Supplement to
“A Nonparametric Goodness-of-fit-based Test for Conditional Heteroskedasticity”

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THIS APPENDIX PROVIDES PROOFS FOR TECHNICAL LEMMAS IN THE ABOVE PAPER.

A Proof of Lemma A.1 in the paper

In this supplement, we prove Lemma A.1 that is used in the proof of the results in Section 3. For notational simplicity, we suppress the dependence ε_{ni}, X_{ni}, and their associated distributions on the sample size n. For example, we will write ε_i for ε_{ni}. **All results proved below hold for the triangular array processes**.

Lemma A.1 *Suppose Assumptions A1-A2 and A5-A6 hold. Let* $S_{qn}(x) \equiv n^{-1}D_h^{-1}\mathbf{X}_{q,x}'\mathbf{W}_x\mathbf{X}_{q,x}D_h^{-1}$, $\bar{S}_{qn}(x) \equiv E[S_{qn}(x)]$, $\gamma_n \equiv n^{-1/2}h^{-p/4}$, *and* $\varsigma_{ij} \equiv \Delta_n(X_{ni})\varepsilon_j + \Delta_n(X_{nj})\varepsilon_i$. *Then*
(i) $R_{n1} \equiv 2n^{-1}h^{p/2}\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}\left[S_{qn}^{-1}(x) - \bar{S}_{qn}^{-1}(x)\right]D_h^{-1}X_{q,jx}K_{jx}dx = o_p(1)$,
(ii) $R_{n2} \equiv 2n^{-1}h^{p/2}\gamma_n\sum_{1\le i<j\le n}\varsigma_{ij}\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}\bar{S}_{qn}^{-1}(x)D_h^{-1}X_{q,jx}K_{jx}dx = o_p(1)$,
(iii) $R_{n3} \equiv 2n^{-1}h^{p/2}\gamma_n\sum_{1\le i<j\le n}\varsigma_{ij}\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}\left[S_{qn}^{-1}(x) - \bar{S}_{qn}^{-1}(x)\right]D_h^{-1}X_{q,jx}K_{jx}dx = o_p(1)$.
(iv) $R_{n4} \equiv 2n^{-2}h^{p}\sum_{1\le i\ne j\le n}\varepsilon_i^2\varepsilon_j^2\{\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}S_{qn}^{-1}(x)D_h^{-1}X_{q,jx}K_{jx}dx\}^2 = \Omega_q + o_p(1)$.

To prove the above lemma, we need some technical lemmas. The following lemma is due to Sun and Chiang (1997, Lemma 2.1).

Lemma A.2 *Let* $\{\xi_i, i \ge 1\}$ *be an* l*-dimensional strong mixing process with mixing coefficient* $\alpha(\cdot)$. *Let* $F_{i_1,\ldots,i_m}$, *denote the distribution function of* $\left(\xi_{i_1},\ldots,\xi_{i_m}\right)$. *For any integer* $m > 1$ *and integers* $(i_1,\ldots,i_m)$ *such that* $1 \le i_1 < i_2 < \cdots < i_m \le n$, *let* ϑ *be a Borel measurable function such that* $max\{\int|\vartheta(v_1,\cdots,v_m)|^{1+\eta}dF_{i_1,\cdots,i_j}(v_1,\cdots,v_j)dF_{i_j+1,\cdots,i_m}(v_{j+1},\cdots,v_m), \int|\vartheta(v_1,\cdots,v_m)|^{1+\eta}dF_{i_1,\cdots,i_m}(v_1,\cdots,v_m)\} \le M_{1n}$ *for some* $\eta > 0$. *Then* $|\int\vartheta(v_1,\cdots,v_m)dF_{i_1,\cdots,i_m}(v_1,\cdots,v_m) - \int\vartheta(v_1,\cdots,v_m)dF_{i_1,\cdots,i_j}(v_1,\cdots,v_j)\,dF_{i_j+1,\cdots,i_m}(v_{j+1},\cdots,v_m)| \le 4M_{1n}^{1/(1+\eta)}\alpha(i_{j+1} - i_j)^{\eta/(1+\eta)}$.

To state the next lemma, we need some additional notation. Let

$$\mathcal{U}_n = \begin{pmatrix} n \\ m \end{pmatrix}^{-1} \sum_{1\le i_1<i_2<\ldots<i_m\le n} \vartheta\left(\xi_{i_1},\ldots,\xi_{i_m}\right)$$

be an mth order U-statistic where ϑ is symmetric in its arguments. Let $\vartheta_0 = \int \cdots \int \vartheta(\xi_1, ..., \xi_m) \Pi_{i=1}^{m} dF(\xi_i)$, and $\vartheta_c(\xi_1, ..., \xi_c) = \int \cdots \int \vartheta(\xi_1, ..., \xi_m) \Pi_{i=c+1}^{m} dF(\xi_i)$ for $c = 1, ..., m$. Let $h^{(1)}(\xi_1) = \vartheta_1(\xi_i) - \vartheta_0$, and

$$h^{(c)}(\xi_1, ..., \xi_c) = \vartheta_c(\xi_1, ..., \xi_c) - \sum_{j=1}^{c-1} \sum_{(c,j)} h^{(j)}\left(\xi_{i_1}, ..., \xi_{i_j}\right) - \vartheta_0 \text{ for } c = 2, ..., m,$$

where the sum $\sum_{(c,j)}$ is taken over all subsets $1 \leq i_1 < i_2 < \cdots < i_j \leq c$ of $\{1, 2, ..., c\}$. Let $\mathcal{H}_n^{(c)} = \left(\begin{array}{c} n \\ c \end{array}\right)^{-1} \sum_{1 \leq i_1 < .. < i_c \leq n} h^{(c)}\left(\xi_{i_1}, ..., \xi_{i_c}\right)$. Then by Theorem 1 in Lee (1990), we have the following Hoeffding decomposition

$$\mathcal{U}_n = \vartheta_0 + \sum_{c=1}^{m} \left(\begin{array}{c} m \\ c \end{array}\right) \mathcal{H}_n^{(c)}. \tag{A.1}$$

Let C signify a generic constant whose exact value may vary from case to case.

Lemma A.3 *Let $\{\xi_i\}$ and ϑ be as in Lemma A.2. Suppose that $\{\xi_i\}$ is strictly stationary with distribution function $F(\cdot)$ and ϑ is symmetric in its arguments.*

(i) If there is $\eta > 0$ such that

$$M_{2n} \equiv \max\left\{\int \left|\vartheta(v_1, \cdots, v_m)\right|^{4+\eta} \Pi_{i=1}^{m} dF(v_i), \max_{1 \leq i_1 < \ldots < i_m \leq n} E\left|\vartheta\left(\xi_{i_1}, ..., \xi_{i_m}\right)\right|^{4+\eta}\right\} < C < \infty,$$

and for some $\tilde{\eta} \in (0, \eta)$, $\alpha(s)^{(2+\tilde{\eta})/[3(4+\tilde{\eta})]} = O\left(s^{-1}\right)$, then we have

$$E\left[\mathcal{H}_n^{(c)}\right]^2 = O\left(n^{-3}\right) \text{ for } 3 \leq c \leq m.$$

(ii) If $\alpha(s)^{\eta/(2+\eta)} = O\left(s^{-2+\epsilon}\right)$ for some $\epsilon > 0$ and $\eta > 0$. Then we have

$$\mathcal{R}_n \equiv \mathcal{U}_n - \vartheta_0 - m\mathcal{H}_n^{(1)} = \mathcal{U}_n - \vartheta_0 - \frac{m}{n} \sum_{i=1}^{n} \left[\vartheta_1(\xi_i) - \vartheta_0\right] = O_p\left(n^{-1+\epsilon/2} s_\eta\right)$$

where $s_\eta = \sup_{1 \leq i_1 < i_2 < \ldots < i_m \leq n} \left[E\left|\vartheta\left(\xi_{i_1}, ..., \xi_{i_m}\right)\right|^{2+\eta}\right]^{1/(2+\eta)}$.

Proof. Yoshihara (1976, Lemma 3) proves (i) for β-mixing processes and Denker and Keller (1983, Proposition 2) prove (ii) for β-mixing processes. Their results extend to the α-mixing processes by applying Lemma A.2 in place of Lemma 1 in Yoshihara (1976). ■

Proof of Lemma A.1

(i) Let $\mathcal{D}_n(x) \equiv \left[S_{qn}^{-1}(x) - \bar{S}_{qn}^{-1}(x)\right]\left[\bar{S}_{qn}(x) - S_{qn}(x)\right]\bar{S}_{qn}^{-1}(x)$ and $\nu_n = n^{-1/2} h^{-p/2} \sqrt{\log n}$. If the kernel function $K(\cdot)$ is compactly supported, we can verify that under Assumptions A1, A2(iv), and A4-6, the conditions in Corollary 2(ii) of Masry (1996) are all satisfied and conclude that $\sup_{x \in \mathcal{X}_n} ||S_{qn}(x) - \bar{S}_{qn}(x)|| = O_p(\nu_n)$. In the case where $K(\cdot)$ is not compactly supported, we can apply Theorem 2 of Hansen (2008) to obtain $\sup_{x \in \mathcal{X}_n} ||S_{qn}(x) - \bar{S}_{qn}(x)|| = O_p(\nu_n)$ under Assumptions A2(iv) and A5. It follows that $\|\mathcal{D}_n(x)\| = O_p(\nu_n^2)$. Now write

$$\begin{aligned}
R_{n1} &= \frac{2h^{p/2}}{n} \sum_{1\le i<j\le n} \varepsilon_i\varepsilon_j \int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}S_{qn}^{-1}(x)\left[\bar{S}_{qn}(x) - S_{qn}(x)\right]\bar{S}_{qn}^{-1}(x)\, D_h^{-1}X_{q,jx}K_{jx}dx \\
&= \frac{2h^{p/2}}{n} \sum_{1\le i<j\le n} \varepsilon_i\varepsilon_j \int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\left[\bar{S}_{qn}(x) - S_{qn}(x)\right]\bar{S}_{qn}^{-1}(x)\, D_h^{-1}X_{q,jx}K_{jx}dx \\
&\quad + \frac{2h^{p/2}}{n} \sum_{1\le i<j\le n} \varepsilon_i\varepsilon_j \int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\mathcal{D}_n(x)\, D_h^{-1}X_{q,jx}K_{jx}dx \\
&\equiv R_{n11} + R_{n12}.
\end{aligned}$$

We first decompose R_{n11} as follows

$$\begin{aligned}
R_{n11} &= \frac{2h^{p/2}}{n^2} \sum_{1\le i<j\le n}\sum_{k=1}^{n} \varepsilon_i\varepsilon_j \int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\, s_k(x)\, \bar{S}_{qn}^{-1}(x)\, D_h^{-1}X_{q,jx}K_{jx}dx \\
&= \frac{2h^{p/2}}{n^2} \sum_{1\le i<j\le n}\sum_{k\ne i,j}^{n} \varepsilon_i\varepsilon_j \int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\, s_k(x)\, \bar{S}_{qn}^{-1}(x)\, D_h^{-1}X_{q,jx}K_{jx}dx \\
&\quad + \frac{2h^{p/2}}{n^2} \sum_{1\le i<j\le n} \varepsilon_i\varepsilon_j \int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\left[s_i(x) + s_j(x)\right]\bar{S}_{qn}^{-1}(x)\, D_h^{-1}X_{q,jx}K_{jx}dx \\
&\equiv R_{n11a} + R_{n11b}, \text{ say,} \qquad \text{(A.2)}
\end{aligned}$$

where $s_k(x) \equiv -K_{kx}D_h^{-1}X_{q,kx}X'_{q,kx}D_h^{-1} + E(K_{kx}D_h^{-1}X_{q,kx}X'_{q,kx}D_h^{-1})$. Let $\vartheta^0(\xi_i,\xi_j,\xi_k) = h^{(3+\eta)p/(4+\eta)}$ $\varepsilon_i\varepsilon_j \int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\, s_k(x)\, \bar{S}_{qn}^{-1}(x)\, D_h^{-1}X_{q,jx}K_{jx}dx$. Define the symmetric version of ϑ^0 as $\vartheta(\xi_i,\xi_j,\xi_k) = [\vartheta^0(\xi_i,\xi_j,\xi_k) + \vartheta^0(\xi_i,\xi_k,\xi_j) + \vartheta^0(\xi_j,\xi_k,\xi_l)]/3$. Then we have

$$R_{n11a} = \frac{6h^{[1/2-(3+\eta)/(4+\eta)]p}}{n^2} \sum_{1\le i<j<k\le n} \vartheta(\xi_i,\xi_j,\xi_k) = \frac{(n-1)(n-2)\,h^{-(2-\eta)p/[2(4+\eta)]}}{n}\mathcal{U}_{n1}$$

where $\mathcal{U}_{n1} \equiv \frac{6}{n(n-1)(n-2)}\sum_{1\le i<j<k\le n}\vartheta(\xi_i,\xi_j,\xi_k)$. For this ϑ, we have $\vartheta_0 = 0$, $\vartheta_1(\xi_1) = 0$, and $\vartheta_2(\xi_1,\xi_2) = 0$, using the notation defined before Lemma A.3. This implies that $\mathcal{U}_{n1}$ is a third order degenerate U-statistic with $\mathcal{H}_n^{(c)} = 0$ for $c = 1, 2$ and $\mathcal{H}_n^{(3)} = \mathcal{U}_{n1}$ by using the notation defined before Lemma A.3 again. In view of

$$M_{2n} \equiv \max\left\{\int |\vartheta(v_1,v_2,v_3)|^{4+\eta}\ \Pi_{i=1}^3 dF(v_i),\ \max_{1\le i_1<i_2<i_3\le n} E\left|\vartheta(\xi_{i_1},\xi_{i_2},\xi_{i_3})\right|^{4+\eta}\right\} \le C < \infty$$

by construction, we apply Lemma A.3(i) to obtain $\mathcal{U}_{n1} = O_p\left(n^{-3/2}\right)$. It follows that

$$R_{n11a} = nh^{-(2-\eta)p/[2(4+\eta)]}O_p\left(n^{-3/2}\right) = O_p\left(n^{-1/2}h^{-(2-\eta)p/[2(4+\eta)]}\right). \qquad \text{(A.3)}$$

Analogously to the determination of the probability order of $\mathcal{U}_{n2}$ below by using Lemma A.3(ii), we can readily show that

$$R_{n11b} = h^{p/2}O_p\left(n^{-1+\epsilon/2}h^{-(1+\eta)p/(2+\eta)}\right) = O_p\left(n^{-1+\epsilon/2}h^{[1/2-(1+\eta)/(2+\eta)]p}\right). \qquad \text{(A.4)}$$

Consequently,

$$R_{n11} = O_p\left(n^{-1/2}h^{-(2-\eta)p/[2(4+\eta)]} + n^{-1+\epsilon/2}h^{[1/2-(1+\eta)/(2+\eta)]p}\right) = o_p(1). \qquad \text{(A.5)}$$

Now we study R_{n12}. Noting that $\text{tr}(AB) = \text{vec}(A')'\text{vec}(B)$ and $\text{vec}(ABD) = (D' \otimes A)\text{vec}(B)$ for any conformable matrices A, B and D [e.g., Bernstein (2005), Propositions 7.1.1 and 7.1.9], we have

$$\begin{aligned} R_{n12} &= \int_{\mathcal{X}_n} \frac{2h^{p/2}}{n} \sum_{1\le i<j\le n} \varepsilon_i\varepsilon_j \text{tr}\left[\mathcal{D}_n(x) D_h^{-1} X_{q,jx} X'_{q,ix} D_h^{-1}\right] K_{ix}K_{jx}dx \\ &= 2h^{p/2}\int_{\mathcal{X}_n} \text{vec}\left(\mathcal{D}_n(x)'\right)' \frac{1}{n}\sum_{1\le i<j\le n} \varepsilon_i\varepsilon_j \text{vec}\left(D_h^{-1}X_{q,jx}X'_{q,ix}D_h^{-1}\right) K_{ix}K_{jx}dx \\ &= (n-1)\, h^{p/2}\int_{\mathcal{X}_n} \text{vec}\left(\mathcal{D}_n(x)'\right)' \varphi_n(x)\, dx, \end{aligned}$$

where $\varphi_n(x) \equiv \frac{2}{n(n-1)}\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j[I_{N_q}\otimes \left(D_h^{-1}X_{q,jx}\right)]\text{vec}\left(D_h^{-1}X_{q,ix}\right) K_{ix}K_{jx}$, and $\otimes$ denotes the Kronecker product. Let $b \in \mathbb{R}^{N_q^2}$ such that $\|b\| = 1$. Then

$$\mathcal{U}_{n2} \equiv b'\varphi_n(x) = \frac{2}{n(n-1)}\sum_{1\le i<j\le n}\vartheta\left(\xi_i, \xi_j\right) = \vartheta_0 + \frac{2}{n}\sum_{i=1}^{n}[\vartheta_1(\xi_i) - \vartheta_0] + \mathcal{R}_n,$$

where $\vartheta\left(\xi_i,\xi_j\right) = [\vartheta^0\left(\xi_i,\xi_j\right)+\vartheta^0\left(\xi_j,\xi_i\right)]/2$, $\vartheta^0\left(\xi_i,\xi_j\right) = \varepsilon_i\varepsilon_j b'[I_{N_q}\otimes\left(D_h^{-1}X_{q,jx}\right)]\text{vec}\left(D_h^{-1}X_{q,ix}\right) K_{ix}K_{jx}$, ϑ_0, ϑ_1, and $\mathcal{R}_n$ are as defined in Lemma A.3(ii). By straightforward calculations, $\vartheta_0 = 0$, $\vartheta_1(\xi_i) = 0$, and $\mathcal{R}_n = O_p\left(n^{-1+\epsilon/2}h^{-(2+2\tilde{\eta})p/(2+\eta)}\right)$ where we use the fact that $\sup_{1\le i<j\le n}\left[E\left|\vartheta\left(\xi_i,\xi_j\right)\right|^{2+\eta}\right]^{1/(2+\eta)} = O\left(h^{-(2+2\tilde{\eta})p/(2+\eta)}\right)$. It follows that $\mathcal{U}_{n2} = O_p\left(n^{-1+\epsilon/2}h^{-(2+2\tilde{\eta})p/(2+\eta)}\right)$. In view of that $\mathcal{D}_n(x) = O_p(\nu_n^2)$, we have

$$R_{n12} = nh^{p/2}O_p(\nu_n^2)O_p\left(n^{-1+\epsilon/2}h^{-(2+2\tilde{\eta})p/(2+\eta)}\right) = O_p\left(n^{-1+\epsilon/2}h^{-[1/2+(2+2\tilde{\eta})/(2+\eta)]p}\ln n\right) = o_p(1). \tag{A.6}$$

(ii) Let $\mathcal{U}_{n3} \equiv \frac{2}{n(n-1)}\sum_{1\le i<j\le n}\vartheta\left(\xi_i,\xi_j\right)$ where $\vartheta\left(\xi_i,\xi_j\right) = \varsigma_{ij}\int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\, D_h^{-1}X_{q,jx}K_{jx}dx$. Then we can write $R_{n2} = (n-1)\, h^{p/2}\gamma_n\mathcal{U}_{n3}$. Let ϑ_0, ϑ_1, and $\mathcal{R}_n$ be as defined in Lemma A.3(ii). Then $\vartheta_0 = 0$ and $\vartheta_1(\xi_i) = \varepsilon_i E_j[\Delta_n(X_j)\int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\; D_h^{-1}X_{q,jx}K_{jx}dx]$. It follows that

$$\mathcal{U}_{n3} = \frac{2}{n}\sum_{i=1}^{n}\vartheta_1(\xi_i) + \mathcal{R}_n.$$

By moment calculations and the Chebyshev inequality, we have $\frac{1}{n}\sum_{i=1}^{n}\vartheta_1(\xi_i) = O_p\left(n^{-1/2}\right)$. By Lemma A.3(ii), $\mathcal{R}_n = O_p(n^{-1+\epsilon/2})$. It follows that

$$R_{n2} = nh^{p/2}\gamma_n O_p\left(n^{-1/2} + n^{-1+\epsilon/2}\right) = O_p\left(h^{p/4}(1 + n^{-1/2+\epsilon/2})\right) = o_p(1).$$

(iii) As in the proof of (i), write

$$\begin{aligned} R_{n3} &= 2n^{-1}h^{p/2}\gamma_n\sum_{1\le i<j\le n}\varsigma_{ij}\int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\left[S_{qn}^{-1}(x) - \bar{S}_{qn}^{-1}(x)\right]D_h^{-1}X_{q,jx}K_{jx}dx \\ &= 2n^{-1}h^{p/2}\gamma_n\sum_{1\le i<j\le n}\varsigma_{ij}\int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\left[\bar{S}_{qn}(x) - S_{qn}(x)\right]\bar{S}_{qn}^{-1}(x)\, D_h^{-1}X_{q,jx}K_{jx}dx \\ &\quad +2n^{-1}h^{p/2}\gamma_n\sum_{1\le i<j\le n}\varsigma_{ij}\int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\mathcal{D}_n(x)\, D_h^{-1}X_{q,jx}K_{jx}dx \\ &\equiv R_{n31} + R_{n32}, \text{ say.} \end{aligned}$$

Further decompose R_{n31} as follows

$$\begin{aligned}
R_{n31} &= \frac{2h^{p/2}}{n^2}\sum_{1\le i<j\le n}\sum_{k=1}^{n}\varsigma_{ij}\int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\,s_k(x)\,\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,jx}K_{jx}dx \\
&= \frac{2h^{p/2}\gamma_n}{n^2}\sum_{1\le i<j\le n}\sum_{k=1,k\neq i,j}^{n}\varsigma_{ij}\int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\,s_k(x)\,\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,jx}K_{jx}dx \\
&\quad +\frac{2h^{p/2}\gamma_n}{n^2}\sum_{1\le i<j\le n}\varsigma_{ij}\int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\,[s_i(x)+s_j(x)]\,\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,jx}K_{jx}dx \\
&\equiv R_{n31a}+R_{n32b},\ \text{say,} \qquad\qquad (\text{A.7})
\end{aligned}$$

where recall $s_k(x)\equiv -K_{kx}D_h^{-1}X_{q,kx}X'_{q,kx}D_h^{-1}+E\left(K_{kx}D_h^{-1}X_{q,kx}X'_{q,kx}D_h^{-1}\right)$. Let $\vartheta^0\left(\xi_i,\xi_j,\xi_k\right)=\varsigma_{ij}$ $\int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\,s_k(x)\,\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,jx}K_{jx}dx$. Define the symmetric version of ϑ^0 as $\vartheta\left(\xi_i,\xi_j,\xi_k\right)$ $=[\vartheta^0\left(\xi_i,\xi_j,\xi_k\right)+\vartheta^0\left(\xi_i,\xi_k,\xi_j\right)+\vartheta^0\left(\xi_j,\xi_k,\xi_2\right)]/3$. Then we have $h^{[1/2-(3+\eta)/(4+\eta)]p}$ $h^{(3+\eta)p/(4+\eta)}$

$$R_{n31a}=\frac{6h^{p/2}\gamma_n}{n^2}\sum_{1\le i<j<k\le n}\vartheta\left(\xi_i,\xi_j,\xi_k\right)=\frac{(n-1)(n-2)\,h^{p/2}\gamma_n}{n}\mathcal{U}_{n4}$$

where $\mathcal{U}_{n4}\equiv\frac{6}{n(n-1)(n-2)}\sum_{1\le i<j<k\le n}\vartheta\left(\xi_i,\xi_j,\xi_k\right)$. For this ϑ, we have $\vartheta_0=0$, $\vartheta_1\left(\xi_i\right)=0$, and

$$\begin{aligned}
\vartheta_2\left(\xi_i,\xi_j\right) &= \frac{1}{3}\left\{\varepsilon_i E_k\left[\Delta_n(X_k)\int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\bar{S}_{qn}^{-1}(x)\,s_j(x)\,\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,kx}K_{kx}dx\right]\right. \\
&\quad \left.+\,\varepsilon_j E_k\left[\Delta_n(X_k)\int_{\mathcal{X}_n} K_{kx}X'_{q,kx}D_h^{-1}\bar{S}_{qn}^{-1}(x)\,s_i(x)\,\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,jx}K_{jx}dx\right]\right\}
\end{aligned}$$

using the notation defined before Lemma A.3. It follows that

$$\mathcal{U}_{n4}=3\mathcal{H}_n^{(2)}+\mathcal{H}_n^{(3)}$$

where $\mathcal{H}_n^{(2)}$ and $\mathcal{H}_n^{(3)}$ are as defined in Lemma A.3 based on the kernel $\vartheta\left(\xi_i,\xi_j,\xi_k\right)$. We can treat $\mathcal{H}_n^{(2)}$ as $\mathcal{R}_n$ in Lemma A.3(ii) and show that $\mathcal{H}_n^{(2)}=O_p\left(n^{-1+\epsilon/2}\right)$. For $\mathcal{H}_n^{(3)}$, we apply Lemma A.3(i) to obtain

$$\begin{aligned}
\mathcal{H}_n^{(3)} &= h^{-(3+\eta)p/(4+\eta)}\left[h^{(3+\eta)p/(4+\eta)}\mathcal{H}_n^{(3)}\right] \\
&= h^{-(3+\eta)p/(4+\eta)}O_p\left(n^{-3/2}\right)=O_p\left(n^{-3/2}h^{-(3+\eta)p/(4+\eta)}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
R_{n31a} &= nh^{p/2}\gamma_n O_p\left(n^{-1+\epsilon/2}+n^{-3/2}h^{-(3+\eta)p/(4+\eta)}\right) \\
&= O_p\left(n^{-1/2+\epsilon/2}h^{p/4}+n^{-1}h^{[1/4-(3+\eta)/(4+\eta)]p}\right)=o_p(1).
\end{aligned}$$

Next, in view of that $E\left|R_{n31b}\right|=O\left(h^{p/2}\gamma_n\right)$, we have $R_{n31b}=O_p\left(h^{p/2}\gamma_n\right)=O_p\left(n^{-1/2}h^{p/4}\right)$ by the Markov inequality. It follows that

$$R_{n31}=O_p\left(n^{-1/2+\epsilon/2}h^{p/4}+n^{-1}h^{[1/4-(3+\eta)/(4+\eta)]p}+n^{-1/2}h^{p/4}\right)=o_p(1).$$

For R_{n32}, using analogous arguments to those the study of R_{n12}, we can easily obtain

$$\begin{aligned}
R_{n32} &= nh^{p/2}\gamma_n O_p(\nu_n^2)O_p\left(n^{-1+\epsilon/2}h^{-(2+2\tilde{\eta})p/(2+\eta)}\right) \\
&= \gamma_n O_p\left(n^{-1+\epsilon/2}h^{-[1/2+(2+2\tilde{\eta})/(2+\eta)]p}\ln n\right)=o_p(1).
\end{aligned}$$

Consequently, $R_{n3}=o_p(1)$.

(iv) Using $S_{qn}^{-1}(x)=\bar{S}_{qn}^{-1}(x)+\left[S_{qn}^{-1}(x)-\bar{S}_{qn}^{-1}(x)\right]$, we have

$$\begin{aligned}
R_{n4} &= 2n^{-2}h^p\sum_{1\le i\ne j\le n}\varepsilon_i^2\varepsilon_j^2\left[\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,jx}K_{jx}dx\right]^2\\
&\quad+2n^{-2}h^p\sum_{1\le i\ne j\le n}\varepsilon_i^2\varepsilon_j^2\left\{\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}\left[S_{qn}^{-1}(x)-\bar{S}_{qn}^{-1}(x)\right]D_h^{-1}X_{q,jx}K_{jx}dx\right\}^2\\
&\quad+4n^{-2}h^p\sum_{1\le i\ne j\le n}\varepsilon_i^2\varepsilon_j^2\left[\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,jx}K_{jx}dx\right]\\
&\quad\times\left[\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}\left[S_{qn}^{-1}(x)-\bar{S}_{qn}^{-1}(x)\right]D_h^{-1}X_{q,jx}K_{jx}dx\right]\\
&\equiv R_{n41}+R_{n42}+R_{n43}.
\end{aligned}$$

We first study R_{n41}. Let $\vartheta(\xi_i,\xi_j)=2h^p\varepsilon_i^2\varepsilon_j^2[\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,jx}K_{jx}dx]^2$. For this ϑ, define

$$\begin{aligned}
\vartheta_0 &= 2h^pE_iE_j\left\{\varepsilon_i^2\varepsilon_j^2\left[\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,jx}K_{jx}dx\right]^2\right\},\\
\vartheta_1(\xi_i) &= 2h^pE_j\left\{\varepsilon_i^2\varepsilon_j^2\left[\int_{\mathcal{X}_n}K_{ix}X_{q,ix}'D_h^{-1}\bar{S}_{qn}^{-1}(x)\,D_h^{-1}X_{q,jx}K_{jx}dx\right]^2\right\}.
\end{aligned}$$

Then we have the following Hoeffding decomposition

$$\begin{aligned}
R_{n41} &= 2n^{-2}\sum_{1\le i<j\le n}\vartheta(\xi_i,\xi_j)\\
&= \vartheta_0+2n^{-1}\sum_{i=1}^n\left[\vartheta_1(\xi_i)-\vartheta_0\right]+\frac{n-1}{n}\frac{2}{n(n-1)}\sum_{1\le i<j\le n}\left[\vartheta(\xi_i,\xi_j)-\vartheta_1(\xi_i)-\vartheta_1(\xi_j)+\vartheta_0\right]\\
&\equiv \vartheta_0+2\mathcal{H}_n^{(1)}+\frac{n-1}{n}\mathcal{H}_n^{(2)}.
\end{aligned}$$

By Lemma A.3(ii) and the fact that $s_\eta\equiv\sup_{1\le i_1<i_2\le n}[E\left|\vartheta(\xi_{i_1},\xi_{i_2})\right|^{2+\tilde{\eta}}]^{1/(2+\tilde{\eta})}=O(1)$, $\mathcal{H}_n^{(2)}=O_p\left(n^{-1+\varepsilon/2}\right)=o_p(1)$. In view of $E[\mathcal{H}_n^{(1)}]=0$, and

$$\begin{aligned}
\operatorname{Var}\left(\mathcal{H}_n^{(1)}\right) &= n^{-2}\sum_{i=1}^n\sum_{j=1}^n\operatorname{Cov}(\vartheta_1(\xi_i),\vartheta_1(\xi_i))\\
&\le C\left\{E\left|\vartheta_1(\xi_1)\right|^{2+\tilde{\eta}}\right\}^{2/(2+\tilde{\eta})}n^{-2}\sum_{i=1}^n\sum_{j=1}^n\alpha(|j-i|)^{\tilde{\eta}/(2+\tilde{\eta})}\\
&\le Cn^{-1}\left\{E\left|\vartheta_1(\xi_1)\right|^{2+\tilde{\eta}}\right\}^{2/(2+\tilde{\eta})}\sum_{\tau=0}^{\infty}\alpha(\tau)^{\tilde{\eta}/(2+\tilde{\eta})}=O\left(n^{-1}\right)
\end{aligned}$$

by the Davydov inequality and Assumption A1, $\mathcal{H}_n^{(1)}=O_p\left(n^{-1/2}\right)$. By the variance calculation in the proof of Theorem 3.1, $\vartheta_0=\Omega_q+o(1)$. It follows that $R_{n41}=\Omega_q+o_p(1)$.

For R_{n42}, we have

$$\begin{aligned} R_{n42} &= 2n^{-2}h^p \sum_{1\leq i\neq j\leq n} \varepsilon_i^2\varepsilon_j^2 \left\{ \int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}\left[S_{qn}^{-1}(x)-\bar{S}_{qn}^{-1}(x)\right]D_h^{-1}X_{q,jx}K_{jx}dx \right\}^2 \\ &\leq 2\sup_{x\in\mathcal{X}_n}\left|S_{qn}^{-1}(x)-\bar{S}_{qn}^{-1}(x)\right| n^{-2}h^p \sum_{1\leq i\neq j\leq n} \varepsilon_i^2\varepsilon_j^2 \left\{ \int_{\mathcal{X}_n} K_{ix}X'_{q,ix}D_h^{-1}D_h^{-1}X_{q,jx}K_{jx}dx \right\}^2 \\ &= O_p(\nu_n)O_p(1)=o_p(1). \end{aligned}$$

By the Cauchy-Schwarz inequality, $R_{n42}\leq 2(R_{n41})^{1/2}(R_{n42})^{1/2}=2O_p(1)\,o_p(1)=o_p(1)$. Consequently, $R_{n4}=\Omega_q+o_p(1)$. ■

B Extension from a nonrandom trimming set to a random one

In the paper we assume that the trimming set $\mathcal{X}_n$ is nonrandom, which greatly facilitates the asymptotic analysis. Here we remark that it is possible to allow $\mathcal{X}_n$ to be random and offer some intuitive arguments.

When $\mathcal{X}_n$ is random, we assume that there exists a nonrandom set $\bar{\mathcal{X}}_n$ such that

(i) $\mathrm{vol}(\bar{\mathcal{X}}_n\backslash\mathcal{X}_n)+\mathrm{vol}(\mathcal{X}_n\backslash\bar{\mathcal{X}}_n)=O_p\left(n^{-1/2}\right)$,

(ii) $\bar{\mathcal{X}}_n\to\mathcal{X}$ as $n\to\infty$.

So the sets $\bar{\mathcal{X}}_n$ and $\mathcal{X}$ will play the role that the nonrandom sets $\mathcal{X}_n$ and $\mathcal{X}$ play in the main paper. Note that condition (i) is satisfied when we use the sample and population quantiles of $\{X_{nt},\ t=1,...,n\}$ to construct the random set $\mathcal{X}_n$ and the nonrandom set $\bar{\mathcal{X}}_n$, respectively.

In various places in the proof of the asymptotic results in the paper, we need to argue that under certain conditions

$$\int_{\mathcal{X}_n} B_n(x)\,dx=o_p(1) \tag{B.1}$$

where $B_n(x)$ can be written as a U-statistic of second or higher order. See, e.g., claims (i)-(iii) in Lemma A.1. Suppose Assumptions A1-A2 and A5-A6 in the text hold. Suppose in addition that for some $\epsilon>0$,

$$n^{-1/2}\int_{\bar{\mathcal{X}}_n^{\epsilon}} B_n(x)^2\,dx=o_p(1), \tag{B.2}$$

where

$$\bar{\mathcal{X}}_n^{\epsilon}\equiv\left\{x\in\mathbb{R}^p:\|x-x^*\|\leq\epsilon \text{ for some } x^*\in\bar{\mathcal{X}}_n\right\}.$$

Then we argue below that (B.1) can be proved.

First, the arguments in the proof of Lemma A.1 can be used to justify that

$$\int_{\bar{\mathcal{X}}_n} B_n(x)\,dx=o_p(1). \tag{B.3}$$

To prove (B.1), (B.3) indicates that it suffices to show that

$$\vartheta_n\equiv\int_{\mathcal{X}_n} B_n(x)\,dx-\int_{\bar{\mathcal{X}}_n} B_n(x)\,dx=o_p(1). \tag{B.4}$$

Now, we decompose ϑ_n as follows:

$$\begin{aligned}
\vartheta_n &= \int B_n(x)\left[1(x \in \mathcal{X}_n) - 1\left(x \in \bar{\mathcal{X}}_n\right)\right] dx \\
&= \int B_n(x)\, 1\left(x \in \bar{\mathcal{X}}_n\right)\left[1(x \in \mathcal{X}_n) - 1\left(x \in \bar{\mathcal{X}}_n\right)\right] dx \\
&\quad + \int B_n(x)\, 1\left(x \in \mathcal{X}_n \backslash \bar{\mathcal{X}}_n\right)\left[1(x \in \mathcal{X}_n) - 1\left(x \in \bar{\mathcal{X}}_n\right)\right] dx \\
&\equiv \vartheta_{1n} + \vartheta_{2n}, \text{ say.} \qquad \text{(B.5)}
\end{aligned}$$

It is sufficient to show that $\vartheta_{sn} = o_p(1)$ for $s = 1, 2$. By the Cauchy-Schwarz inequality, (B.2), and condition (i),

$$\begin{aligned}
\vartheta_{1n}^2 &\leq \int_{\bar{\mathcal{X}}_n} B_n(x)^2\, dx \int_{\bar{\mathcal{X}}_n} \left|1(x \in \mathcal{X}_n) - 1\left(x \in \bar{\mathcal{X}}_n\right)\right| dx \\
&\leq \left\{n^{-1/2} \int_{\bar{\mathcal{X}}_n} B_n(x)^2\, dx\right\}\left\{n^{1/2}\left[\operatorname{vol}\left(\mathcal{X}_n \backslash \bar{\mathcal{X}}_n\right) + \operatorname{vol}\left(\bar{\mathcal{X}}_n \backslash \mathcal{X}_n\right)\right]\right\} \\
&= o_p(1)\, O_p(1) = o_p(1).
\end{aligned}$$

Condition (i) also implies that with probability tending to one, $\mathcal{X}_n \subset \bar{\mathcal{X}}_n^{\epsilon}$ for any fixed $\epsilon > 0$. Fix $\epsilon > 0$. Then with probability tending to one we have

$$\begin{aligned}
\vartheta_{2n}^2 &= \left\{\int B_n(x)\, 1\left(x \in \mathcal{X}_n \backslash \bar{\mathcal{X}}_n\right)\left[1(x \in \mathcal{X}_n) - 1\left(x \in \bar{\mathcal{X}}_n\right)\right] dx\right\}^2 \\
&\leq \left\{n^{-1/2} \int_{\mathcal{X}_n \backslash \bar{\mathcal{X}}_n} B_n(x)^2\, dx\right\}\left\{n^{1/2} \int_{\bar{\mathcal{X}}_n} \left|1(x \in \mathcal{X}_n) - 1\left(x \in \bar{\mathcal{X}}_n\right)\right| dx\right\} \\
&\leq n^{-1/2} \int_{\bar{\mathcal{X}}_n^{\epsilon}} B_n(x)^2\, dx \left\{n^{1/2}\left[\operatorname{vol}\left(\mathcal{X}_n \backslash \bar{\mathcal{X}}_n\right) + \operatorname{vol}\left(\bar{\mathcal{X}}_n \backslash \mathcal{X}_n\right)\right]\right\} \\
&= o_p(1)\, O_p(1) = o_p(1).
\end{aligned}$$

As a result, one only needs to prove (B.2). Noting that $\bar{\mathcal{X}}_n^{\epsilon}$ is a nonrandom set, one can prove (B.2) by following similar arguments as used in the proof of Lemma A.1(i)-(iii).

For Lemma A.1(iv), one can only show that it continues to hold when $\mathcal{X}_n$ is random and the two conditions stated at the beginning of this appendix hold.

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