_n \varepsilon_n \equiv \sum_{k=1}^3 T_{n5k}.
$$

Noting that D_{n1} is nonrandom, we have $E\left[T_{n51}\right]=0$ and $\mathrm{Var}(T_{n51})=n^{-2}D_{n1}'\overrightarrow{\mathcal{A}}_n\Sigma_n\overrightarrow{\mathcal{A}}_n'D_{n1}'$ $=n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}d_{n1,i}d_{n1,j}\overrightarrow{\sigma}_{n,ij}\leq n^{-1}\max_{1\leq i\leq n}d_{n1,i}^2 n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\overrightarrow{\sigma}_{n,ij}\right|=o(n^{-3/2})$, where $\overrightarrow{\sigma}_{n,ij}$ is the (i,j)th element of $\overrightarrow{\Sigma}_n\equiv\overrightarrow{\mathcal{A}}_n\Sigma_n\overrightarrow{\mathcal{A}}_n'$. So $T_{n51}=o_p(n^{-3/4})$ by the Chebyshev inequality. Now, $T_{n52}=n^{-1}D_{n2}'\overrightarrow{\mathcal{A}}_n\varepsilon_n=n^{-1}\varepsilon_n'\overrightarrow{\mathbf{S}}_n\varepsilon_n=n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon_{n,i}\varepsilon_{n,j}\overrightarrow{\mathbf{s}}_{n,ij}$, where $\overrightarrow{\mathbf{s}}_{n,ij}$ is the (i,j)th element of $\overrightarrow{\mathbf{S}}_n\equiv\left(I_n-\gamma_n^0W_{2n}'\right)^{-1}\mathbf{S}_{\widetilde{h}\widetilde{\lambda}}'\overrightarrow{\mathcal{A}}_n$. By Assumption 6, Lemma A.2, the property of $\overrightarrow{\mathcal{A}}_n$ discussed above, and Fact 1, the rows and columns of $\overrightarrow{\mathbf{S}}_n$ are uniformly summable in absolute value for sufficiently large n. Noting that $\mathbf{s}_{\widetilde{h}\widetilde{\lambda},ij}=O(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1})$ uniformly in (i,j), then the elements $\overrightarrow{\mathbf{s}}_{n,ij}$ of $\overrightarrow{\mathbf{S}}_n$ are also uniformly $O(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1})$ by Fact 2. Hence $E\left[T_{n52}\right]=n^{-1}\sum_{i=1}^{n}\sigma_{n,i}^2\overrightarrow{\mathbf{s}}_{n,ii}=O(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1})=o(n^{-1/2})$, and

$$\begin{aligned}E\left[T_{n52}\right]^2&=n^{-2}E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\varepsilon_{n,i}\varepsilon_{n,j}\overrightarrow{\mathbf{s}}_{n,ij}\varepsilon_{n,k}\varepsilon_{n,l}\overrightarrow{\mathbf{s}}_{n,kl}\right]\\&=n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left[\varepsilon_{n,i}^2\varepsilon_{n,j}^2\right](\overrightarrow{\mathbf{s}}_{n,ij}^2+\overrightarrow{\mathbf{s}}_{n,ii}\overrightarrow{\mathbf{s}}_{n,jj}+\overrightarrow{\mathbf{s}}_{n,ji}^2)+n^{-2}\sum_{i=1}^{n}E\left[\varepsilon_{n,i}^4\right]\overrightarrow{\mathbf{s}}_{n,ii}^2\\&=O(n^{-2}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-2})+O(n^{-3}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-2})=o(n^{-1}).\end{aligned}$$

It follows from the Chebyshev inequality that $T_{n52}=o_p(n^{-1/2})$. Now, let $\widetilde{\mathbf{S}}_n=\mathbf{S}_{\widetilde{h}\widetilde{\lambda}}'\overrightarrow{\mathcal{A}}_n$. Then by Lemma A.2, the property of $\overrightarrow{\mathcal{A}}_n$ discussed above and Facts 1-2, the row and column sums of $\widetilde{\mathbf{S}}_n$ are also uniformly bounded in absolute value for sufficiently large n, and the elements $\widetilde{\mathbf{s}}_{n,ij}$ of $\widetilde{\mathbf{S}}_n$ are uniformly $O(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1})$. Write

$$\begin{aligned}T_{n53}&=-n^{-1}\delta_{\rho n}\overline{Y}_n'\mathbf{S}_{\widetilde{h}\widetilde{\lambda}}'\overrightarrow{\mathcal{A}}_n\varepsilon_n=-n^{-1}\delta_{\rho n}\overline{Y}_n'\widetilde{\mathbf{S}}_n\varepsilon_n\\&=-\delta_{\rho n}[n^{-1}\mathbf{m}\left(X_n\right)'G_{1n}'\widetilde{\mathbf{S}}_n\varepsilon_n-n^{-1}\varepsilon_n'\widetilde{G}_{1n}\widetilde{\mathbf{S}}_n\varepsilon_n]\equiv-\delta_{\rho n}\left(T_{n53,a}+T_{n53,b}\right),\end{aligned}$$

where $\widetilde{G}_{1n}\equiv\left(I_n-\gamma_n^0W_{2n}'\right)^{-1}G_{1n}'$. Analogously to the proof of T_{n3a}, one can show that $T_{n53,a}=o_p(1)$. Analogously to the proof of T_{n3b}, we can show that $T_{n53,b}=E\left[T_{n53,b}\right]+O_p(n^{-1/2})$, where

$$\begin{aligned}E\left[T_{n53,b}\right]&=n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}E\left[\varepsilon_{n,i}g_{1n,ij}\overrightarrow{\mathbf{s}}_{n,jk}\varepsilon_{n,k}\right]=n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{n,i}^2\widetilde{g}_{1n,ij}\widetilde{\mathbf{s}}_{n,ji}\\&\leq n^{-1}\max_{i,j}\left|\widetilde{\mathbf{s}}_{n,ji}\right|\overline{\sigma}^2\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\widetilde{g}_{1n,ij}\right|=O(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1})=o(1).\end{aligned}$$

Hence, $T_{n53}=o_p\left(n^{-1/2}\right)$ and

$$T_{n5}=o_p(n^{-1/2}). \tag{B.6}$$

Fourth, write $T_{n6}=n^{-1}\delta_{\rho n}D_n'\overline{G}_{1n}'U_n+n^{-1}\delta_{\rho n}D_n'\overline{G}_{1n}'\mathbf{m}\left(X_n\right)\equiv\delta_{\rho n}(T_{n61}+T_{n62})$. Analogously to the proof of T_{n5}, we can show that $T_{n61}=o_p\left(n^{-1/2}\right)$. Noting that $\delta_{\rho n}=O_p(n^{-1/2})$, it remains to show that $T_{n62}=o_p\left(1\right)$, we write $T_{n62}=\sum_{j=1}^{3}n^{-1}D_{nj}'\overline{G}_{1n}'\mathbf{m}\left(X_n\right)\equiv\sum_{j=1}^{3}T_{n62,j}$. It is straightforward to show that $T_{n62,1}=o(n^{-1/4})$. Write $T_{n62,2}=n^{-1}\varepsilon_n'H_n\mathbf{m}\left(X_n\right)$, where $H_n=\left(I_n-\gamma_n^0W_{2n}'\right)^{-1}\mathbf{S}_{\widetilde{h}\widetilde{\lambda}}'\overline{G}_{1n}'$. Noting the row and column sums of H_n are uniformly bounded for sufficiently large n, it is easy to show that $E[T_{n62,2}]=0$, $\mathrm{Var}(T_{n62,2})=O(n^{-1})$ and hence $T_{n62,2}=$

$O_p(n^{-1/2}) = o_p(1)$. Similarly, write $T_{n62,3} = -\delta_{\rho n} n^{-1} \overline{Y}'_n \mathbf{S}'_{\widetilde{h}\widetilde{\lambda}} \overline{G}'_{1n} \mathbf{m}(X_n) = -\delta_{\rho n}(T_{n62,3a} + T_{n62,3b})$, where $T_{n62,3a} \equiv n^{-1}\mathbf{m}(X_n)' G'_{1n}\overline{G}'_{1n}\mathbf{m}(X_n)$ and $T_{n62,3b} \equiv n^{-1}U'_n G_{1n}\overline{G}'_{1n}\mathbf{m}(X_n)$. It is easy to show that $T_{n62,3a} = O(1)$, $T_{n62,3b} = O_p(n^{-1/2})$, and hence $T_{n62,3} = O_p(n^{-1/2}) = o_p(1)$. Consequently,

$$T_{n6} = o_p(n^{-1/2}). \tag{B.7}$$

Fifth, write $T_{n7} = n^{-1}D'_n\mathcal{A}_n D_n = \sum_{k=1}^3\sum_{l=1}^3 T_{n7,kl}$, where $T_{n7,kl} = n^{-1}D'_{nk}\mathcal{A}_n D_{nl}$ for $k,l = 1,2,3$. Clearly, $T_{n7,11} \leq C\sup_i d^2_{n1,i} = o(n^{-1/2})$. Write $T_{n7,12} = n^{-1}D'_{n1}\mathcal{A}_n\mathbf{S}_{\widetilde{h}\widetilde{\lambda}}U_n$. Then $E[T_{n7,12}] = 0$, $\text{Var}(T_{n7,12}) = o(n^{-3/2})$, and hence $T_{n7,12} = o_p(n^{-3/4})$. Similarly, one can show that $T_{n7,13} = -\delta_{\rho n}n^{-1}D'_{n1}\mathcal{A}_n\mathbf{S}_{\widetilde{h}\widetilde{\lambda}}\overline{Y}_n = o_p(n^{-3/4})$, $T_{n7,22} = n^{-1}U'_n\mathbf{S}'_{\widetilde{h}\widetilde{\lambda}}\mathcal{A}_n\mathbf{S}_{\widetilde{h}\widetilde{\lambda}}U_n = O_p(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1}) = o_p(n^{-1/2})$, $T_{n7,23} = -\delta_{\rho n}n^{-1}U'_n\mathbf{S}'_{\widetilde{h}\widetilde{\lambda}}\mathcal{A}_n\mathbf{S}_{\widetilde{h}\widetilde{\lambda}}\overline{Y}_n = O_p(n^{-1})$, $T_{n7,33} = \delta^2_{\rho n}n^{-1}\overline{Y}'_n\mathbf{S}'_{\widetilde{h}\widetilde{\lambda}}\mathcal{A}_n\mathbf{S}_{\widetilde{h}\widetilde{\lambda}}\overline{Y}_n = O_p(n^{-1})$. By symmetric arguments, $T_{n7,21} = o_p(n^{-3/4})$, $T_{n7,31} = o_p(n^{-3/4})$, and $T_{n7,32} = O_p(n^{-1})$. Hence

$$T_{n7} = o_p(n^{-1/2}). \tag{B.8}$$

Combining (B.2)-(B.4) and (B.6)-(B.8) yields $\widetilde{\vartheta}_n - \vartheta_n = -n^{-1}\delta_{\rho n}E[U'_n\overline{G}_{12n}\varepsilon_n] + o_p(n^{-1/2})$. This concludes the proof. ■

C Proof of Lemmas D.1 and D.2

Lemma D.1 *Let* $\boldsymbol{\sigma}^2_n = (\sigma^2_{n,1},\cdots,\sigma^2_{n,n})'$, $\boldsymbol{\varepsilon}^2_n = (\varepsilon^2_{n,1},\cdots,\varepsilon^2_{n,n})'$, *and* $\widetilde{\boldsymbol{\varepsilon}}^2_n = (\widetilde{\varepsilon}^2_{n,1},\cdots,\widetilde{\varepsilon}^2_{n,n})'$. *Let* $\Lambda_n = n^{-1}\left(\boldsymbol{\sigma}^2_n\right)'\mathbf{A}_n\boldsymbol{\sigma}^2_n$, $\overline{\Lambda}_n = n^{-1}(\boldsymbol{\varepsilon}^2_n)'\mathbf{A}_n\boldsymbol{\varepsilon}^2_n$, *and* $\widetilde{\Lambda}_n = n^{-1}(\widetilde{\boldsymbol{\varepsilon}}^2_n)'\mathbf{A}_n\widetilde{\boldsymbol{\varepsilon}}^2_n$, *where* $\mathbf{A}_n$ *are* $n\times n$ *real symmetric and nonstochastic matrices. Suppose that the diagonal elements of* A_n *are zero and that the row and column sums are uniformly bounded in absolute value by* $c_{\mathbf{a}}$. *Then*

(a) $E[\overline{\Lambda}_n] = \Lambda_n = O(1)$, *Var*$(\overline{\Lambda}_n) = o(1)$, *and hence* $\overline{\Lambda}_n - \Lambda_n = o_p(1)$;

(b) $\widetilde{\Lambda}_n - \overline{\Lambda}_n = o_p(1)$.

Proof. The proof of part (a) is straightforward and is thus omitted (or see Lemma C.3 of Kelejian and Prucha (2010)). To prove (b), write

$$\begin{aligned}
\widetilde{\Lambda}_n - \overline{\Lambda}_n &= n^{-1}\sum_{i=1}^n\sum_{j=1}^n \mathbf{a}_{n,ij}\left[\widetilde{\varepsilon}^2_{n,i}\widetilde{\varepsilon}^2_{n,j} - \varepsilon^2_{n,i}\varepsilon^2_{n,j}\right] \\
&= n^{-1}\sum_{i=1}^n\sum_{j=1}^n \mathbf{a}_{n,ij}\left[\widetilde{\varepsilon}^2_{n,i} - \varepsilon^2_{n,i}\right]\varepsilon^2_{n,j} + n^{-1}\sum_{i=1}^n\sum_{j=1}^n \mathbf{a}_{n,ij}\varepsilon^2_{n,i}\left[\widetilde{\varepsilon}^2_{n,j} - \varepsilon^2_{n,j}\right] \\
&\quad + n^{-1}\sum_{i=1}^n\sum_{j=1}^n \mathbf{a}_{n,ij}\left[\widetilde{\varepsilon}^2_{n,i} - \varepsilon^2_{n,i}\right]\left[\widetilde{\varepsilon}^2_{n,j} - \varepsilon^2_{n,j}\right] \\
&\equiv \Delta_{1n} + \Delta_{2n} + \Delta_{3n}.
\end{aligned}$$

We will only prove $\Delta_{1n} = o_p(1)$, since the proof of $\Delta_{2n} = o_p(1)$ is identical and that of $\Delta_{3n} = o_p(1)$ is analogous.

By (B.5) and the definition of $\widetilde{\varepsilon}_n$, $\widetilde{\varepsilon}_n = (I_n - \widetilde{\gamma}_n W_{2n}) \left(U_n - D_n - \delta_{\rho_n} \overline{Y}_n \right) = -(I_n - \widetilde{\gamma}_n W_{2n}) D_{n1} + (I_n - \widetilde{\gamma}_n W_{2n})(I_n - \mathbf{S}_{\widetilde{h}\widetilde{\lambda}}) U_n - \delta_{\rho_n} (I_n - \widetilde{\gamma}_n W_{2n})(I_n - \mathbf{S}_{\widetilde{h}\widetilde{\lambda}}) \overline{Y}_n$. Hence,

$$\begin{aligned}
& \widetilde{\varepsilon}_n - \varepsilon_n \\
= \;& \widetilde{\varepsilon}_n - (I_n - \gamma_n^0 W_{2n}) U_n \\
= \;& -(I_n - \widetilde{\gamma}_n W_{2n}) D_{n1} - (C_{3n} + \delta_{\gamma n} C_{4n}) \varepsilon_n - \delta_{\rho_n} (I_n - \widetilde{\gamma}_n W_{2n})(I_n - \mathbf{S}_{\widetilde{h}\widetilde{\lambda}}) \overline{Y}_n \\
\equiv \;& -\eta_{1n} - \eta_{2n} - \eta_{3n}, \qquad \text{(C.1)}
\end{aligned}$$

where $\delta_{\gamma n} \equiv \widetilde{\gamma}_n - \gamma_n^0$,

$$C_{3n} \equiv (I_n - \gamma_n^0 W_{2n}) \mathbf{S}_{\widetilde{h}\widetilde{\lambda}} (I_n - \gamma_n^0 W_{2n})^{-1}, \text{ and } C_{4n} \equiv W_{2n} (I_n - \mathbf{S}_{\widetilde{h}\widetilde{\lambda}})(I_n - \gamma_n^0 W_{2n})^{-1}. \qquad \text{(C.2)}$$

By Assumption 6, the property of $\mathbf{S}_{\widetilde{h}\widetilde{\lambda}}$, and Facts 1-2, the elements $c_{3n,ij}$ of C_{3n} are uniformly $O(n^{-1} \Pi_{s=1}^{p_c} \widetilde{h}_s^{-1})$, and the row and column sums of C_{3n} and C_{4n} are uniformly bounded in absolute value (say by c_c) for sufficiently large n.

Let $\eta_{kn,i}$ denote the ith element of η_{kn} for $k = 1, 2, 3$. Then

$$\begin{aligned}
\Delta_{1n} \;=\; & n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{a}_{n,ij} \varepsilon_{n,j}^2 \{ \eta_{1n,i}^2 + \eta_{2n,i}^2 + \eta_{3n,i}^2 - 2\eta_{1n,i}\varepsilon_{n,i} - 2\eta_{2n,i}\varepsilon_{n,i} - 2\eta_{3n,i}\varepsilon_{n,i} \\
& + 2\eta_{1n,i}\eta_{2n,i} + 2\eta_{1n,i}\eta_{3n,i} + 2\eta_{2n,i}\eta_{3n,i} \} \equiv \sum_{k=1}^{9} \Delta_{1nk}, \qquad \text{(C.3)}
\end{aligned}$$

where, for example, $\Delta_{1n1} \equiv n^{-1} \sum_{i=1}^n \sum_{j=1}^n \mathbf{a}_{n,ij} \varepsilon_{n,j}^2 \eta_{1n,i}^2$. $\Delta_{1n} = o_p(1)$ provided $\Delta_{1nk} = o_p(1)$ for $k = 1, \cdots, 9$.

First, recall that $d_{n1,i} = d_{n1}(x_{n,i})$ denotes the ith element of D_{n1} (see the definitions preceding (B.5)). Noting that $\sup_i |d_{n1,i}| = o(n^{-1/4})$, it is easy to show that $\sup_i |\eta_{1n,i}| = o_p(n^{-1/4})$ by Assumption 6, Fact 2 and Theorem 4.1. Thus

$$|\Delta_{1n1}| \leq \sup_i \eta_{1n,i}^2 \max_{1 \leq j \leq n} \sum_{i=1}^{n} |\mathbf{a}_{n,ij}| \, n^{-1} \sum_{j=1}^{n} \varepsilon_{n,j}^2 = o_p(n^{-1/2}) O(1) O_p(1) = o_p(1). \qquad \text{(C.4)}$$

Second, by the triangle and Cauchy-Schwarz inequalities,

$$\begin{aligned}
|\Delta_{1n2}| \;\leq\; & 2n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} |\mathbf{a}_{n,ij}| \varepsilon_{n,j}^2 \left(\sum_{k=1}^{n} c_{3n,ik} \varepsilon_{n,k} \right)^2 + 2\delta_{\gamma n}^2 n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} |\mathbf{a}_{n,ij}| \varepsilon_{n,j}^2 \left(\sum_{k=1}^{n} c_{4n,ik} \varepsilon_{n,k} \right)^2 \\
\equiv\; & 2(\Delta_{1n2,a} + \delta_{\gamma n}^2 \Delta_{1n2,b}).
\end{aligned}$$

Observe that $E|\Delta_{1n2,a}| = n^{-1} \sum_{i=1}^n \sum_{j=1}^n |\mathbf{a}_{n,ij}| \sum_{k=1}^n c_{3n,ik}^2 E\left[\varepsilon_{n,j}^2 \varepsilon_{n,k}^2 \right] \leq \overline{\mu}_4 \sup_l n^{-1} \sum_{i=1}^n |c_{3n,il}| \times \sum_{j=1}^n |\mathbf{a}_{n,ij}| \sum_{k=1}^n |c_{3n,ik}| = O(n^{-1})$, where $\sup_i E\left[\varepsilon_{n,i}^4 \right] \leq \overline{\mu}_4$. It follows from the Markov inequality that $\Delta_{1n2,a} = O_p(n^{-1})$. Similarly, $\Delta_{1n2,b} = O_p(n^{-1})$. Hence

$$\Delta_{1n2} = O_p(n^{-1}) + O_p(n^{-2}) = o_p(1). \qquad \text{(C.5)}$$

Third, noting that $\overline{Y}_n = G_{1n}\mathbf{m}(X_n) + G_{12n}\varepsilon_n$ with $G_{12n} \equiv G_{1n}\left(I_n - \gamma_n^0 W_{2n}\right)^{-1}$, we rewrite $\eta_{3n} = \delta_{\rho_n} C_{5n}\mathbf{m}(X_n) + \delta_{\rho_n} C_{6n}\varepsilon_n - \delta_{\rho_n}\delta_{\gamma_n} C_{7n}\mathbf{m}(X_n) - \delta_{\rho_n}\delta_{\gamma_n} C_{8n}\varepsilon_n \equiv \sum_{l=1}^4 \eta_{3nl}$, where $C_{5n} \equiv (I_n - \gamma_n^0 W_{2n})(I_n - \mathbf{S}_{\widetilde{h}\widetilde{\lambda}})G_{1n}$, $C_{6n} \equiv (I_n - \gamma_n^0 W_{2n})(I_n - \mathbf{S}_{\widetilde{h}\widetilde{\lambda}})G_{12n}$, $C_{7n} \equiv W_{2n}(I_n - \mathbf{S}_{\widetilde{h}\widetilde{\lambda}})G_{1n}$, and $C_{8n} \equiv W_{2n}(I_n - \mathbf{S}_{\widetilde{h}\widetilde{\lambda}})G_{12n}$. Clearly, the row and column sums of C_{kn}, $k = 5, 6, 7, 8$, are uniformly bounded in absolute value (say also by c_c) for sufficiently large n. Observe that $|\Delta_{1n3}| \leq 4\sum_{l=1}^4 n^{-1}\sum_{i=1}^n \sum_{j=1}^n |\mathbf{a}_{n,ij}|\,\varepsilon_{n,j}^2 \eta_{3nl,i}^2 \equiv 4\sum_{l=1}^4 \Delta_{1n3l}$, where $\Delta_{1n3l} = n^{-1}\sum_{i=1}^n \sum_{j=1}^n |\mathbf{a}_{n,ij}|\,\varepsilon_{n,j}^2 \eta_{3nl,i}^2$ and $\eta_{3nl,i}$ is the ith element of η_{3nl}. Clearly,

$$\Delta_{1n31} = \delta_{\rho_n}^2 n^{-1}\sum_{i=1}^n \sum_{j=1}^n |\mathbf{a}_{n,ij}|\,\varepsilon_{n,j}^2 \left(\sum_{k=1}^n c_{5n,ik} m(x_{n,k})\right)^2 \leq c_c^2 c_m^2 c_{\mathbf{a}} \delta_{\rho_n}^2 n^{-1}\sum_{j=1}^n \varepsilon_{n,j}^2 = O_p(n^{-1}),$$

where $c_m \equiv \sup_{x\in\mathcal{X}} m(x)$, $\sup_i \sum_{k=1}^n |c_{5n,ik}| \leq c_c$. Next, let $\overline{\Delta}_{1n32} \equiv n^{-1}\sum_{i=1}^n \sum_{j=1}^n |\mathbf{a}_{n,ij}|\,\varepsilon_{n,j}^2 \left(\sum_{k=1}^n c_{6n,ik}\varepsilon_{n,k}\right)^2$. Noting that $E|\overline{\Delta}_{1n32}| = n^{-1}\sum_{i=1}^n \sum_{j=1}^n |\mathbf{a}_{n,ij}| \sum_{k=1}^n c_{6n,ik}^2 E\left[\varepsilon_{n,j}^2 \varepsilon_{n,k}^2\right] = O(1)$, it follows from the Markov inequality that $\Delta_{1n32} = \delta_{\rho_n}^2 \overline{\Delta}_{1n32} = O_p(n^{-1})$. Similarly, $\Delta_{1n3k} = O_p(n^{-2})$ for $k = 3, 4$. Hence

$$\Delta_{1n3} = O_p\left(n^{-1}\right) = o_p(1). \tag{C.6}$$

Fourth, by the assumption on $\mathbf{A}_n$, Assumption 2, and the triangle and Hölder inequalities,

$$\begin{aligned}
|\Delta_{1n4}| &\leq 2\max_i \left|\eta_{1n,i}\right| n^{-1}\sum_{j=1}^n \varepsilon_{n,j}^2 \sum_{i=1}^n |\mathbf{a}_{n,ij}|\,|\varepsilon_{n,i}| \\
&\leq 2n^{1/4}\max_i \left|\eta_{1n,i}\right| n^{-1}\sum_{j=1}^n \varepsilon_{n,j}^2 \left(\sum_{i=1}^n |\mathbf{a}_{n,ij}|^{4/3}\right)^{3/4} \left(n^{-1}\sum_{i=1}^n \varepsilon_{n,i}^4\right)^{1/4} \\
&= n^{1/4} o_p(n^{-1/4}) O_p(1) O(1)\, O_p(1) = o_p(1). \qquad \text{(C.7)}
\end{aligned}$$

Fifth, write $\Delta_{1n5} = -2n^{-1}\sum_{i=1}^n \sum_{j=1}^n \mathbf{a}_{n,ij}\varepsilon_{n,j}^2 \varepsilon_{n,i} \sum_{k=1}^n c_{3n,ik}\varepsilon_{n,k} - 2\delta_{\gamma n} n^{-1}\sum_{i=1}^n \sum_{j=1}^n \mathbf{a}_{n,ij}\varepsilon_{n,j}^2 \varepsilon_{n,i} \sum_{k=1}^n c_{4n,ik}\varepsilon_{n,k} \equiv -2(\Delta_{1n51} + \delta_{\gamma n}\Delta_{1n52})$. Noting that the diagonal elements $\mathbf{a}_{n,ii}$ of $\mathbf{A}_n$ are zero, we have

$$\begin{aligned}
\Delta_{1n51} &= n^{-1}\sum_{i=1}^n \sum_{j=1}^n \mathbf{a}_{n,ij}\varepsilon_{n,j}^2 \varepsilon_{n,i}^2 c_{3n,ii} + n^{-1}\sum_{i=1}^n \sum_{j\neq i}^n \mathbf{a}_{n,ij} c_{3n,ij}\varepsilon_{n,j}^3 \varepsilon_{n,i} \\
&\quad + n^{-1}\sum_{i=1}^n \sum_{j\neq i}^n \mathbf{a}_{n,ij} \sum_{k\neq j,i}^n c_{3n,ik}\varepsilon_{n,j}^2 \varepsilon_{n,k}\varepsilon_{n,i} \\
&\equiv \Delta_{1n51,a} + \Delta_{1n51,b} + \Delta_{1n51,c}, \text{ say.}
\end{aligned}$$

Since $E|\Delta_{1n51,a}| \leq \overline{\mu}_4 \sup_i |c_{3n,ii}|\, n^{-1}\sum_{i=1}^n \sum_{j=1}^n |\mathbf{a}_{n,ij}| = O(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1})$, $\Delta_{1n51,a} = O_p(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1})$ by the Markov inequality. By the triangle and Hölder inequalities, $E|\Delta_{1n51,b}| \leq n^{-1}\sum_{i=1}^n \sum_{j\neq i}^n |\mathbf{a}_{n,ij}||c_{3n,ij}| \left\{E\left[\varepsilon_{n,j}^4\right]\right\}^{3/4} \left\{E\left[\varepsilon_{n,i}^4\right]\right\}^{1/4} \leq n^{-1}\overline{\mu}_4 c_{\mathbf{a}} c_c = O(n^{-1})$, and hence $\Delta_{1n51,b} = O_p(n^{-1})$.

Note that $E[\Delta_{1n51,c}]=0$ and

$$
\begin{aligned}
\mathrm{Var}(\Delta_{1n51,c}) &= n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\mathbf{a}_{n,ij}\sum_{k\neq j,i}^{n}\sum_{i'=1}^{n}\sum_{j'\neq i'}^{n}\mathbf{a}_{n,i'j'}\sum_{k'\neq j',i'}^{n}c_{3n,ik}c_{3n,i'k'}E\left[\varepsilon_{n,j}^{2}\varepsilon_{n,k}\varepsilon_{n,i}\varepsilon_{n,j'}^{2}\varepsilon_{n,k'}\varepsilon_{n,i'}\right]\\
&= n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\mathbf{a}_{n,ij}\sum_{k\neq j,i}^{n}\sum_{j'\neq i}^{n}\mathbf{a}_{n,ij'}c_{3n,ik}c_{3n,ik}E\left[\varepsilon_{n,j}^{2}\varepsilon_{n,k}^{2}\varepsilon_{n,i}^{2}\varepsilon_{n,j'}^{2}\right]\\
&\quad +n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\mathbf{a}_{n,ij}\sum_{k\neq j,i}^{n}\sum_{j'\neq k}^{n}\mathbf{a}_{n,kj'}c_{3n,ik}c_{3n,ki}E\left[\varepsilon_{n,j}^{2}\varepsilon_{n,k}^{2}\varepsilon_{n,i}^{2}\varepsilon_{n,j'}^{2}\right]\\
&\equiv \Delta_{1n51,c1}+\Delta_{1n51,c2},
\end{aligned}
$$

where $|\Delta_{1n51,c1}|\leq\overline{\mu}_4^2 n^{-2}\sup_k\sum_{i=1}^n|c_{3n,ik}|\sum_{j=1}^n|\mathbf{a}_{n,ij}|\sum_{k=1}^n|c_{3n,ik}|\sum_{j'=1}^n|\mathbf{a}_{n,ij'}|=O(n^{-2})$, and similarly, $|\Delta_{1n51,c2}|=O(n^{-2})$. Then $\Delta_{1n51,c}=O_p\left(n^{-1}\right)$ by the Chebyshev inequality. Hence $\Delta_{1n51}=O_p(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1})$. Similarly, we can show that $\Delta_{1n52}=O_p\left(1\right)$. It follows that

$$\Delta_{1n5}=O_p(n^{-1}\Pi_{s=1}^{p_c}\widetilde{h}_s^{-1})+O_p(n^{-1/2})=o_p(1). \tag{C.8}$$

Sixth, write $\Delta_{1n6}=-2\sum_{l=1}^4 n^{-1}\sum_{i=1}^n\sum_{j=1}^n\mathbf{a}_{n,ij}\varepsilon_{n,j}^2\eta_{3nl,i}\varepsilon_{n,i}\equiv-2\sum_{l=1}^4\Delta_{1n6j}$. Analogously to the proof of Δ_{1n5}, we can show that $\Delta_{1n61}=O_p(n^{-1/2})$, $\Delta_{1n62}=O_p(n^{-1/2})$, $\Delta_{1n63}=O_p(n^{-1})$, and $\Delta_{1n64}=O_p(n^{-1})$. Hence

$$\Delta_{1n6}=O_p(n^{-1/2})=o_p(1). \tag{C.9}$$

Last, by the proof of Δ_{1nk} $(k=1,2,3)$ and the Cauchy-Schwarz inequality, we have

$$\Delta_{1nk}=o_p\left(1\right),\ \text{for } k=7,8,9. \tag{C.10}$$

Combining (C.3)-(C.10) yields $\Delta_{1n}=o_p\left(1\right)$. ■

Lemma D.2 *Let $\mathbf{c}_n$ and $\mathbf{d}_n$ be $n\times1$ vectors whose elements are uniformly bounded in absolute value by c. Let $\widetilde{F}_n$, F_n, $\widetilde{\boldsymbol{\alpha}}_{kn}$, and $\boldsymbol{\alpha}_{kn}$ be as defined in the proof of Theorem 5.1. Recall $\Sigma_n=diag(\boldsymbol{\sigma}_n^2)$ and $\widetilde{\Sigma}_n=diag(\widetilde{\boldsymbol{\varepsilon}}_n^2)$, where $\boldsymbol{\sigma}_n^2$ and $\widetilde{\boldsymbol{\varepsilon}}_n^2$ are as defined in Lemma D.1. Then*

(a) $n^{-1}\mathbf{c}_n'\Sigma_n\mathbf{d}_n=O\left(1\right)$, and $n^{-1}\mathbf{c}_n'(\widetilde{\Sigma}_n-\Sigma_n)\mathbf{d}_n=O_p(n^{-1/2})=o_p\left(1\right)$;

(b) $n^{-1}F_n'\Sigma_nF_n=O\left(1\right)$, and $n^{-1}\widetilde{F}_n'\widetilde{\Sigma}_n\widetilde{F}_n'-n^{-1}F_n'\Sigma_nF_n=o_p\left(1\right)$;

(c) $\widetilde{\boldsymbol{\alpha}}_{kn}-\boldsymbol{\alpha}_{kn}=o_p\left(1\right)$ for $k=1,2$.

Proof. (a) The first part is obvious. For the second part, define $\boldsymbol{\tau}_n=n^{-1}\mathbf{c}_n'\Sigma_n\mathbf{d}_n$, $\overline{\boldsymbol{\tau}}_n=n^{-1}\mathbf{c}_n'\overline{\Sigma}_n\mathbf{d}_n$, $\widetilde{\boldsymbol{\tau}}_n=n^{-1}\mathbf{c}_n'\widetilde{\Sigma}_n\mathbf{d}_n$, where $\overline{\Sigma}_n=\mathrm{diag}(\boldsymbol{\varepsilon}_n^2)$. By the triangle inequality,

$$|\widetilde{\boldsymbol{\tau}}_n-\boldsymbol{\tau}_n|\leq|\widetilde{\boldsymbol{\tau}}_n-\overline{\boldsymbol{\tau}}_n|+|\overline{\boldsymbol{\tau}}_n-\boldsymbol{\tau}_n|.$$

First, $E\left[\overline{\boldsymbol{\tau}}_n-\boldsymbol{\tau}_n\right]=n^{-1}\sum_{i=1}^n\mathbf{c}_{n,i}\mathbf{d}_{n,i}E\left(\varepsilon_{n,i}^2-\sigma_{n,i}^2\right)=0$ and $\mathrm{Var}(\overline{\boldsymbol{\tau}}_n-\boldsymbol{\tau}_n)=n^{-2}\sum_{i=1}^n\mathbf{c}_{n,i}^2\mathbf{d}_{n,i}^2\mathrm{Var}\left(\varepsilon_{n,i}^2\right)$ $=O(n^{-1})$. So $\overline{\boldsymbol{\tau}}_n-\boldsymbol{\tau}_n=O_p(n^{-1/2})$ by the Chebyshev inequality. Noting that $\widetilde{\varepsilon}_n=(I_n-\widetilde{\gamma}_nW_{2n})\widetilde{U}_n$, we can rewrite

$$
\begin{aligned}
\overline{\boldsymbol{\tau}}_n &= n^{-1}\varepsilon_n'\mathbf{E}_n\varepsilon_n=n^{-1}U_n'\mathbf{E}_nU_n-2\gamma_n^0n^{-1}U_n'W_{2n}'\mathbf{E}_nU_n+(\gamma_n^0)^2n^{-1}U_n'W_{2n}'\mathbf{E}_nW_{2n}U_n,\\
\widetilde{\boldsymbol{\tau}}_n &= n^{-1}\widetilde{\varepsilon}_n'\mathbf{E}_n\widetilde{\varepsilon}_n=n^{-1}\widetilde{U}_n'\mathbf{E}_n\widetilde{U}_n-2\widetilde{\gamma}_nn^{-1}\widetilde{U}_n'W_{2n}'\mathbf{E}_n\widetilde{U}_n+\widetilde{\gamma}_n^2n^{-1}\widetilde{U}_n'W_{2n}'\mathbf{E}_nW_{2n}\widetilde{U}_n,
\end{aligned}
$$

where $\mathbf{E}_n = \text{diag}(\mathbf{c}_n \odot \mathbf{d}_n)$. Then by the triangle inequality

$$\begin{aligned}
|\widetilde{\boldsymbol{\tau}}_n - \overline{\boldsymbol{\tau}}_n| &\leq |n^{-1}\widetilde{U}_n'\mathbf{E}_n\widetilde{U}_n - n^{-1}U_n'\mathbf{E}_nU_n| \\
&\quad +2|\widetilde{\gamma}_n n^{-1}\widetilde{U}_n'W_{2n}'\mathbf{E}_n\widetilde{U}_n - \gamma_n^0 n^{-1}U_n'W_{2n}'\mathbf{E}_nU_n| \\
&\quad +|\widetilde{\gamma}_n^2 n^{-1}\widetilde{U}_n'W_{2n}'\mathbf{E}_nW_{2n}\widetilde{U}_n - \left(\gamma_n^0\right)^2 n^{-1}U_n'W_{2n}'\mathbf{E}_nW_{2n}U_n| \\
&\equiv \tau_{1n} + 2\tau_{2n} + \tau_{3n}.
\end{aligned}$$

By Lemma C.1, $|n^{-1}\widetilde{U}_n'\mathbf{E}_n\widetilde{U}_n - n^{-1}U_n'\mathbf{E}_nU_n| = O_p(n^{-1/2})$. By Theorem 4.1, Lemma C.1 and the triangle inequality,

$$\begin{aligned}
\tau_{2n} &\leq |\widetilde{\gamma}_n|\,|(n^{-1}\widetilde{U}_n'W_{2n}'\mathbf{E}_n\widetilde{U}_n - n^{-1}U_n'W_{2n}'\mathbf{E}_nU_n)| + \left|\widetilde{\gamma}_n - \gamma_n^0\right| \left|n^{-1}U_n'W_{2n}'\mathbf{E}_nU_n\right| \\
&= O_p(1)\,O_p(n^{-1/2}) + O_p(n^{-1/2})O_p(1) = O_p(n^{-1/2}).
\end{aligned}$$

Similarly, we can show that $\tau_{3n} = O_p(n^{-1/2})$. Thus $|\widetilde{\boldsymbol{\tau}}_n - \overline{\boldsymbol{\tau}}_n| = O_p(n^{-1/2})$. This completes the proof of part (a).

(*b*) Let $\overline{Z}_n' \equiv (B'\Omega B)^{-1}B'\Omega Z_n$ and $C_n \equiv (I_n - \mathbf{S}_{h\gamma})\left(I_n - \gamma_n^0 W_{2n}\right)^{-1}\Sigma_n\left(I_n - \gamma_n^0 W_{2n}'\right)^{-1}(I_n - \mathbf{S}_{h\gamma}')$. By Assumption 5, the elements $\overline{z}_{n,i}$ of $\overline{Z}_n$ are uniformly bounded by $c_{\overline{z}}$, say. By Lemma A.2 and Assumptions 2 and 6, the row and column sums of C_n are uniformly bounded in absolute value for sufficiently large n. Hence $n^{-1}F_n'\Sigma_nF_n = n^{-1}\overline{Z}_n'C_n\overline{Z}_n = n^{-1}\sum_{i=1}^n\sum_{j=1}^n \overline{z}_{n,i}\overline{z}_{n,j}c_{n,ij} \leq c_{\overline{z}}^2 \max_i \sum_{j=1}^n |c_{n,ij}| = O(1)$. Now, write

$$\begin{aligned}
&n^{-1}\widetilde{F}_n'\widetilde{\Sigma}_n\widetilde{F}_n - n^{-1}F_n'\Sigma_nF_n \\
= \;& n^{-1}(\widetilde{F}_n - F_n)'(\widetilde{\Sigma}_n - \Sigma_n)F_n + n^{-1}F_n'(\widetilde{\Sigma}_n - \Sigma_n)(\widetilde{F}_n - F_n) + n^{-1}F_n'(\widetilde{\Sigma}_n - \Sigma_n)F_n \\
&+n^{-1}(\widetilde{F}_n - F_n)'(\widetilde{\Sigma}_n - \Sigma_n)(\widetilde{F}_n - F_n) + n^{-1}(\widetilde{F}_n - F_n)'\Sigma_nF_n + n^{-1}F_n'\Sigma_n(\widetilde{F}_n - F_n) \\
&+n^{-1}(\widetilde{F}_n - F_n)'\Sigma_n(\widetilde{F}_n - F_n) \equiv \sum_{j=1}^7 \xi_{jn}, \text{ say.}
\end{aligned}$$

Let $\widetilde{\pi}_n' = (B_n'\Omega_nB_n)^{-1}B_n'\Omega_n$ and $\overline{\pi}' = (B'\Omega B)^{-1}B'\Omega$. Then by Assumption 5, $\widetilde{\pi}_n' - \overline{\pi}' = o_p(1)$. Since for sufficiently large n (say $n \geq N_1$), the row and column sums of $\mathbf{S}_{h\gamma}$ are uniformly bounded in absolute value, this implies that the row and column sums of $(I_n - \mathbf{S}_{h\gamma})\left(I_n - \gamma_n^0 W_{2n}\right)^{-1}$ are uniformly bounded in absolute value for $n \geq N_1$ by Assumption 6 and Fact 1. Hence the elements of $\mathbf{e}_n' \equiv Z_n'(I_n - \mathbf{S}_{h\gamma})\left(I_n - \gamma_n^0 W_{2n}\right)^{-1}$ are uniformly bounded for $n \geq N_1$ by Assumption 5 and Fact 2. Noting that $\widetilde{F}_n' - F_n' = (\widetilde{\pi}_n - \overline{\pi})'Z_n'(I_n - \mathbf{S}_{h\gamma})\left(I_n - \gamma_n^0 W_{2n}\right)^{-1} + \widetilde{\pi}_n'Z_n'(I_n - \mathbf{S}_{h\gamma})\,[(I_n - \widetilde{\gamma}_nW_{2n})^{-1} - \left(I_n - \gamma_n^0 W_{2n}\right)^{-1}]$, we can write $\xi_{1n} = \xi_{1n,a} + \xi_{1n,b}$, where

$$\begin{aligned}
\xi_{1n,a} &\equiv n^{-1}(\widetilde{\pi}_n - \overline{\pi})'\mathbf{e}_n'(\widetilde{\Sigma}_n - \Sigma_n)F_n, \\
\xi_{1n,b} &\equiv n^{-1}\widetilde{\pi}_n'Z_n'(I_n - \mathbf{S}_{h\gamma})[(I_n - \widetilde{\gamma}_nW_{2n})^{-1} - \left(I_n - \gamma_n^0 W_{2n}\right)^{-1}](\widetilde{\Sigma}_n - \Sigma_n)F_n.
\end{aligned}$$

By (a), $n^{-1}\mathbf{e}_n'(\widetilde{\Sigma}_n - \Sigma_n)F_n = o_p(1)$. It follows that $\xi_{1n,a} = o_p(1)\,o_p(1) = o_p(1)$. We next show that $\xi_{1n,b} = o_p(1)$.

For later use in the proof of part (c), let

$$L_n = 2(I_n - \gamma_n^0 W_{2n}')A_{kn}\left(I_n - \gamma_n^0 W_{2n}'\right), \text{ and } \varsigma_{sn} = \rho_*^s n^{-1}U_n'(W_{1n}^{s+1})'L_nU_n, s = 1, 2, \cdots.$$

Note that the row and column sums of $\rho_*^s(W_{1n}^{s+1})'L_n$ and $\overline{L}_{sn} \equiv \rho_*^s(I_n - \gamma_n^0 W_{2n}')(W_{1n}^{s+1})'L_n\ (I_n - \gamma_n^0 W_{2n})$ are uniformly bounded in absolute value for each s, say by $c_{\overline{l}}$, where $c_{\overline{l}}$ does not depend on s. Lemma C.1(a) implies that $\varsigma_{sn} = \varsigma_{sn}^* + O_p(n^{-1/2})$, with $\varsigma_{sn}^* \equiv E\left[\rho_*^s n^{-1} U_n'(W_{1n}^{s+1})'L_n U_n\right]$. One can readily show that $\sup_s |\varsigma_{sn}^*| \leq c_{\overline{l}}\overline{\sigma}^2$.

In the following, we will frequently use the fact that if a sequence of random vectors b_n converges in probability to b if and only if every subsequence $\{b_{n_l} : l \in \mathbb{N}\}$ contains a further subsequence $\{b_{n_{l(j)}} : j \in \mathbb{N}\}$ which converges almost surely to b. See Theorem 18.6 in Davidson (1994) (c.f. Theorem 2.24 in White (2001)). For notational simplicity, hereafter we will write $\{n_l'\}$ as the subsequence of $\{n_l\}$.

Let $\{n_l\}$ denote some subsequence. Then by the above analysis and the consistency of $\widetilde{\gamma}_n$ and $\widetilde{\rho}_n$, there exists a subsequence $\{n_l'\}$ such that for $\omega \in S_1$ with $P(S_1) = 1$,

$$\begin{aligned} \left|\widetilde{\gamma}_{n_l'}(\omega) - \gamma_{n_l'}^0\right| &\to 0,\ \left|\widetilde{\rho}_{n_l'}(\omega) - \rho_{n_l'}^0\right| \to 0,\ \left\|\widetilde{\pi}_{n_l'} - \overline{\pi}\right\| \to 0, \text{ and} \\ |\varsigma_{sn} - \varsigma_{sn}^*| &\to 0,\ s = 1, 2, \cdots, \text{as } n_l' \to \infty. \end{aligned}$$

Hence there exists $N_{2\omega}$ such tat for all $n_l' \geq N_{2\omega}$,

$$\begin{aligned} \left|\widetilde{\gamma}_{n_l'}(\omega)\right| &\leq \gamma_{**} < \gamma_*,\ \left|\widetilde{\rho}_{n_l'}(\omega)\right| \leq \rho_{**} < \rho_*, \\ \left\|\widetilde{\pi}_{n_l'}(\omega)\right\| &\leq 2\left\|\overline{\pi}\right\|, \text{ and } \left|\varsigma_{sn_l'}(\omega)\right| \leq 2c_{\overline{l}}\overline{\sigma}^2, \end{aligned}$$

where $\gamma_{**} = (\sup_n \left|\gamma_n^0\right| + \gamma_*)/2$, and $\rho_{**} = (\sup_n \left|\rho_n^0\right| + \rho_*)/2$. In the following, we assume that $\omega \in S_1$ and $n_l' \geq \max(N_1, N_{2\omega})$. Noting that $|\widetilde{\gamma}_{n_l'}(\omega)| < \gamma_*$, the row sums of $\widetilde{\gamma}_{n_l'}(\omega) W_{2n_l'}$ are less than 1 in absolute value and it follows from Corollary 5.6.16 of Horn and Johnson (1985) that $I_{n_l'} - \widetilde{\gamma}_{n_l'}(\omega) W_{2n_l'}$ is invertible, that the row and column sums of $(I_{n_l'} - \widetilde{\gamma}_{n_l'}(\omega) W_{2n_l'})^{-1}$ are uniformly bounded in absolute value, and that

$$(I_{n_l'} - \widetilde{\gamma}_{n_l'}(\omega) W_{2n_l'})^{-1} - (I_{n_l'} - \gamma_{n_l'}^0 W_{2n_l'})^{-1} = \sum_{s=1}^{\infty}[\widetilde{\gamma}_{n_l'}^s(\omega) - (\gamma_{n_l'}^0)^s] W_{2n_l'}^s. \tag{C.11}$$

Hence $\xi_{1n_l',b}(\omega) = \sum_{s=1}^{\infty}\{[\widetilde{\gamma}_{n_l'}^s(\omega) - (\gamma_{n_l'}^0)^s]/\gamma_*^s\}\varkappa_{n_l',s}(\omega)$, where $\varkappa_{n_l',s}(\omega) = n_l'^{-1}\gamma_*^s\widetilde{\pi}_{n_l'}'(\omega) Z_{n_l'}'\ (I_{n_l'} - \mathbf{S}_{h\gamma})W_{2n_l'}^s(\widetilde{\Sigma}_{n_l'}(\omega) - \Sigma_{n_l'})F_{n_l'}$. Observing that $|\widetilde{\gamma}_{n_l'}^s(\omega) - (\gamma_{n_l'}^0)^s|/\gamma_*^s \leq 2(\gamma_{**}/\gamma_*)^s$ and $|\varkappa_{n_l',s}(\omega)| \leq C$ for some constant by part (a), we have $\left|\xi_{1n_l',b}(\omega)\right| \leq 2C\sum_{s=1}^{\infty}(\gamma_{**}/\gamma_*)^s < \infty$. Consequently, $\xi_{1n_l',b}(\omega) \to 0$ as $n_l' \to \infty$ by the dominated convergence theorem, implying $\xi_{1n,b} = o_p(1)$. Hence $\xi_{1n} = o_p(1)$. Analogous arguments yield $\xi_{jn} = o_p(1)$ for $j = 2, 3, \cdots, 7$. Hence $n^{-1}\widetilde{F}_n'\widetilde{\Sigma}_n\widetilde{F}_n - n^{-1}F_n'\Sigma_n F_n = o_p(1)$.

(*c*) Let $\widetilde{\boldsymbol{\zeta}}_n = 2(I_n - \widetilde{\rho}_n W_{1n}')^{-1}W_{1n}'(I_n - \widetilde{\gamma}_n W_{2n}')A_{kn}(I_n - \widetilde{\gamma}_n W_{2n})$ and $\boldsymbol{\zeta}_n = 2(I_n - \rho_n^0 W_{1n}')^{-1} W_{1n}'(I_n - \gamma_n^0 W_{2n}')A_{kn}\left(I_n - \gamma_n^0 W_{2n}'\right)$. Then

$$\begin{aligned} -(\widetilde{\boldsymbol{\alpha}}_{kn} - \boldsymbol{\alpha}_{kn}) &= n^{-1}\widetilde{U}_n'\widetilde{\boldsymbol{\zeta}}_n\widetilde{U}_n - n^{-1}U_n'\boldsymbol{\zeta}_n U_n \\ &= n^{-1}(\widetilde{U}_n'\boldsymbol{\zeta}_n\widetilde{U}_n - U_n'\boldsymbol{\zeta}_n U_n) + n^{-1}U_n'(\widetilde{\boldsymbol{\zeta}}_n - \boldsymbol{\zeta}_n)U_n \\ &\quad + n^{-1}U_n'(\widetilde{\boldsymbol{\zeta}}_n - \boldsymbol{\zeta}_n)(\widetilde{U}_n - U_n) + n^{-1}(\widetilde{U}_n - U_n)'(\widetilde{\boldsymbol{\zeta}}_n - \boldsymbol{\zeta}_n)U_n \\ &\quad + n^{-1}(\widetilde{U}_n - U_n)'(\widetilde{\boldsymbol{\zeta}}_n - \boldsymbol{\zeta}_n)(\widetilde{U}_n - U_n) \\ &\equiv \chi_{1n} + \chi_{2n} + \chi_{3n} + \chi_{4n} + \chi_{5n}. \end{aligned}$$

By Lemma C.1, $\chi_{1n} = n^{-1}\delta_{\rho n} E[U'\overline{G}_{12n}\varepsilon_n] + o_p(n^{-1/2})$, where

$$\overline{G}_{12n} = G'_{1n}(\boldsymbol{\zeta}_n + \boldsymbol{\zeta}'_n)\left(I_n - \gamma_n^0 W'_{2n}\right)^{-1}.$$

By Assumptions 2 and 6 and Fact 1, the row and column sums of $\boldsymbol{\zeta}_n$ and $\overline{G}_{12n}$ are uniformly bounded in absolute value. With this, one can readily show that $\chi_{1n} = O_p(n^{-1/2}) = o_p(1)$. It remains to show that $\chi_{jn} = o_p(1)$ for $j = 2, 3, 4, 5$.

By the above subsequence arguments and Corollary 5.6.16 of Horn and Johnson (1985), $I_{n'_l} - \widetilde{\rho}_{n'_l}(\omega) W_{1n'_l}$ is invertible, the row and column sums of $(I_{n'_l} - \widetilde{\rho}_{n'_l}(\omega) W_{1n'_l})^{-1}$ are uniformly bounded in absolute value, and

$$(I_{n'_l} - \widetilde{\rho}_{n'_l}(\omega) W_{1n'_l})^{-1} - (I_{n'_l} - \rho^0_{n'_l} W_{1n'_l})^{-1} = \sum_{s=1}^{\infty} [\widetilde{\rho}^s_{n'_l}(\omega) - (\rho^0_{n'_l})^s] W^s_{1n'_l}. \tag{C.12}$$

Let $\widetilde{L}_n = 2(I_n - \widetilde{\gamma}_n W'_{2n}) A_{kn} (I_n - \widetilde{\gamma}_n W'_{2n})$. Then w.p.a.1, we can write

$$\begin{aligned}
\chi_{2n} &= n^{-1} U'_n (I_n - \rho^0_n W'_{1n})^{-1} W'_{1n} (\widetilde{L}_n - L_n) U_n \\
&\quad + n^{-1} U'_n [(I_n - \widetilde{\rho}_n W'_{1n})^{-1} - (I_n - \rho^0_n W'_{1n})^{-1}] W'_{1n} \widetilde{L}_n U_n \\
&\equiv \chi_{2n,a} + \chi_{2n,b}, \text{ say.}
\end{aligned}$$

Note that $\widetilde{L}_n - L_n = 2\delta^2_{\rho n} W'_{2n} A_{kn} W'_{2n} + 2\delta_{\rho n} W'_{2n} A_{kn} (I_n - \gamma^0_n W'_{2n}) + 2\delta_{\rho n} (I_n - \gamma^0_n W'_{2n}) A_{kn} W'_{2n}$, where recall $\delta_{\rho n} = \widetilde{\rho}_n - \rho^0_n = O_p(n^{-1/2})$. By Assumptions 2 and 5, Fact 1 and Lemma C.1(a), it is straightforward to show that $\chi_{2n,a} = O_p(n^{-1/2})$. In addition, we see that the elements of $\widetilde{L}_{n'_l}(\omega) - L_{n'_l}$ are uniformly $O(n^{-1/2})$ and the row and column sums of $L_{n'_l}$ and $\widetilde{L}_{n'_l}(\omega)$ are uniformly bounded in absolute value. Now on a set of probability one, we have

$$\chi_{2n'_l,b} = n^{-1} \sum_{s=1}^{\infty} [\widetilde{\rho}^s_{n'_l} - (\rho^0_{n'_l})^s] U'_{n'_l} (W^{s+1}_{1n'_l})' L_{n'_l} U_{n'_l} + n^{-1} \sum_{s=1}^{\infty} [\widetilde{\rho}^s_{n'_l} - (\rho^0_{n'_l})^s] U'_{n'_l} (W^{s+1}_{1n'_l})' (\widetilde{L}_{n'_l} - L_{n'_l}) U_{n'_l}.$$

Observing the second term in the above expression is of smaller order, it suffices to show that the first term is $o(1)$ for ω on a set of probability 1. Recall $\varsigma_{sn} = \rho^s_* n^{-1} U'_n (W^{s+1}_{1n})' L_n U_n$. Define $\eta_{n'_l}(\omega) = n^{-1} \sum_{s=1}^{\infty} \{[\widetilde{\rho}^s_{n'_l}(\omega) - (\rho^0_{n'_l})^s]/\rho^s_*\} \varsigma_{sn'_l}(\omega)$. Noting that $|\widetilde{\rho}^s_{n'_l}(\omega) - (\rho^0_{n'_l})^s|/\rho^s_* \leq 2(\rho_{**}/\rho_*)^s$ and $|\varsigma_{sn'_l}(\omega)| \leq 2c_{\overline{l}}\overline{\sigma}^2$, we have that for $n'_l \geq \max\{N_1, N_{2\omega}\}$ and $\omega \in S_1$,

$$|\eta_{n'_l}(\omega)| \leq 2n^{-1} \sum_{s=1}^{\infty} (\rho_{**}/\rho_*)^s \left|\varsigma_{sn'_l}(\omega)\right| \leq 4c_{\overline{l}}\overline{\sigma}^2 n^{-1} \sum_{s=1}^{\infty} (\rho_{**}/\rho_*)^s < \infty.$$

It follows from the dominated convergence theorem that $\eta_{n'_l}(\omega) \to 0$ as $n'_l \to \infty$. Hence $\chi_{2n,b} = o_p(1)$ and $\chi_{2n} = o_p(1)$. Analogously, we can show that $\chi_{jn} = o_p(1)$ for $j = 3, 4, 5$. Hence $\widetilde{\boldsymbol{\alpha}}_{kn} - \boldsymbol{\alpha}_{kn} = o_p(1)$. ■

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