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Testing for parameter stability in quantile regression models

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ABSTRACT

We propose a test for structural change of conditional distribution in dynamic regression models. The test is constructed based on time series regression quantile estimates and complements conventional parameter instability tests in least-square type regression models. Asymptotic distribution for our test under the null hypothesis is derived.

1. Introduction

Since Page (1954), there has been a large literature on testing for structural changes, see Csörgő and Horváth (1997) for an excellent review on this topic. An important subfield in testing for structural changes is testing for structural change at the distributional level, see, inter alia, Picard (1985), Bai (1994), Inoue (2001), and Lee and Na (2004). We believe that quantile regression can provide a useful alternative approach in testing for changes in distribution or conditional distribution. Being the inverse of a conditional distribution function, the conditional quantile function is a natural object to examining distributional changes.

In this paper, we study testing for structural change in regression quantiles with unknown timing. The proposed test is constructed based on a sequence of quantile regression estimators using the ideas of Koenker and Basset (1978). We demonstrate the weak convergence of these estimators as a two-parameter process. Then we propose a sup-Wald test for the null hypothesis of no structural change and show that it is asymptotically pivotal under the null.

There are several key features that are associated with our test. First, it is based on a two-parameter quantile process instead of one-parameter quantile process that has been widely studied (see Koenker (2005)). Second, if the regressor contains only a constant term, our test reduces to testing for unconditional distributional change as was studied in the aforementioned papers. Third, in general case, our test can be regarded as a test for a change in conditional distribution.

The rest of the paper is organized as follows. In Section 2 we introduce our hypotheses and the sequential quantile regression estimators (SQREs). In Section 3 we study the asymptotic properties of the SQREs and propose an asymptotically distribution-free test for the null. All proofs are relegated to the Appendix.

2. Hypothesis and sequential quantile regression estimators

Let $\{y_t, x_t\}_{t=1}^n$ denote a time series sequence of random vectors. Let $\mathcal{F}_0 = \sigma\text{-field}\ \{x_1\}$, $\mathcal{F}_{t-1} = \sigma\text{-field}\ (y_{t-1}, \ldots, y_1, x_t, \ldots, x_1)$ for $t \geq 2$. For simplicity, we assume that the τth conditional quantile function of y_t given $\mathcal{F}_{t-1}$ is linear in x_t :

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = \beta(\tau, t)' x_t, \tag{1}$$

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where $\beta(\tau, t)$ is a $p \times 1$ parameter vector. For example, if $x_t = (1, y_{t-1}, \ldots, y_{t-p-1})'$ and $\beta(\tau, t)$ does not depend on t, we get the quantile autoregression (QAR) model of Koenker and Xiao (2006):

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = \beta_1(\tau) + \beta_2(\tau) y_{t-1} + \cdots + \beta_p(\tau) y_{t-p-1} = \beta(\tau)' x_t, \tag{2}$$

where $\beta(\tau) = (\beta_1(\tau), \beta_2(\tau), \cdots, \beta_p(\tau))'$.

We are interested in testing whether the quantile regression parameters remain stable over time:

$$H_0 : \beta(\tau, t) = \beta(\tau) \quad \text{for some } \beta(\tau) \in \mathcal{B} \subset \mathbb{R}^p. \tag{3}$$

Why are we interested in regression quantile instability? As we mentioned above, stability of conditional distribution is an important issue in many statistical analysis. Suppose that we are interested in testing the null hypothesis that, in a random sample of size n, the conditional distribution of y_t given x_t has not changed. Let the conditional distribution function be $F_t(y, x_t) = \Pr(y_t \le y|x_t)$, then, the null hypothesis can be written as

$$H_0 : F_t(y, x_t) = F(y, x_t).$$

Since the inverse of a conditional distribution function is the conditional quantile function, we can equivalently express H_0 as

$$H_0 : Q_t(\tau, x_t) = Q(\tau, x_t),$$

where $Q_t(\tau, x_t)$ and $Q(\tau, x_t)$ are conditional quantile functions of y_t given x_t obtained from solving

$$F_t(y, x_t) = \tau \quad \text{and} \quad F(y, x_t) = \tau, \text{ respectively for } \tau \in [0, 1].$$

If we consider a linear parametric model such that

$$Q_t(\tau, x_t) = \beta(\tau, t)' x_t,$$

we can then write the inference problem in terms of (3).

The alternative hypothesis of interest may be of several forms. In this paper we focus on the one-time structural change alternative with unknown change point $r \in (0, 1)$. Since n is the sample size, $\lceil nr \rfloor$ (where $\lceil \cdot \rfloor$ denotes the integer part) is the time of change, and for simplicity, we will refer to r as the change point. The one-time change alternative with change point r is

$$H_{1n}(r) : \beta(\tau, t) = \begin{cases} \beta(\tau) & \text{for } t = 1, \ldots, \lceil nr \rfloor \\ \beta(\tau) + \delta(\tau) & \text{for } t = \lceil nr \rfloor + 1, \ldots, \end{cases} \tag{4}$$

where $\beta(\tau) \in \mathcal{B}$ and $\beta(\tau) + \delta(\tau) \in \mathcal{B}$. In other words, $\delta(\tau)$ indicates the size of the structural change.

Since the change point r is usually unknown in practice, we have to endogenize it. For this purpose, we define a dummy variable $I_{r,t} = 1\,(t \ge \lceil nr \rfloor + 1)$, where $1\,(.)$ is the indicator function. We consider the sequential quantile regression model

$$Q_{y_t}(\tau|x_t, I_{r,t}) = \beta(\tau)' x_t + \delta(\tau)' (x_t I_{r,t}), \tag{5}$$

then testing the null of no structural change reduces to testing

$$H_0 : \delta_0(\tau) = 0 \quad \text{for all } \tau, \tag{6}$$

where $\delta_0(\tau)$ is the true parameter value of $\delta(\tau)$ in (5).

To proceed, let $z_{rt} = (x_t', x_t' I_{r,t})'$ and $\theta(\tau) = (\beta(\tau)', \delta(\tau)')'$. Based on $\{y_t, z_{rt}\}_{t=1}^n$, the sequential quantile regression estimators (SQREs) of $\theta(\tau)$ are given by

$$\widehat{\theta}(\tau, r) = \arg\min_{\beta \in \mathbb{R}^{2p}} \rho_\tau\left(y_t - \theta(\tau)' z_{rt}\right), \tag{7}$$

where $\rho_\tau(u) = u[\tau - 1(u < 0)]$. $\widehat{\theta}(\tau, r) = (\widehat{\beta}(\tau, r)', \widehat{\delta}(\tau, r)')' \in \mathbb{R}^p \times \mathbb{R}^p$. Intuitively, under the null hypothesis we expect that $\widehat{\delta}(\tau, r)$ should be small for all τ and r.

3. Main results

To state the main results, we make the following set of assumptions:

Assumptions

A1. $\{y_t, x_t\}$ is a strictly stationary and ergodic sequence such that $n^{-1/2}\max_{1\le t\le n}\|x_t\| = o_P(1)$, where $\|\cdot\|$ is the Euclidean norm.

A2. Let $F(.|\mathcal{F}_{t-1})$ denote the conditional distribution function of y_t given $\mathcal{F}_{t-1}$. $F(.|\mathcal{F}_{t-1}) = F(.|x_t) = F_t(.)$. $F_t(.)$ has Lebesgue density $f_t(.) = f(.|x_t)$ a.s. such that

(i) $\inf_{c_1 \le \tau \le 1-c_1} f\left(F^{-1}(\tau|x)\,|x\right) \ge 1/C_0 > 0$ for each unit vector x like $(1, 0, \ldots, 0)'$,
(ii) $\sup_{\{0<F_t(y)<1\}} f_t(y) \le C_{1t}$ for all t,
(iii) $|f_t(y_1) - f_t(y_2)| \le C_{2t}|y_1 - y_2|$ for all t,

where $0 < C_0 < \infty$, $c_1 \in (0, 1/2)$, $E[C_{1t} \|x_t\|^2] < \infty$, and $E[C_{2t} \|x_t\|^2] < \infty$.

A3. $E[\psi_\tau(y_t - \beta_0(\tau)' x_t)|\mathcal{F}_{t-1}] = 0$ a.s. for some unique $\beta_0(\tau) \in \mathcal{B} \subset \mathbb{R}^p$, where $\psi_\tau(u) = \tau - 1(u < 0)$ and $\beta_0(\tau)$ is an interior point of the compact set $\mathcal{B}$ for each τ.

A4. There exist a random variable ξ_n and a constant κ_1 $(0 \le \kappa_1 < 1/2)$ such that for all $0 \le r_1 \le r_2 \le 1$, $n^{-1} \sum_{t=\lceil nr_1 \rfloor}^{\lceil nr_2 \rfloor} \|x_t\| \le (r_2 - r_1) \xi_n n^{\kappa_1}$ a.s. In addition, $\sup_n E(\xi_n^{\kappa_2}) \le C < \infty$ for some $\kappa_2 > 2$.

A5. There exist $\kappa_3 \ge \kappa_4 > 1$, and $C_3 < \infty$ such that for all $0 \le r_1 < r_2 \le 1$, and all $n > 1$, $n^{-1} \sum_{t=\lceil nr_1 \rfloor+1}^{\lceil nr_2 \rfloor} E(x_t' x_t)^{\kappa_3} \le C_3 (r_2 - r_1)$, $E\left[n^{-1} \sum_{t=\lceil nr_1 \rfloor+1}^{\lceil nr_2 \rfloor} x_t' x_t\right]^{\kappa_3}$
$\le C_3 (r_2 - r_1)^{\kappa_4}$, and $(\kappa_3 - 1)/(\kappa_4 - 1) > 1 + 2\kappa_1$.

A6. (i) $\sup_{0<r\le 1} \left| n^{-1} \sum_{t=1}^{\lceil nr \rfloor} x_t x_t' - rQ \right| = o_P(1)$, where Q is a finite, symmetric and positive definite matrix.

(ii) $\sup_{0\le\tau\le 1} \sup_{0<r\le 1} \left| n^{-1} \sum_{t=1}^{\lceil nr \rfloor} f_t(\beta_0(\tau)' x_t) x_t x_t' - rH^*(\tau) \right| = o_P(1)$, where $H^*(\tau)$ is a finite, symmetric and positive definite matrix for each τ.

Assumption A1 is typical in dynamic regression models. Assumption A2 assumes that $F(.|\mathcal{F}_{t-1}) = F(.|x_t)$ and imposes a uniform and smoothness condition on $f_t(y)$. Assumption A3 imposes the null hypothesis and is an unique identification condition for $\beta_0(\tau)$. Assumption A4 and A5 are identical to (A.5) and (A.6) in Bai (1996) and are needed to ensure the tightness of certain sequential weighted empirical processes. As Bai (1996) remarked, we can take $\kappa_1 = 1/\kappa_2$ in Assumption A4 if $E\|x_t\|^{\kappa_2} < C_4 < \infty$ for some $\kappa_2 > 2$ and for all t; and we can take $\kappa_3 = 2$, $\kappa_4 = 3/2$ in Assumption A5 if $E(x_t' x_t)^2 \le C_5 < \infty$ for all t. Assumption A6 is fairly general and it is assumed to facilitate the derivation of the asymptotic results.

Under Assumptions A1–A5, we first show that the SQREs $\{\widehat{\theta}(\tau, r)\}$ have a Bahadur representation uniformly in both τ and r. To do so, define

$$H_n(\tau, r) = n^{-1} \sum_{t=1}^{n} f_t(\theta_0(\tau)' z_{rt}) z_{rt} z_{rt}', \quad \text{and} \quad J_n(\tau, r) = n^{-1/2} \sum_{t=1}^{n} \psi_\tau(y_t - \theta_0(\tau)' z_{rt}) z_{rt}, \tag{8}$$

where $\theta_0(\tau) = (\beta_0(\tau)', \delta_0(\tau)')'$ signifies the true parameter vector under the null, i.e., $\delta_0(\tau) = 0$.

Theorem 3.1. *Suppose Assumptions* A1–A5 *hold. Then*

$$\sup_{\tau\in\mathcal{T}} \sup_{r\in\mathcal{A}} \left\| \sqrt{n} \left(\widehat{\theta}(\tau, r) - \theta_0(\tau) \right) - H_n(\tau, r)^{-1} J_n(\tau, r) \right\| = o_P(1),$$

where $\mathcal{T} \equiv [c_1, 1 - c_1]$ *and* $\mathcal{A} \equiv [c_2, 1 - c_2]$, $c_i \in (0, 1/2)$.

Theorem 3.1 generalizes the Bahadur representation results in the literature. It serves as the basis for testing $H_0^*: R\theta_0(\tau) = c$ for all $\tau \in \mathcal{T}$, where R is a $q \times 2p$ matrix with rank $q \le 2p$ and c is a constant that may be known or unknown. We assume that τ and r on $[c_i, 1 - c_i] \subset (0, 1)$ as other work of this type since the uniform Bahadur representation holds only on a compact subinterval of $[0, 1]$ (unless some other conditions are imposed on the tail behavior of the quantiles.[2]), see, for example, Portnoy (1984) and Portnoy and Koenker (1989) for related discussions. In addition, focusing on compact subintervals of $[0, 1]$ can also accommodate the estimation of the covariance matrix. Of particular interest are hypotheses of the form $H_0: \delta_0(\tau) = 0$ for all τ. To test for H_0, we need weak convergence $\sqrt{n}\,\widehat{\delta}(\tau, r)$, as is shown in the next theorem.

Theorem 3.2. *Suppose Assumptions* A1–A6 *hold. Then*

$$\sqrt{n}\,\widehat{\delta}(\tau, r) \Rightarrow (r(1 - r))^{-1} H^*(\tau)^{-1} Q^{1/2} W(\tau, r),$$

where $\Rightarrow$ *denotes weak convergence in the Skorohod space* $\mathcal{T} \times \mathcal{A}$, $W(\tau, r) = rW^*(\tau, 1) - W^*(\tau, r)$, *and* $\{W^*(\tau, r) : (\tau, r) \in [0, 1]^2\}$ *is a Kiefer process with* $E[W^*(\tau, r)] = 0$ *and* $E[W^*(\tau_1, r_1) W^*(\tau_2, r_2)] = (r_1 \wedge r_2)(\tau_1 \wedge \tau_2 - \tau_1\tau_2) I_p$.

By definition, $\{W(\tau, r) : (\tau, r) \in [0, 1]^2\}$ is a p-vector of tied-down Kiefer process such that each component is independent of each other, $E[W(\tau, r)] = 0$ and $E[W(\tau_1, r_1) W(\tau_2, r_2)] = (\tau_1 \wedge \tau_2 - \tau_1\tau_2)(r_1 \wedge r_2 - r_1 r_2) I_p$. For discussions on tied-down Kiefer processes, see Csörgő and Horváth (1997, p. 384). For fixed τ and r, we can find the asymptotic covariance matrix of $\sqrt{n}\,\widehat{\delta}(\tau, r)$ given by

$$\Omega(\tau, r) \equiv \frac{\tau(1 - \tau)}{r(1 - r)} H^*(\tau)^{-1} Q H^*(\tau)^{-1}.$$

Let $\widehat{\Omega}(\tau, r)$ be a consistent estimator of $\Omega(\tau, r)$ uniformly in $\tau \in \mathcal{T}$ and $r \in \mathcal{A}$. The sup-Wald statistic for testing $H_{0a}: \delta_0(\tau) = 0$ for all τ is given by

$$\sup W_n \equiv \sup_{\tau\in\mathcal{T}} \sup_{r\in\mathcal{A}} W_n(\tau, r) \quad \text{with } W_n(\tau, r) = n\widehat{\delta}(\tau, r)' \,\widehat{\Omega}(\tau, r)^{-1} \,\widehat{\delta}(\tau, r). \tag{9}$$

We reject H_0 for large values of $\sup W_n$.

[2] Generally there is no good way to control the process in the tails. In addition, even when assumptions are made for the tails it would require some form of extreme value theory and the standard Brownian bridge theory would not apply.

Theorem 3.3. *Suppose Assumptions* A1–A6 *hold. Suppose* $\sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}}\|\widehat{\Omega}(\tau,r)-\Omega(\tau,r)\|=o_P(1)$. *Then*

$$\sup W_n \xrightarrow{d} \sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}} W(\tau,r)'W(\tau,r)/[\tau(1-\tau)r(1-r)].$$

Theorem 3.3 indicates that the limiting distribution of W_n is pivotal and its critical values can easily be tabulated. The key to implementing the supW_n test is to estimate $\Omega(\tau,r)$ consistently, which in turn requires consistent estimation of Q and $H^*(\tau)$. Assumption A6(i) implies that $Q_n\equiv n^{-1}\sum_{t=1}^{n}x_tx_t'$ is consistent with Q. Koenker (2005) discusses several ways to estimate $H^*(\tau)$ consistently.

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Appendix

We use C to signify a generic constant whose exact value may vary from case-to-case. Let E_i and Var_i denote expectation and variance conditional on $\mathcal{F}_{i-1}$, respectively.

Proof of Theorem 3.1. Recall $\theta(\tau)=(\beta(\tau)',\delta(\tau)')'$. Let $\widehat{\theta}_{\tau r}=\widehat{\theta}(\tau,r)=(\widehat{\beta}_{\tau r}',\widehat{\delta}_{\tau r}')'$, where $\widehat{\beta}_{\tau r}=\widehat{\beta}(\tau,r)$, and $\widehat{\delta}_{\tau r}=\widehat{\delta}(\tau,r)$. We frequently write $\theta_{0\tau}=\theta_0(\tau)=(\beta_{0\tau}',\delta_{0\tau}')'$. Define

$$\widehat{\Delta}_{\tau r}=\begin{pmatrix}\sqrt{n}(\widehat{\beta}_{\tau r}-\beta_{0\tau})\\ \sqrt{n}\left(\widehat{\delta}_{\tau r}-\delta_{0\tau}\right)\end{pmatrix},\qquad \Delta_\tau=\begin{pmatrix}\sqrt{n}\,(\beta(\tau)-\beta_{0\tau})\\ \sqrt{n}\,(\delta(\tau)-\delta_{0\tau})\end{pmatrix}.$$

Let $y_i^*=y_i-\theta_{0\tau}'z_{ri}$, and $y_{n,ri}^*(\Delta_\tau)=y_i^*-\Delta_\tau'z_{ri}/\sqrt{n}=y_i-\theta(\tau)'z_{ri}$. It follows from (7) that

$$\widehat{\Delta}_{\tau r}=\arg\min_{\Delta_\tau\in\mathbb{R}^{2p}}\sum_{i=1}^{n}\rho_\tau\left(y_{n,ri}^*(\Delta_\tau)\right). \tag{10}$$

Set

$$V_n(\tau,r;\Delta_\tau)=n^{-1/2}\sum_{i=1}^{n}\psi_\tau\left(y_{n,ri}^*(\Delta_\tau)\right)z_{ri}=n^{-1/2}\sum_{i=1}^{n}\psi_\tau\left(y_i-(\theta_{0\tau}+n^{-1/2}\Delta_\tau)'z_{ri}\right)z_{ri},\quad\text{and}$$

$$\overline{V}_n(\tau,r;\Delta_\tau)=n^{-1/2}\sum_{i=1}^{n}E_i\left[\psi_\tau\left(y_i-(\theta_{0\tau}+n^{-1/2}\Delta_\tau)'z_{ri}\right)\right]z_{ri}.$$

Noting that $-\Delta_\tau'V_n(\tau,r;\lambda\Delta_\tau)$ is an increasing function of $\lambda\geq 1$, Theorem 3.1 then follows from the following three lemmas by Lemma A.4 of Koenker and Zhao (1996).

Lemma 1. *Suppose Assumptions* A1–A5 *hold. Then*

$$\sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}}\sup_{\|\Delta\|\leq M}\|V_n(\tau,r;\Delta)-V_n(\tau,r;0)-\left[\overline{V}_n(\tau,r;\Delta)-\overline{V}_n(\tau,r;0)\right]\|=o_P(1).$$

Proof. We first establish a pointwise convergence result. Let

$$S_n(\tau,r;\Delta)=-\{V_n(\tau,r;\Delta)-V_n(\tau,r;0)-[\overline{V}_n(\tau,r;\Delta)-\overline{V}_n(\tau,r;0)]\}=n^{-1/2}\sum_{i=1}^{n}\widetilde{s}_{n,i}(\tau,r;\Delta),$$

where $\widetilde{s}_{n,i}(\tau,r;\Delta)=s_{n,i}(\tau,r;\Delta)-E_i[s_{n,i}(\tau,r;\Delta)]$, and

$$s_{n,i}(\tau,r;\Delta)=[1(y_i\leq\left(\theta_{0\tau}+n^{-1/2}\Delta\right)'z_{ri})-1(y_i\leq\theta_{0\tau}'z_{ri})]z_{ri}.$$

We want to show

$$\|S_n(\tau,r;\Delta)\|=o_P(1)\quad\text{for each fixed }\tau,r,\Delta, \tag{11}$$

which would hold if

$$S_{n,k}(\tau,r;\Delta)=o_P(1)\quad\text{for each fixed }\tau,r,\Delta,\ \text{and }k=1,\ldots,2p, \tag{12}$$

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where $S_{n,k}$ is the kth element of $S_n(\tau, r; \Delta)$. Let z_{rik}, $s_{n,ik}$ and $\widetilde{s}_{n,ik}$ be the kth element of z_{ri}, $s_{n,i}$ and $\widetilde{s}_{n,i}$, respectively. By Assumptions A1, A2(ii) and A3, the variance decomposition formula, the Jensen inequality and the martingale difference property of $\{(\widetilde{s}_{n,ik}, \mathcal{F}_i), 1 \le i \le n\}$, $E[S_{n,k}(\tau, r; \Delta)] = 0$, and

$$\begin{aligned}
\operatorname{Var}(S_{n,k}(\tau, r; \Delta)) &= n^{-1} \sum_{i=1}^{n} E[\operatorname{Var}_i(s_{n,ik}(\tau, r; \Delta))] \le n^{-1} \sum_{i=1}^{n} E\left[E_i\left(s_{n,ik}^2(\tau, r; \Delta)\right)\right] \\
&\le n^{-1} \sum_{i=1}^{n} E\left[\left|F_i\left((\theta_{0\tau} + n^{-1/2}\Delta)' z_{ri}\right) - F_i(\theta_{0\tau}' z_{ri})\right| z_{rik}^2\right] \\
&\le n^{-3/2} \sum_{i=1}^{n} E\left[C_{1i}\left|\Delta' z_{ri}\right| z_{rik}^2\right] \\
&\le C \max_{1 \le i \le n}(n^{-1/2}\|x_i\|) n^{-1} \sum_{i=1}^{n} E\left[C_{1i}\|x_i\|^2\right] = o(1).
\end{aligned}$$

Hence $|S_{n,k}(\tau, r; \Delta)| = o_P(1)$ by Chebyshev's inequality.

We next show that (12) holds uniformly over $\Gamma \equiv \{\Delta : \|\Delta \le M\|\}$, $\tau \in \mathcal{T}$ and $r \in \mathcal{A}$, where $M \in (0, \infty)$. This will hold by the triangle inequality provided

$$\sup_{\tau \in \mathcal{T}} \sup_{r \in \mathcal{A}} \sup_{\|\Delta\| \le M} \left|S_{n,k}^{+}(\tau, r; \Delta)\right| = o_P(1), \quad \text{and} \quad \sup_{\tau \in \mathcal{T}} \sup_{r \in \mathcal{A}} \sup_{\|\Delta\| \le M} \left|S_{n,k}^{-}(\tau, r; \Delta)\right| = o_P(1), \tag{13}$$

where $S_{n,k}^{+}$ and $S_{n,k}^{-}$ are defined analogously to $S_{n,k}$ but with z_{rik} replaced by $z_{rik}^{+} \equiv \max(z_{rik}, 0)$ and $z_{rik}^{-} \equiv \max(-z_{rik}, 0)$, respectively. We will only show the first part of (13) since the other case is similar. Define for every $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\widetilde{S}_{n,k}^{+}(\tau, r; \Delta, \lambda) &= n^{-1/2} \sum_{i=1}^{n} \Big\{1\left(y_i \le (\theta_{0\tau} + n^{-1/2}\Delta)' z_{ri} + \lambda\|n^{-1/2} z_{ri}\|\right) \\
&\quad - F_i\left((\theta_{0\tau} + n^{-1/2}\Delta)' z_{ri} + \lambda\|n^{-1/2} z_{ri}\|\right) - 1\left(y_i \le \theta_{0\tau}' z_{ri}\right) + F_i\left(\theta_{0\tau}' z_{ri}\right)\Big\} z_{rik}^{+}.
\end{aligned}$$

Note that $\widetilde{S}_{n,k}^{+}(\tau, r; \Delta, 0) = S_{n,k}^{+}(\tau, r; \Delta)$. We follow Koul (1991) and Bai (1994) and show that the first part of (13) is a consequence of the following result

$$\sup_{\tau \in \mathcal{T}} \sup_{r \in \mathcal{A}} \left|\widetilde{S}_{n,k}^{+}(\tau, r; \Delta, \lambda)\right| = o_P(1) \quad \text{for every given } \Delta \text{ and } \lambda. \tag{14}$$

Since Γ is compact, we can partition it into a finite number $N(\sigma)$ of subsets $\{\Gamma_1, \ldots, \Gamma_{N(\sigma)}\}$ such that the diameter of each subset is not greater than σ. Fix $s \in \{1, \ldots, N(\sigma)\}$ and $\Delta_s \in \Gamma_s$. Noting that $\Delta' z_{ri} \le \Delta_s' z_{ri} + \sigma\|z_{ri}\|$ for any $\Delta \in \Gamma_s$, it follows from the monotonicity of the indicator function $1(y_i \le \cdot)$ and the nonnegativity of z_{rik}^{+} that for any $\Delta \in \Gamma_s$,

$$\begin{aligned}
S_{n,k}^{+}(\tau, r; \Delta) &\le \widetilde{S}_{n,k}^{+}(\tau, r; \Delta_s, \sigma) + n^{-1/2} \sum_{i=1}^{n} \Big\{F_i\left((\theta_{0\tau} + n^{-1/2}\Delta_s)' z_{ri} + \sigma\|n^{-1/2} z_{ri}\|\right) \\
&\quad - F_i\left((\theta_{0\tau} + n^{-1/2}\Delta_s)' z_{ri}\right) z_{rik}^{+}\Big\}.
\end{aligned}$$

A reverse inequality holds with σ replaced by $-\sigma$ for all $\Delta_s \in \Gamma_s$. Note that

$$\begin{aligned}
&\sup_{\tau \in \mathcal{T}} \left|n^{-1/2} \sum_{i=1}^{n} \left\{F_i\left((\theta_{0\tau} + n^{-1/2}\Delta_s)' z_{ri} + \sigma\|n^{-1/2} z_{ri}\|\right) - F_i\left((\theta_{0\tau} + n^{-1/2}\Delta_s)' z_{ri}\right)\right\} z_{rik}^{+}\right| \\
&\quad \le \sigma n^{-1} \sum_{i=1}^{n} C_{1i}\|z_{ri}\| z_{rik}^{+} = \sigma O_P(1),
\end{aligned}$$

where $O_P(1)$ is uniform for all $r \in (0, 1)$ by the definition of z_{ri}, and $\Delta_s \in \Gamma_s$. Consequently

$$\sup_{\tau \in \mathcal{T}} \sup_{r \in \mathcal{A}} \sup_{\|\Delta\| \le M} \left|S_{n,k}^{+}(\tau, r; \Delta)\right| \le \sup_{s \le N(\sigma)} \sup_{\tau \in \mathcal{T}} \sup_{r \in \mathcal{A}} \left|\widetilde{S}_{n,k}^{+}(\tau, r; \Delta_s, \sigma)\right| + \sup_{s \le N(\sigma)} \sup_{\tau \in \mathcal{T}} \sup_{r \in \mathcal{A}} \left|\widetilde{S}_{n,k}^{+}(\tau, r; \Delta_s, -\sigma)\right| + \sigma O_P(1).$$

By the compactness of Γ, the term σ can be made arbitrarily small and $N(\sigma)$ is finite. So we can prove (13) by proving (14).

To show (14), we also use a chaining argument. Let Δ and λ be fixed. Let $N_1 \equiv N_1(n)$ be an integer such that $N_1 = \lceil n^{1/2+d} \rceil + 1$ for some $d \in (0, 1/2)$. We divide the interval $\mathcal{T}$ into N_1 subintervals by points $c_1 = \tau_0 < \tau_1 < \cdots < \tau_{N_1} = 1 - c_1$. The length of each interval is denoted as $\delta^* = (1 - 2c_1)/N_1$. By Assumption A2(i), for all $\tau_i, \tau_j \in \mathcal{T}$ such that $|\tau_i - \tau_j| \le \delta^*$, we have: $\left\|\theta_{0\tau_j} - \theta_{0\tau_i}\right\| \le pC_0|\tau_j - \tau_i| \le pC_0\delta^* \equiv C^*$. By the monotonicity of $1(y_i \le \cdot)$ and $F_i(\cdot)$ and the nonnegativity of z_{rij}^{+}, we have that for all τ with $\tau_s \le \tau \le \tau_{s+1}$,

$$\begin{aligned}&\widetilde{S}^{+}_{n,k}(\tau,r;\Delta,\lambda)-\widetilde{S}^{+}_{n,k}(\tau_{s+1},r;\Delta,\lambda)\\&\quad\le n^{-1/2}\sum_{i=1}^{n}\{F_i((\theta_{0\tau_{s+1}}+n^{-1/2}\Delta)'z_{ri}+\lambda\|n^{-1/2}z_{ri}\|)-F_i((\theta_{0\tau}+n^{-1/2}\Delta)'z_{ri}+\lambda\|n^{-1/2}z_{ri}\|)\}\\&\qquad+n^{-1/2}\sum_{i=1}^{n}\{1(y_i\le\theta'_{0\tau_{s+1}}z_{ri})-F_i(\theta'_{0\tau_{s+1}}z_{ri})-1\,(y_i\le\theta'_{0\tau}z_{ri})+F_i\,(\theta'_{0\tau}z_{ri})\}z^{+}_{rik},\end{aligned}$$

because $\theta'_{0\tau}z_{ri}=\beta'_{0\tau}x_i\le\beta'_{0\tau_{s+1}}x_i=\theta'_{0\tau_{s+1}}z_{ri}$ by the fact that $\beta'_{0\tau}x_i$ is the τth quantile of y_i given $\mathcal{F}_{i-1}$. A reverse inequality holds with $\theta_{0\tau_{s+1}}$ replaced by $\theta_{0\tau_s}$. Therefore,

$$\sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}}\left|\widetilde{S}^{+}_{n,k}(\tau,r;\Delta,\lambda)\right|\le\max_{0\le s\le N_1}\sup_{r\in\mathcal{A}}\left|\widetilde{S}^{+}_{n,k}(\tau_s,r;\Delta,\lambda)\right| \tag{15}$$

$$+\max_{0\le s\le N_1-1}\sup_{r\in\mathcal{A}}n^{-1/2}\left|\sum_{i=1}^{n}\{F_i\,(\theta_{0\tau_{s+1}}+\eta_{n,ri})-F_i\,(\theta_{0\tau_s}+\eta_{n,ri})\}\,z^{+}_{rik}\right| \tag{16}$$

$$+\sup_{\tau_l,\tau_m}\sup_{r\in\mathcal{A}}n^{-1/2}\left|\sum_{i=1}^{n}\left\{1(y_i\le\theta'_{0\tau_l}z_{ri})-F_i(\theta'_{0\tau_l}z_{ri})-1(y_i\le\theta'_{0\tau_m}z_{ri})+F_i(\theta'_{0\tau_m}z_{ri})\right\}z^{+}_{rik}\right|, \tag{17}$$

where $\eta_{n,ri}=n^{-1/2}(\Delta'z_{ri}+\lambda\|z_{ri}\|)$ and $\sup_{\tau_l,\tau_m}$ is an abbreviation for $\sup_{\tau_l,\tau_m\in\mathcal{T}:|\tau_l-\tau_m|\le\delta^*}$. The expression (16) is $o_p(1)$ by a mean valuation expression (see also Lemma 2.1 of Koul (1991)). Under the null hypothesis, the expression (17) is equal to

$$\sup_{\tau_l,\tau_m\in\mathcal{T}:|\tau_l-\tau_m|\le\delta^*}\sup_{r\in\mathcal{A}}\left|n^{-1/2}\sum_{i=1}^{n}\{1(F_i\,(y_i)\le\tau_l)-\tau_l-1\,(F_i\,(y_i)\le\tau_m)+\tau_m\}\,z^{+}_{rik}\right|. \tag{18}$$

Since $z^{+}_{rik}=x^{+}_{ik}$ for all r for $k\le p$ and $z^{+}_{rik}=x^{+}_{i,k-p}1\,(i\ge\lceil nr\rfloor+1)$ for all r for $k=p+1,\ldots,2p$. (18) will be $o_P(1)$ if we can prove

$$\sup_{\tau_l,\tau_m\in\mathcal{T}:|\tau_l-\tau_m|\le\delta^*}\sup_{r\in\mathcal{A}}\left|n^{-1/2}\sum_{i=1}^{\lceil nr\rfloor}x^{+}_{ik}\,\{1(F_i\,(y_i)\le\tau_l)-\tau_l-1\,(F_i\,(y_i)\le\tau_m)+\tau_m\}\right|=o_P(1)\,. \tag{19}$$

Note that $F_i\,(y_i)$ are i.i.d. $U\,(0,1)$ by Diebold et al. (1998). So (19) is ensured due to the stochastic equicontinuity property of the sequential weighted empirical process under Assumptions A1–A5 by Theorem A.1 of Bai (1996). It remains to show that the expression (15) is $o_P(1)$.

Let $\epsilon>0$. Note that

$$P\left(\max_{0\le s\le N_1}\sup_{r\in\mathcal{A}}\left|\overline{S}^{+}_{n,k}(\tau_s,r;\Delta,\lambda)\right|>\epsilon\right)\le(N_1+1)\max_{0\le s\le N_1}P\left(\sup_{r\in\mathcal{A}}\left|\overline{S}^{+}_{n,k}(\tau_s,r;\Delta,\lambda)\right|>\epsilon\right). \tag{20}$$

Recall $z_{ri}=(x'_i,x'_iI_{r,i})'$. Write $\Delta=(\Delta'_1,\Delta'_2)'$, where Δ_1 is a p-vector. Let $\overline{\eta}_{n,1i}=n^{-1/2}(\Delta'_1x_i+\lambda\|x_i\|)$, and $\overline{\eta}_{n,2i}=n^{-1/2}(\Delta'_1x_i+\Delta'_2x_i+\sqrt{2}\lambda\|x_i\|)$. Then $\eta_{n,ri}=\overline{\eta}_{n,1i}$ if $i\le\lceil nr\rfloor$ and $\eta_{n,ri}=\overline{\eta}_{n,2i}$ if $i\ge\lceil nr\rfloor+1$. Let

$$s^{*}_{n,ji}=1(y_i\le\beta'_{0\tau_s}x_i+\overline{\eta}_{n,ji})-F_i(\beta'_{0\tau_s}x_i+\overline{\eta}_{n,ji})-1\left(y_i\le\beta'_{0\tau_s}x_i\right)+F_i(\beta'_{0\tau_s}x_i),\quad j=1,2.$$

We shall bound the probability on the right-hand side of (20) by considering two cases: (a) $k\le p$, and (b) $k=p+1,\ldots,2p$. In the first case, $z^{+}_{rik}=x^{+}_{ik}$ for all r and we have

$$\overline{S}^{+}_{n,k}(\tau_s,r;\Delta,\lambda)=n^{-1/2}\sum_{i=1}^{\lceil nr\rfloor}s^{*}_{n,1i}x^{+}_{ik}+n^{-1/2}\sum_{i=\lceil nr\rfloor+1}^{n}s^{*}_{n,2i}x^{+}_{ik}.$$

Hence

$$P\left(\sup_{r\in\mathcal{A}}\left|\overline{S}^{+}_{n,k}(\tau_s,r;\Delta,\lambda)\right|>\epsilon\right)\le P\left(\sup_{r\in\mathcal{A}}\left|n^{-1/2}\sum_{i=1}^{\lceil nr\rfloor}s^{*}_{n,1i}x^{+}_{ik}\right|>\epsilon/2\right)+P\left(\sup_{r\in\mathcal{A}}\left|n^{-1/2}\sum_{i=\lceil nr\rfloor+1}^{n}s^{*}_{n,2i}x^{+}_{ik}\right|>\epsilon/2\right).$$

Note that $\left\{(s^{*}_{n,1i}x^{+}_{ik},\mathcal{F}_i),1\le i\le n\right\}$ is a martingale difference sequence. By Doob's inequality (e.g., Hall and Heyde (1980, pp. 14–15)),

$$P\left(\sup_{r\in\mathcal{A}}\left|n^{-1/2}\sum_{i=1}^{\lceil nr\rfloor}s^{*}_{n,1i}x^{+}_{ik}\right|>\epsilon/2\right)\le\frac{16}{n^2\epsilon^4}E\left|\sum_{i=1}^{n}s^{*}_{n,1i}\right|^4.$$

By Rosenthal's inequality (e.g., Hall and Heyde (1980, p.23)), we have

$$T_n \equiv E\left|\sum_{i=1}^{n} s^*_{n,1i}x^+_{ik}\right|^4$$
$$\leq C\sum_{i=1}^{n} E\left[\left(s^*_{n,1i}x^+_{ik}\right)^4\right] + CE\left\{\sum_{i=1}^{n} E\left[\left(s^*_{n,1i}x^+_{ik}\right)^2 |\mathcal{F}_{i-1}\right]\right\}^2 \equiv T_{n1} + T_{n2}.$$

It is easy to show that $T_{n1} = O(n^{1/2})$. For T_{n2}, noting that x^+_{ik} is measurable with respect to $\mathcal{F}_{i-1}$ and $E[(s^*_{n,1i})^2|\mathcal{F}_{i-1}] \leq |F_i(\beta'_{0\tau_s}x_i + \overline{\eta}_{n,1i}) - F_i(\beta'_{0\tau_s}x_i)| \leq C_{1i}\overline{\eta}_{n,1i}$, we have $T_{n2} \leq CE[\sum_{i=1}^{n} C_{1i}\overline{\eta}_{n,1i}(x^+_{ik})^2]^2 = O\,(n)$. Consequently, $T_n = O\,(n)$ and

$$P\left(\sup_{r\in\mathcal{A}}\left|n^{-1/2}\sum_{i=1}^{\lceil nr\rfloor} s^*_{n,1i}x^+_{ik}\right| > \epsilon/2\right) = O\left(n^{-1}\right)$$

and this order is free of τ_s. Similarly, we can show that

$$P\left(\sup_{r\in\mathcal{A}}\left|n^{-1/2}\sum_{i=\lceil nr\rfloor+1}^{n} s^*_{n,2i}x^+_{ik}\right| > \epsilon/2\right) = O\left(n^{-1}\right).$$

Hence for $k = 1, \ldots, p$

$$P\left(\max_{0\leq s\leq N_1}\sup_{r\in\mathcal{A}}\left|\overline{S}^+_{n,k}(\tau_s, r; \Delta, \lambda)\right| > \epsilon\right) = O\left(N_1 n^{-1}\right) = o\,(1).$$

In the second case, $z^+_{rik} = x^+_{i,k-p}$ and it is easy to verify that

$$\overline{S}^+_{n,k}(\tau_s, r; \Delta, \lambda) = n^{-1/2}\sum_{i=\lceil nr\rfloor+1}^{n} s^*_{n,2i}x^+_{i,k-p}.$$

One can follow the above arguments and show that for $k = p+1, \ldots, 2p$,

$$P\left(\max_{0\leq s\leq N_1}\sup_{r\in\mathcal{A}}\left|\overline{S}^+_{n,k}(\tau_s, r; \Delta, \lambda)\right| > \epsilon\right) = O\left(N_1 n^{-1}\right) = o\,(1).$$

Consequently, we have shown that expression (15) is $o_p\,(1)$ by Chebyshev's inequality. □

Lemma 2. *Suppose Assumptions* A1–A5 *hold. Then* $\sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}}\sup_{\|\Delta\|\leq M}\|\overline{V}_n(\tau, r; \Delta) - \overline{V}_n(\tau, r; 0) + H_n(\tau, r)\Delta\| = o_P(1)$.

Proof. By Assumption A1 and A2(iii),

$$\sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}}\sup_{\|\Delta\|\leq M}\left\|\overline{V}_n(\tau, r; \Delta) - \overline{V}_n(\tau, r; 0) + H_n(\tau, r)\Delta\right\|$$
$$= \sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}}\sup_{\|\Delta\|\leq M}\left\|n^{-1/2}\sum_{i=1}^{n}\left[F_i\left((\theta_{0\tau} + n^{-1/2}\Delta)'z_{ri}\right) - F_i\left(\theta'_{0\tau}z_{ri}\right)\right]z_{ri} - H_n(\tau, r)\Delta\right\|$$
$$= \sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}}\sup_{\|\Delta\|\leq M}\left\|n^{-1}\sum_{i=1}^{n}\int_0^1\left[f_i\left((\theta_{0\tau} + n^{-1/2}\Delta)'z_{ri}\right) - f_i\left(\theta'_{0\tau}z_{ri}\right)\right]\mathrm{d}s z_{ri}z'_{ri}\Delta\right\|$$
$$\leq \sup_{r\in\mathcal{A}}\sup_{\|\Delta\|\leq M} n^{-1}\sum_{i=1}^{n}\left\|C_{2i}(n^{-1/2}\Delta'z_{ri})z_{ri}z'_{ri}\Delta\right\|$$
$$\leq 2M^2\max_{1\leq i\leq n}(n^{-1/2}\|x_i\|)n^{-1}\sum_{i=1}^{n} C_{2i}\|x_i\|^2 = o_P(1). \quad \square$$

Lemma 3. *Suppose Assumptions* A1–A5 *hold. Then* $\sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}}\|V_n(\tau, r; \widehat{\Delta}_{\tau r})\| = o_p(1)$.

Proof. By the proof of Lemma A2 in Ruppert and Carroll (1980), $\sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}}\|V_n(\tau, r; \widehat{\Delta}_{\tau r})\| = \sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}}\|n^{-1/2}\sum_{i=1}^{n}\psi_\tau(y_i - \widehat{\theta}'_{\tau r}z_{ri})z_{ri}\| \leq \sup_{\tau\in\mathcal{T}}\sup_{r\in\mathcal{A}} n^{-1/2}\sum_{i=1}^{n} 1(y_i - \widehat{\theta}'_{\tau r}z_{ri} = 0)\,\|z_{ri}\| \leq 2\sqrt{2}pn^{-1/2}\max_{1\leq i\leq n}\|x_i\| = o_p(1)$. □

Proof of Theorem 3.2. Let $H^*_n(\tau, r) = n^{-1}\sum_{i=1}^{\lceil nr\rfloor} f_i\,(\beta'_{0\tau}x_i)\,x_ix'_i$ and $J^*_n(\tau, r) = n^{-1/2}\sum_{i=1}^{\lceil nr\rfloor}\psi_\tau\,(y_i - \beta'_{0\tau}x_i)\,x_i = n^{-1/2}\sum_{i=1}^{\lceil nr\rfloor}[\tau - 1\,(\eta_i < \tau)]\,x_i$, where $\eta_i = F_i\,(y_i)$. Then by Theorem 3.1, we have

$$\sqrt{n}\,\widehat{\delta}\,(\tau, r) = \overline{H}^*_n(\tau, r)^{-1}\overline{J}^*_n(\tau, r) - H^*_n(\tau, r)^{-1}J^*_n(\tau, r) + o^*_P(1),$$

where $\overline{H}^*_n(\tau, r) = H^*_n(\tau, 1) - H^*_n(\tau, r), \overline{J}^*_n(\tau, r) = J^*_n(\tau, 1) - J^*_n(\tau, r)$, and $o^*_P(1)$ denotes $o_P(1)$ uniformly in $(\tau, r) \in \mathcal{T}\times\mathcal{A}$. Assumption A6(ii) ensures that $H^*_n(\tau, r) = rH^*(\tau) + o^*_P(1)$. We next show that

$$J^*_n(\tau, r) \Rightarrow Q^{1/2}W^*(\tau, r), \qquad (21)$$

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where $W^*(\tau, r)$ is defined in Theorem 3.2. To show (21), by Theorem 7.1 of Billingsley (1999, p. 80), we need to prove the weak convergence of the finite dimensional distributions and tightness. Noting that $\{J_n^*(\tau, r)\}$ is a sequential weighted empirical process and $\{\eta_i\}$ is an i.i.d. $U(0, 1)$ sequence by Diebold et al. (1998), the tightness follows from Theorem A.1 of Bai (1996). The finite dimensional convergence to a normal distribution follows from the central limit theorem for a martingale difference sequence and the Cramer–Wold device. So it suffices to calculate the covariance matrix of the limiting Gaussian process. Let $r_1 < r_2$. Then by the martingale difference property of $\{((\tau - 1(\eta_i < \tau_1))x_i, \mathcal{F}_i), 1 \le i \le n\}$ and double expectations,

$$E\left\{J_n^*(\tau_1, r_1) J_n^*(\tau_2, r_2)'\right\} = n^{-1} \sum_{i=1}^{\lceil nr_1 \rfloor} E\left\{[\tau_1 - 1(\eta_i < \tau_1)][\tau_2 - 1(\eta_i < \tau_2)] x_i x_i'\right\}$$

$$= (\tau_1 \wedge \tau_2 - \tau_1 \tau_2) n^{-1} \sum_{i=1}^{\lceil nr_1 \rfloor} E\left[x_i x_i'\right] \xrightarrow{P} (\tau_1 \wedge \tau_2 - \tau_1 \tau_2) r_1 Q.$$

Now it is easy to verify that

$$\overline{H}_n^*(\tau, r)^{-1} \overline{J}_n^*(\tau, r) - H_n^*(\tau, r)^{-1} J_n^*(\tau, r) = \frac{1}{r(1-r)} H^*(\tau)^{-1} Q^{1/2} W(\tau, r) + o_P^*(1),$$

where $W(\tau, r) = rW^*(\tau, 1) - W^*(\tau, r)$ is a tied-down Kiefer process with $E[W(\tau, r)] = 0$ and $E[W(\tau_1, r_1) W(\tau_2, r_2)] = (\tau_1 \wedge \tau_2 - \tau_1 \tau_2)(r_1 \wedge r_2 - r_1 r_2) I_p$. The result follows. □

Proof of Theorem 3.3. The result follows from Theorem 3.2, Slutsky's theorem, and the continuous mapping theorem. □

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