

Supplement to
 “Sieve Estimation of Panel Data Models with Cross Section
 Dependence”

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THIS APPENDIX PROVIDES PROOFS FOR TECHNICAL LEMMAS IN THE ABOVE PAPER.

Let C signify a generic constant whose exact value may vary from case to case. Let $\mathcal{D} = \{(x_{it}, f_{1t}, f_{2t}) : i = 1, \dots, n, t = 1, \dots, T\}$. Let $E_{\mathcal{D}}(\cdot)$ and $\text{Var}_{\mathcal{D}}(\cdot)$ denote the conditional expectation and conditional variance given \mathcal{D} , respectively.

A Lemmas used in the proofs for section 4

In this appendix, we prove several lemmas that are used in the proof of the main results in Section 4. Note that $T^{-1}p'_i m_h p_i = T^{-1}p'_i p_i - T^{-1}p'_i h (h'h)^{-1} p_i h$. Recall $Q_{ipp} = E(p_{it} p'_{it})$, $Q_{iph} = E(p_{it} h'_t)$, $Q_{hh} = E(h_t h'_t)$, and $Q_i = Q_{ipp} - Q_{iph} Q_{hh}^{-1} Q'_{iph}$. For notational simplicity, we have suppressed the dependence of Q_i , Q_{ipp} , and Q_{iph} on K, T , or n . The following lemma studies the properties of $p'_i p_i$, $p'_i h$, and $p'_i m_h p_i$.

Lemma A.1 *Suppose Assumptions 1-2 and 3(iv) hold, then*

- (i) $E\|T^{-1}p'_i p_i - Q_{ipp}\|^2 = O(K^2/T)$;
- (ii) $E\|T^{-1}p'_i h - Q_{iph}\|^2 = O(K/T)$;
- (iii) $\|T^{-1}p'_i m_h p_i - Q_i\| = O_p(K/\sqrt{T})$;
- (iv) $\lambda_{\max}(p'_i m_h p_i/T) = \lambda_{\max}(Q_i) + O_p(K/\sqrt{T}) = O_p(1)$;
- (v) $\lambda_{\min}(p'_i m_h p_i/T) \geq \lambda_{\min}(Q_i)/2$ with probability approaching 1 (w.p.a.1).

Proof. (i) Let $Q_{ipp,jk}$ denote the (j, k) th element of Q_{ipp} . Then

$$\begin{aligned} & E\|T^{-1}p'_i p_i - Q_{ipp}\|^2 \\ &= \sum_{j=1}^K \sum_{k=1}^K E \left(\frac{1}{T} \sum_{t=1}^T p_j(x_{it}) p_k(x_{it}) - Q_{ipp,jk} \right)^2 \\ &= \frac{1}{T} \sum_{j=1}^K \sum_{k=1}^K E [p_j(x_{i1}) p_k(x_{i1}) - Q_{ipp,jk}]^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{2}{T} \sum_{j=1}^K \sum_{k=1}^K \sum_{\tau=1}^{T-1} \left(1 - \frac{\tau}{T}\right) \text{Cov} (p_j(x_{i1})p_k(x_{i1}), p_j(x_{i,1+\tau})p_k(x_{i,1+\tau})) \\
& \equiv A_{i1} + 2A_{i2}.
\end{aligned}$$

Noting that $E[p_j(x_{it})p_k(x_{it})] = Q_{ipp,jk}$, by Assumption 3(iv) and the Cauchy-Schwarz inequality

$$\begin{aligned}
A_{i1} & \leq \frac{1}{T} \sum_{j=1}^K \sum_{k=1}^K E\{[p_j(x_{i1})p_k(x_{i1})]^2\} \leq \frac{1}{T} E \left(\sum_{j=1}^K \sum_{k=1}^K \{E[p_j(x_{i1})]^4\}^{1/2} \{E[p_k(x_{i1})]^4\}^{1/2} \right) \\
& \leq CK^2/T = O(K^2/T).
\end{aligned}$$

By Assumptions 1(i)-(ii) and 2(i)-(ii), $\{x_{it}, t \geq 1\}$ is also α -mixing with mixing coefficients $\bar{\alpha}_i(j) \equiv \max\{\alpha_i(j), \alpha_0(j)\}$. Then by Assumption 3(iv) and the Davydov inequality (e.g., Bosq, 1996, pp. 19-20) for α -mixing processes, we have

$$\begin{aligned}
|A_{i2}| & = \frac{1}{T} \sum_{j=1}^K \sum_{k=1}^K \sum_{\tau=1}^{T-1} \left(1 - \frac{\tau}{T}\right) \text{Cov} (p_j(x_{i1})p_k(x_{i1}), p_j(x_{i,1+\tau})p_k(x_{i,1+\tau})) \\
& \leq \frac{c_\eta}{T} \sum_{j=1}^K \sum_{k=1}^K \sum_{\tau=1}^{T-1} \left|1 - \frac{\tau}{T}\right| \bar{\alpha}_i(\tau)^{\eta/(4+\eta)} \left\{E|p_j(x_{i1})p_k(x_{i1})|^{2+\eta/2}\right\}^{4/(4+\eta)} \\
& \leq \frac{c_\eta}{T} \sum_{j=1}^K \sum_{k=1}^K \sum_{\tau=1}^{T-1} \bar{\alpha}_i(\tau)^{\eta/(4+\eta)} \left\{E|p_j(x_{i1})|^{4+\eta} E|p_k(x_{i1})|^{4+\eta}\right\}^{2/(4+\eta)} \\
& \leq \frac{CK^2}{T} \sum_{\tau=1}^{\infty} \bar{\alpha}_i(\tau)^{\eta/(4+\eta)} = O(K^2/T),
\end{aligned}$$

where $c_\eta = 2^{(4+2\eta)/(4+\eta)}(4+\eta)/\eta$. Hence $E\|T^{-1}p'_i p_i - Q_{ipp}\|^2 = O(K^2/T)$.

(ii) The proof is similar to the case of (i) and thus we only sketch it. Let $Q_{iph,jk}$ denote the (j, k) th element of Q_{iph} and $h_{t,k}$ denote the k th element of the $(q_1 + d + 1)$ -vector h_t . By the remark after Assumption 2, h_t has finite $(4 + \eta)$ th moments. Then by Assumption 3(iv), the Cauchy-Schwarz inequality, and the Davydov inequality

$$\begin{aligned}
& E\|T^{-1}p'_i h - Q_{iph}\|^2 \\
& = \sum_{j=1}^K \sum_{k=1}^{q_1+d+1} E \left(\frac{1}{T} \sum_{t=1}^T p_j(x_{it})h_{t,k} - Q_{iph,jk} \right)^2 \\
& = \frac{1}{T} \sum_{j=1}^K \sum_{k=1}^{q_1+d+1} E [p_j(x_{i1})h_{1,k} - Q_{iph,jk}]^2 \\
& \quad + \frac{2}{T} \sum_{j=1}^K \sum_{k=1}^{q_1+d+1} \sum_{\tau=1}^{T-1} \left(1 - \frac{\tau}{T}\right) \text{Cov} (p_j(x_{i1})h_{1,k}, p_j(x_{i,1+\tau})h_{1+\tau,k}) \\
& = O(K/T) + O(K/T) = O(K/T).
\end{aligned}$$

(iii) By repeated use of the triangle inequality and $m_h = I_T - h(h'h)^{-1}h'$, we have

$$\begin{aligned}
\|T^{-1}p'_i m_h p_i - Q_i\| &= \left\| (T^{-1}p'_i p_i - Q_{ipp}) - \left(T^{-1}p'_i h (h'h)^{-1} h' p_i - Q_{iph} Q_{hh}^{-1} Q'_{iph} \right) \right\| \\
&\leq \|T^{-1}p'_i p_i - Q_{ipp}\| + \left\| T^{-1}p'_i h (h'h)^{-1} h' p_i - Q_{iph} Q_{hh}^{-1} Q'_{iph} \right\| \\
&\leq \|T^{-1}p'_i p_i - Q_{ipp}\| + \left\| (T^{-1}p'_i h - Q_{iph}) (h'h)^{-1} h' p_i \right\| \\
&\quad + \|Q_{iph} [(T^{-1}h'h)^{-1} - Q_{hh}^{-1}] T^{-1} h' p_i\| + \|Q_{iph} Q_{hh}^{-1} (T^{-1} h' p_i - Q'_{iph})\| \\
&\equiv A_{i3} + A_{i4} + A_{i5} + A_{i6}.
\end{aligned}$$

By (i) and Assumption 3(iv), $E[A_{i3}] = O(K/\sqrt{T})$, so $A_{i3} = O_p(K/\sqrt{T})$. Noting that $E\|T^{-1}p'_i h\|^2 = O(K)$,

$$\begin{aligned}
A_{i4} &\leq \left\| (T^{-1}p'_i h - Q_{iph}) \right\| \left\| (T^{-1}h'h)^{-1} \right\| \|T^{-1}p'_i h\| \\
&= O_p(\sqrt{K/T}) O_p(1) O_p(\sqrt{K}) = O_p(K/\sqrt{T}).
\end{aligned}$$

Similarly, using the fact that $T^{-1}h'h - Q_{hh} = O_p(T^{-1/2})$ under Assumption 1 and that $\|T^{-1}p'_i h\| = O_p(\sqrt{K})$, we can show that $A_{i5} = O_p(K/\sqrt{T})$ and $A_{i6} = O_p(K/\sqrt{T})$. It follows that $\|T^{-1}p'_i m_h p_i - Q_i\| = O_p(K/\sqrt{T})$.

(iv) By (iii) and Assumption 3(iv),

$$\begin{aligned}
\lambda_{\max}(p'_i m_h p_i / T) &= \max_{\|\varkappa\|=1} \{ \varkappa' Q_i \varkappa + \varkappa' (p'_i m_h p_i / T - Q_i) \varkappa \} \\
&\leq \lambda_{\max}(Q_i) + \|p'_i m_h p_i / T - Q_i\| \\
&= \lambda_{\max}(Q_i) + O_p(K/\sqrt{T}) = O_p(1).
\end{aligned}$$

(v) By (iii) and Assumption 3(iv),

$$\begin{aligned}
\lambda_{\min}(p'_i m_h p_i / T) &= \min_{\|\varkappa\|=1} \{ \varkappa' Q_i \varkappa + \varkappa' (p'_i m_h p_i / T - Q_i) \varkappa \} \\
&\geq \lambda_{\min}(Q_i) - \|p'_i m_h p_i / T - Q_i\| \\
&= \lambda_{\min}(Q_i) - O_p(K/\sqrt{T}) \geq \lambda_{\min}(Q_i) / 2 \text{ w.a.p.1.}
\end{aligned}$$

■

Lemma A.2 Suppose Assumptions 3(i)-(iii) hold, then $T^{-1}E\|\mathbf{g}_i - p_i a_{g_i}\|^2 = O(K^{-2\lambda_i/d})$.

Proof. For a square integrable function $a : \mathcal{X}_i \rightarrow \mathbb{R}$, define the Hilbert norm $\|a\|_2 \equiv \{\int a(x)^2 f_i(x) dx\}^{1/2}$, where recall $f_i(x)$ is the density of x_{it} . By Assumptions 3(i) and (iii), for $g_i(\cdot) \in \Lambda_c^{\lambda_i}(\mathcal{X}_i, \omega_i)$, there exists $\Pi_{\infty K} g_i(\cdot) = \alpha'_{g_i} p^K(\cdot) \in \Lambda_c^{\lambda_i}(\mathcal{X}_i, \omega_i)$ such that for any fixed $\bar{\omega}_i > \omega_i + \lambda_i$ we have

$$\|g_i(\cdot) - \Pi_{\infty K} p^K(\cdot)\|_{\infty, \bar{\omega}_i} = \sup_{x \in \mathcal{X}_i} \left| [g_i(x) - \alpha'_{g_i} p^K(x)] [1 + \|x\|^2]^{-\bar{\omega}_i/2} \right| \leq cK^{-\lambda_i/d}.$$

Then by Assumption 3(iii) with $\bar{\omega}_i = \omega_i + \lambda_i + \epsilon_i$ for a small number $\epsilon_i > 0$,

$$\begin{aligned}
& T^{-1} E \|\mathbf{g}_i - p_i a_{g_i}\|^2 \\
&= T^{-1} \sum_{t=1}^T E [g_i(x_{it}) - \alpha'_{g_i} p^K(x_{it})]^2 = \int [g_i(x) - \alpha'_{g_i} p^K(x)]^2 f_i(x) dx \\
&= \int \left\{ [g_i(x) - \alpha'_{g_i} p^K(x)] [1 + \|x\|^2]^{-\bar{\omega}_i/2} \right\}^2 f_i(x) [1 + \|x\|^2]^{\bar{\omega}_i} dx \\
&\leq \left(\|g_i(\cdot) - \Pi_{\infty K} p^K(\cdot)\|_{\infty, \bar{\omega}_i} \right)^2 \int [1 + \|x\|^2]^{\bar{\omega}_i} f_i(x) dx \leq CK^{-2\lambda_i/d}.
\end{aligned}$$

■

Lemma A.3 Suppose Assumptions 1(i), (v), (vi), and (viii) hold, then $nE[\bar{v}\bar{v}'] \leq CI_T$ and $nE[\bar{\varepsilon}\bar{\varepsilon}'] \leq CI_T$ for some $C < \infty$, where $\bar{v} = (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_T)'$ and $\bar{\varepsilon} = (\bar{\varepsilon}_1, \bar{\varepsilon}_2, \dots, \bar{\varepsilon}_T)'$.

Proof. We only prove the first part since the other case is similar. Without loss of generality, we assume $d = 1$ here. Then by Assumption 1(v), $E(\bar{v}_t \bar{v}'_s) = n^{-2} \sum_{i=1}^n E(v_{it} v_{is})$ for $t, s = 1, 2, \dots, T$. By Assumptions 1(i), (vi), and (viii) and the Davydov inequality, with $\gamma = 4 + \eta$,

$$\begin{aligned}
\sum_{s=1}^T n^{-1} \sum_{i=1}^n |E(v_{it} v_{is})| &\leq \frac{2^{2-2/\gamma} \gamma}{\gamma - 2} \sum_{s=1}^T n^{-1} \sum_{i=1}^n \alpha_i (|t - s|)^{1-2/\gamma} \{E|v_{i1}|^\gamma\}^{2/\gamma} \\
&\leq \frac{2^{2-2/\gamma} \gamma}{\gamma - 2} \max_{1 \leq j \leq n} \{E|v_{j1}|^\gamma\}^{2/\gamma} n^{-1} \sum_{i=1}^n \sum_{\tau=0}^{\infty} \alpha_i (\tau)^{1-2/\gamma} \equiv c_\gamma < \infty.
\end{aligned}$$

Let $C \geq c_\gamma$. Let $a = (a_1, a_2, \dots, a_T)'$ be an arbitrary T -vector. Then

$$\begin{aligned}
a' (CI_T - nE[\bar{v}\bar{v}']) a &= C \sum_{t=1}^T a_t^2 - \sum_{t=1}^T \sum_{s=1}^T a_t a_s n^{-1} \sum_{i=1}^n E(v_{it} v_{is}) \\
&\geq C \sum_{t=1}^T a_t^2 - \frac{1}{2} \sum_{t=1}^T \sum_{s=1}^T (a_t^2 + a_s^2) n^{-1} \sum_{i=1}^n |E(v_{it} v_{is})| \\
&= C \sum_{t=1}^T a_t^2 - \sum_{t=1}^T a_t^2 \sum_{s=1}^T n^{-1} \sum_{i=1}^n |E(v_{it} v_{is})| \\
&\geq C \sum_{t=1}^T a_t^2 - c_\gamma \sum_{t=1}^T a_t^2 \geq 0.
\end{aligned}$$

That is, $CI_T - nE[\bar{v}\bar{v}']$ is positive semidefinite (p.s.d.) for some finite number C . ■

Lemma A.4 Suppose Assumptions 1(iii), (iv) and (vii), 2(i)-(ii), and 3(iv) hold, then $E \|T^{-1} p'_i m_h \varepsilon_i\|^2 = O(K/T)$.

Proof. First, by Assumptions 1(iii)-(iv) and 2(i)-(ii), ε_i is independent of x_{it} , $t = 1, 2, \dots, T$, and thus of p_i . Combined with the fact that $\text{tr}(AB) \leq \lambda_{\max}(A)\text{tr}(B)$ for p.s.d. matrix B and symmetric matrix A and Assumptions 1(vii) and 3(iv), this implies

$$\begin{aligned} E [\varepsilon_i' p_i p_i' \varepsilon_i] &= E[\text{tr}(p_i p_i' E_{\mathcal{D}}(\varepsilon_i \varepsilon_i'))] = E[\text{tr}(p_i p_i' \Psi_{i,T})] \\ &\leq \lambda_{\max}(\Psi_{i,T}) E \|p_i\|^2 = O(KT). \end{aligned}$$

Noting that $\lambda_{\max}(m_h) = 1$, we have

$$\begin{aligned} E \|T^{-1} p_i' m_h \varepsilon_i\|^2 &= T^{-2} E [\text{tr}(\varepsilon_i' m_h p_i p_i' m_h \varepsilon_i)] = T^{-2} E [\text{tr}(m_h p_i p_i' m_h \varepsilon_i \varepsilon_i')] \\ &\leq T^{-2} E [\text{tr}(p_i p_i' m_h \varepsilon_i \varepsilon_i')] = T^{-2} E [\text{tr}(m_h \varepsilon_i \varepsilon_i' p_i p_i')] \\ &\leq T^{-2} E [\text{tr}(\varepsilon_i \varepsilon_i' p_i p_i')] = T^{-2} E [\varepsilon_i' p_i p_i' \varepsilon_i] = O(K/T). \end{aligned}$$

■

Recall $h_t \equiv (f_{1t}', \bar{x}_t', \bar{y}_t)'$. Let $z_t \equiv (f_{1t}', f_{2t}')'$. Then by (2.4) in the text, we have

$$h_t = \Gamma' z_t + \bar{v}_t^*, \quad (\text{A.1})$$

where

$$\Gamma_{(q_1+q_2) \times (q_1+d+1)} = \begin{pmatrix} I_{q_1} & \bar{\Gamma}_1 & \bar{\gamma}_1 \\ 0 & \bar{\Gamma}_2 & \bar{\gamma}_2 \end{pmatrix}, \quad \bar{v}_t^*{}_{1 \times (q_1+d+1)} = \begin{pmatrix} 0' & \bar{v}_t' & \bar{g}_t + \bar{\varepsilon}_t \end{pmatrix}. \quad (\text{A.2})$$

Let $z \equiv (z_1, z_2, \dots, z_T)'$, and $\bar{v}^* \equiv (\bar{v}_1^*, \bar{v}_2^*, \dots, \bar{v}_T^*)'$. We can rewrite (A.1) in matrix form

$$h = z\Gamma + \bar{v}^*. \quad (\text{A.3})$$

The following two lemmas study the approximation error due to the replacement of $z\Gamma$ by h .

Lemma A.5 *Suppose Assumptions 1, 2(i)-(ii) and 3(iv) hold, then*

- (i) $E \|T^{-1} p_i' h - T^{-1} p_i' z\Gamma\| = O(\sqrt{K/n})$;
- (ii) $E \|T^{-1} f_2' h - T^{-1} f_2' z\Gamma\| = O(1/\sqrt{n})$;
- (iii) $E \|T^{-1} (h'h - \Gamma' z' z\Gamma)\| = O(1/\sqrt{n})$.

If in addition Assumption 2(iii) holds, then

- (iv) $m_b f_2 = 0$ w.p.a.1 as $n \rightarrow \infty$, where $b \equiv z\Gamma$ and $m_b \equiv I_T - b(b'b)^{-1} b'$;
- (v) $\|T^{-1} (p_i' m_h f_2 - p_i' m_b f_2) \gamma_{2i}\| = O_p(\sqrt{K/n})$;
- (vi) $\|T^{-1} (p_i' m_h \varepsilon_i - p_i' m_b \varepsilon_i)\| = O_p(\sqrt{K/nT})$.

Proof. (i) By (A.3),

$$T^{-1} p_i' h = T^{-1} p_i' z\Gamma + T^{-1} p_i' \bar{v}^*. \quad (\text{A.4})$$

Recall $\bar{v} = (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_T)'$ and $\bar{\varepsilon} = (\bar{\varepsilon}_1, \bar{\varepsilon}_2, \dots, \bar{\varepsilon}_T)'$. Let $\bar{g} \equiv (\bar{g}_1, \bar{g}_2, \dots, \bar{g}_T)'$. Noting that $\bar{v}_t^* = (0', \bar{v}_t', \bar{g}_t + \bar{\varepsilon}_t)'$, we prove the claim in (i) by showing that

$$E \|T^{-1} p_i' \bar{\zeta}\|^2 = O_p(K/n^2 + K/(nT)) \text{ for } \bar{\zeta} = \bar{v}, \bar{\varepsilon}, \text{ and } E \|T^{-1} p_i' \bar{g}\| = O_p(\sqrt{K/n}). \quad (\text{A.5})$$

By Assumptions 1(iii)-(v) and 2(i)-(ii), we have

$$\begin{aligned}
& E \|T^{-1}p'_i\bar{v}\|^2 \\
&= T^{-2}E [\text{tr}(\bar{v}'p_i p'_i\bar{v})] = T^{-2}E \text{tr} \left(\sum_{t=1}^T \sum_{s=1}^T \bar{v}_t p'_{it} p_{is} \bar{v}'_s \right) \\
&= (nT)^{-2} E \text{tr} \left(\sum_{t=1}^T \sum_{s=1}^T \sum_{j=1}^n \sum_{k=1}^n v_{jt} p'_{it} p_{is} v'_{ks} \right) = (nT)^{-2} \text{tr} \left(\sum_{t=1}^T \sum_{s=1}^T \sum_{j=1}^n E(v_{jt} p'_{it} p_{is} v'_{js}) \right) \\
&= (nT)^{-2} \text{tr} \left(\sum_{t=1}^T \sum_{s=1}^T E(v_{it} p'_{it} p_{is} v'_{is}) \right) + (nT)^{-2} \text{tr} \left(\sum_{t=1}^T \sum_{s=1}^T \sum_{j \neq i}^n E(v_{jt} p'_{it} p_{is} v'_{js}) \right) \\
&\equiv B_{i1} + B_{i2}.
\end{aligned}$$

Under Assumptions 1(i)-(ii) and 2(i)-(ii), $\{(x_{it}, v_{jt}), t \geq 1\}$ is also strong mixing. So we can apply the Cauchy-Schwarz and Davydov inequalities to show that $B_{i1} = O(K/n^2)$ and $B_{i2} = O(K/(nT))$ under Assumptions 1(vi) and 3(iv). Hence $E \|T^{-1}p'_i\bar{v}\|^2 = O(K/n^2 + K/(nT))$. By the same arguments, $E \|T^{-1}p'_i\bar{\varepsilon}\|^2 = O(K/n^2 + K/(nT))$. Now, $E \|T^{-1}p'_i\bar{g}\| \leq \{T^{-1}E \|\bar{g}\|^2\}^{1/2} \{T^{-1}E \|p_i\|^2\}^{1/2} = O(n^{-1/2}) O(K^{1/2}) = O(\sqrt{K/n})$. This proves (i).

(ii) By (A.3), $\|T^{-1}f'_2 h - T^{-1}f'_2 z\Gamma\| \leq T^{-1} \|f'_2 \bar{v}\| + T^{-1} \|f'_2 \bar{\varepsilon}\| + T^{-1} \|f'_2 \bar{g}\|$. Noting that $T^{-2}E \|f'_2 \bar{\zeta}\|^2 = T^{-2}E [\text{tr}(f'_2 E(\bar{\zeta}\bar{\zeta}') f_2)] \leq Cn^{-1}T^{-2}E [\text{tr}(f'_2 f_2)] = O(n^{-1}T^{-1})$ for $\bar{\zeta} = \bar{v}$ and $\bar{\varepsilon}$ by the independence between $(\bar{v}'_t, \bar{\varepsilon}'_t)'$ and f_{2t} and Lemma A.3, we have $T^{-1}E \|f'_2 \bar{v}\| + T^{-1}E \|f'_2 \bar{\varepsilon}\| = O(1/\sqrt{nT})$. $T^{-1}E \|f'_2 \bar{g}\| \leq \{T^{-1}E \|f_2\|^2\}^{1/2} \{T^{-1}E \|\bar{g}\|^2\}^{1/2} = O(1) O(1/\sqrt{n}) = O(1/\sqrt{n})$. It follows that $E \|T^{-1}f'_2 h - T^{-1}f'_2 z\Gamma\| = O(1/\sqrt{n})$.

(iii) By (A.3),

$$\begin{aligned}
T^{-1}(h'h - \Gamma'z'z\Gamma) &= T^{-1}(z\Gamma + \bar{v}^*)'(z\Gamma + \bar{v}^*) - T^{-1}\Gamma'z'z\Gamma \\
&= T^{-1}\bar{v}^*'\bar{v}^* + T^{-1}\Gamma'z'z\bar{v}^* + T^{-1}\bar{v}^*z\Gamma.
\end{aligned} \tag{A.6}$$

Noting that $\bar{v}_t^* = (0', \bar{v}'_t, \bar{g}_t + \bar{\varepsilon}_t)'$, it is straightforward to show that $T^{-1}E \|\bar{v}^*'\bar{v}^*\| = O(n^{-1})$ and $T^{-1}E \|\Gamma'z'z\bar{v}^*\| = O(n^{-1/2})$. Consequently, $E \|T^{-1}(h'h - \Gamma'z'z\Gamma)\| = O(n^{-1/2})$.

(iv) First, note that $\text{rank}(\Gamma) = q_1 + \text{rank}(\bar{\Gamma}_2^*)$. By Assumption 2(iii), the almost sure convergence of $\bar{\Gamma}_2^* \equiv (\bar{\Gamma}_2, \bar{\gamma}_2)$ to $\bar{\Gamma}_2 \equiv (\Gamma_2, \gamma_2)$ as $n \rightarrow \infty$ where $\Gamma_2 \equiv E(\Gamma_{2i})$ and $\gamma_2 \equiv E(\gamma_{2i})$, and Theorem 7.2.2 of Wang, Wei, and Qiao (2004), we have $\text{rank}(\bar{\Gamma}_2^*) = q_2$ w.p.a.1 as $n \rightarrow \infty$. Hence $\text{rank}(\Gamma) = q_1 + q_2 \leq q_1 + d + 1$ w.p.a.1. Let A and B be $m_1 \times m_2$ and $m_2 \times m_3$ matrices, respectively. Then $(AB)^- = B^-A^- = B'(BB')^{-1}(A'A)^{-1}A'$ provided $\text{rank}(A) = \text{rank}(B) = m_2$ (e.g., Bernstein (2005, Fact 6.4.8)). It follows that

$$\begin{aligned}
m_b &= I_T - b(b'b)^{-1}b' = I_T - z\Gamma(\Gamma'z'z\Gamma)^{-1}\Gamma'z' \\
&= I_T - z\Gamma(z'z\Gamma)^-z' = I_T - z(z'z)^-z' \equiv m_z.
\end{aligned}$$

Consequently, $m_b f_2 = m_z f_2 = 0$ a.s. because $f_2 \subset z$.

(v) Let $b_t \equiv \Gamma' z_t$. Noting that $h = b + \bar{v}^*$ and $h'h - b'b = \bar{v}^* \bar{v}^* + b' \bar{v}^* + \bar{v}^* b$, we have

$$\begin{aligned}
& (p'_i m_h f_2 - p'_i m_b f_2) \gamma_{2i} / T \\
&= \frac{1}{T} p'_i \left\{ (b + \bar{v}^*) (h'h)^{-1} (b' + \bar{v}^*) - b (b'b)^{-1} b' \right\} f_2 \gamma_{2i} \\
&= \frac{1}{T} p'_i \bar{v}^* (h'h)^{-1} b' f_2 \gamma_{2i} + \frac{1}{T} p'_i b (h'h)^{-1} \bar{v}^* f_2 \gamma_{2i} \\
&\quad + \frac{1}{T} p'_i \bar{v}^* (h'h)^{-1} \bar{v}^* f_2 \gamma_{2i} - \frac{1}{T} p'_i b (h'h)^{-1} (hh' - b'b) (b'b)^{-1} b' f_2 \gamma_{2i} \\
&= \frac{1}{T} \left\{ p'_i \bar{v}^* (h'h)^{-1} b' f_2 \gamma_{2i} - p'_i b (h'h)^{-1} b' \bar{v}^* (b'b)^{-1} b' f_2 \gamma_{2i} \right\} \\
&\quad + \frac{1}{T} \left\{ p'_i b (h'h)^{-1} \bar{v}^* f_2 \gamma_{2i} - p'_i b (h'h)^{-1} \bar{v}^* b (b'b)^{-1} b' f_2 \gamma_{2i} \right\} \\
&\quad + \frac{1}{T} \left\{ p'_i \bar{v}^* (h'h)^{-1} \bar{v}^* f_2 \gamma_{2i} - p'_i b (h'h)^{-1} \bar{v}^* \bar{v}^* (b'b)^{-1} b' f_2 \gamma_{2i} \right\} \\
&\equiv B_{i3} + B_{i4} + B_{i5}, \text{ say.} \tag{A.7}
\end{aligned}$$

First,

$$B_{i4} = \frac{1}{T} p'_i b (h'h)^{-1} \bar{v}^* m_b f_2 \gamma_{2i} = 0 \text{ w.p.a.1 by (iv).} \tag{A.8}$$

Next, write $B_{i5} = \frac{1}{T} p'_i \bar{v}^* (h'h)^{-1} \bar{v}^* f_2 \gamma_{2i} - \frac{1}{T} p'_i b (h'h)^{-1} \bar{v}^* \bar{v}^* (b'b)^{-1} b' f_2 \gamma_{2i} \equiv B_{i5a} - B_{i5b}$. Noting that $\|T^{-1} p'_i \bar{v}^*\| = O_p(\sqrt{K/n})$ and $(b'b)^{-1} b' f_2 = \bar{\Gamma}_2^{\alpha'}$ where $\bar{\Gamma}_2^\alpha = (0_{q_2 \times q_1}, \bar{\Gamma}_2, \bar{\gamma}_2)$, we have

$$\begin{aligned}
\|B_{i5a}\| &\leq \left[\lambda_{\min} \left(\frac{h'h}{T} \right) \right]^{-1} \left\| \frac{1}{T^2} p'_i \bar{v}^* \bar{v}^* f_2 \gamma_{2i} \right\| \\
&\leq \left[\lambda_{\min} \left(\frac{h'h}{T} \right) \right]^{-1} \left\| \frac{1}{T} p'_i \bar{v}^* \right\| \left\| \frac{1}{T} \bar{v}^* f_2 \gamma_{2i} \right\| \\
&= O_p(1) O_p(\sqrt{K/n}) O_p(\sqrt{1/n}) = O_p(\sqrt{K/n}),
\end{aligned}$$

and similarly

$$\begin{aligned}
\|B_{i5b}\| &\leq \left[\lambda_{\min} \left(\frac{h'h}{T} \right) \right]^{-1} \left\| \frac{p'_i b}{T} \right\| \left\| \frac{\bar{v}^* \bar{v}^* \bar{\Gamma}_2^{\alpha'} \gamma_{2i}}{T} \right\| \\
&= O_p(1) O_p(\sqrt{K}) O_p(1/n) = O_p(\sqrt{K/n}).
\end{aligned}$$

It follows that

$$\|B_{i5}\| = O_p(\sqrt{K/n}). \tag{A.9}$$

Now, write

$$\begin{aligned}
B_{i3} &= \frac{1}{T} \left\{ p'_i \bar{v}^* (h'h)^{-1} b' f_2 \gamma_{2i} - p'_i b (h'h)^{-1} b' \bar{v}^* (b'b)^{-1} b' f_2 \gamma_{2i} \right\} \\
&= \frac{1}{T} \left\{ p'_i \bar{v}^* - p'_i b (b'b)^{-1} b' \bar{v}^* \right\} (b'b)^{-1} b' f_2 \gamma_{2i} + \frac{1}{T} p'_i \bar{v}^* \left\{ (h'h)^{-1} - (b'b)^{-1} \right\} b' f_2 \gamma_{2i} \\
&\quad + \frac{1}{T} p'_i b \left\{ (b'b)^{-1} - (h'h)^{-1} \right\} b' \bar{v}^* (b'b)^{-1} b' f_2 \gamma_{2i} \\
&\equiv B_{i3a} + B_{i3b} + B_{i3c}, \text{ say.} \tag{A.10}
\end{aligned}$$

By (iii) and the fact that $\|T^{-1}p'_i\bar{v}^*\| = O_p(\sqrt{K/n})$,

$$\begin{aligned}\|B_{i3b}\| &\leq \left\| \frac{p'_i\bar{v}^*}{T} \right\| \left\| \left(\frac{h'h}{T} \right)^{-1} - \left(\frac{b'b}{T} \right)^{-1} \right\| \left\| \frac{b'f_2\gamma_{2i}}{T} \right\| \\ &= O_p(\sqrt{K/n})O_p(\sqrt{1/n})O_p(1) = O_p(\sqrt{K/n^2}),\end{aligned}\tag{A.11}$$

and similarly $\|B_{i3c}\| = O_p(\sqrt{K/n^2})$. Combining (A.7)-(A.11) yields

$$\begin{aligned}&(p'_i m_h f_2 - p'_i m_b f_2) \gamma_{2i} / T \\ &= \frac{1}{T} p'_i m_b \bar{v}^* (b'b)^{-1} b' f_2 \gamma_{2i} + O_p(\sqrt{K/n^2}) \\ &= \frac{1}{T} p'_i m_b \bar{v}^* \bar{\Gamma}_2^{a'} \gamma_{2i} + O_p(\sqrt{K/n^2}) \\ &= \frac{1}{T} p'_i m_b \bar{v}^* \Gamma_2^{a'} \gamma_{2i} + O_p(\sqrt{K/n^2})\end{aligned}\tag{A.12}$$

where $\Gamma_2^a = E(\bar{\Gamma}_2^a)$ and the last equality follows from the triangle inequality, the fact that $\bar{\Gamma}_2^a - \Gamma_2^a = O_p(\sqrt{1/n})$, and the fact that $\|T^{-1}p'_i m_b \bar{v}^*\| = O_p(\sqrt{K/n})$. Consequently $\|T^{-1}(p'_i m_h f_2 - p'_i m_b f_2) \gamma_{2i}\| \leq \|T^{-1}p'_i m_b \bar{v}^*\| \|\Gamma_2^{a'} \gamma_{2i}\| + O_p(\sqrt{K/n^2}) = O_p(\sqrt{K/n})$.

(vi) By the triangle inequality,

$$\begin{aligned}&\frac{1}{T} \|p'_i m_h \varepsilon_i - p'_i m_b \varepsilon_i\| \\ &= \frac{1}{T} \|p'_i b (b'b)^{-1} b' \varepsilon_i - p'_i h (h'h)^{-1} h' \varepsilon_i\| \\ &\leq \frac{1}{T} \|(p'_i b - p'_i h) (b'b)^{-1} b' \varepsilon_i\| + \frac{1}{T} \|p'_i h [(b'b)^{-1} - (h'h)^{-1}] b' \varepsilon_i\| \\ &\quad + \frac{1}{T} \|p'_i h (h'h)^{-1} [b' \varepsilon_i - h' \varepsilon_i]\| \\ &\leq \|T^{-1}(p'_i b - p'_i h)\| \|(T^{-1}b'b)^{-1}\| \|T^{-1}b' \varepsilon_i\| \\ &\quad + \|T^{-1}p'_i h\| \|(T^{-1}b'b)^{-1} - (T^{-1}h'h)^{-1}\| \|T^{-1}b' \varepsilon_i\| \\ &\quad + \|T^{-1}p'_i h\| \|(T^{-1}h'h)^{-1}\| \|T^{-1}(b' \varepsilon_i - h' \varepsilon_i)\| \\ &= O_p(\sqrt{K/n})O_p(1)O_p(\sqrt{1/T}) + O_p(\sqrt{K})O_p(\sqrt{1/n})O_p(\sqrt{1/T}) + O_p(\sqrt{K})O_p(1)O_p(1/\sqrt{nT}) \\ &= O_p(\sqrt{K/nT}),\end{aligned}$$

where we have used the fact that $\|T^{-1}b' \varepsilon_i\| = O_p(\sqrt{1/T})$ and $\|T^{-1}(h' \varepsilon_i - b' \varepsilon_i)\|^2 = \|T^{-1}\bar{v}^* \varepsilon_i\|^2 = \|T^{-1}\bar{v} \varepsilon_i\|^2 + \|T^{-1}\bar{g}' \varepsilon_i\|^2 + \|T^{-1}\bar{\varepsilon}' \varepsilon_i\|^2 = O_p(1/(nT))$. ■

Let $\mathbf{h} \equiv \text{diag}(\|h_1\|, \dots, \|h_T\|)$ and $\mathbf{p}_i \equiv \text{diag}(\|p_{i1}\|, \dots, \|p_{iT}\|)$.

Lemma A.6 *Suppose Assumptions 1, 2(i)-(ii), and 3(iv) hold, then*

(i) $E\|T^{-1}(h - z\Gamma)\| = O(\sqrt{1/nT})$, $E\|T^{-1}\mathbf{h}(h - z\Gamma)\| = O(\sqrt{1/nT})$, and $E\|T^{-1}\mathbf{p}_i(h - z\Gamma)\| = O(\sqrt{K/nT})$;

(ii) $\|T^{-1}(m_h f_2 - m_b f_2)\| = O_p(\sqrt{1/nT})$, $\|T^{-1}\mathbf{h}(m_h f_2 - m_b f_2)\| = O_p(\sqrt{1/nT})$, and $\|T^{-1}\mathbf{p}_i(m_h f_2 - m_b f_2)\| = O_p(\sqrt{K/nT})$.

Proof. (i) The proof is similar to but simpler than that of Lemma A.5(i).

(ii) The proof is analogous to that of Lemma A.5(v). ■

The next lemma is used in the proof of Theorem 4.2.

Lemma A.7 *Let $x \in \mathcal{X}_i$ be given and $\|p^K(x)\| > c$ for some constant $c > 0$. Suppose Assumptions 1, 2, 3(iv), and 4 hold. $T_{inT} \equiv A_{inT} p^K(x)' (p'_i m_h p_i)^- p'_i m_h \varepsilon_i \xrightarrow{d} N(0, 1)$.*

Proof. Let $\bar{p}^K(x) = p^K(x) / \|p^K(x)\|$. Then $T_{inT} = T^{-1/2} \bar{p}^K(x)' (p'_i m_h p_i / T)^- p'_i m_h \varepsilon_i / \sqrt{T V_{inT} / \|p^K(x)\|^2}$. Define

$$\begin{aligned} \bar{T}_{inT} &\equiv \frac{T^{-1/2} \bar{p}^K(x)' Q_i^- p'_i m_h \varepsilon_i}{\sqrt{T^{-1} \bar{p}^K(x)' Q_i^- p'_i m_h \Psi_{i,T} m_h p_i Q_i^- \bar{p}^K(x)}}, \text{ and} \\ T_{inT}^* &\equiv \frac{T^{-1/2} \bar{p}^K(x)' Q_i^- p'_i m_b \varepsilon_i}{\sqrt{T^{-1} \bar{p}^K(x)' Q_i^- p'_i m_b \Psi_{i,T} m_b p_i Q_i^- \bar{p}^K(x)}}. \end{aligned}$$

By Assumption 1(vii) and Lemma A.1,

$$\begin{aligned} T V_{inT} / \|p^K(x)\|^2 &= T \bar{p}^K(x)' (p'_i m_h p_i)^- p'_i m_h \Psi_{i,T} m_h p_i (p'_i m_h p_i)^- \bar{p}^K(x) \\ &\geq \lambda_{\min}(\Psi_{i,T}) \bar{p}^K(x)' (p'_i m_h p_i / T)^- \bar{p}^K(x) \\ &\geq \lambda_{\min}(\Psi_{i,T}) [\lambda_{\max}(p'_i m_h p_i / T)]^{-1} \geq C > 0 \text{ w.p.a.1.} \end{aligned}$$

The proof is complete if we can show that

$$T_{inT} - \bar{T}_{inT} = o_p(1), \quad (\text{A.13})$$

$$\bar{T}_{inT} - T_{inT}^* = o_p(1), \text{ and} \quad (\text{A.14})$$

$$T_{inT}^* \xrightarrow{d} N(0, 1). \quad (\text{A.15})$$

First, we prove (A.13) by showing that $\zeta_{1inT} \equiv T V_{inT} / \|p^K(x)\|^2 - T^{-1} \bar{p}^K(x)' Q_i^- p'_i m_h \Psi_{i,T} m_h p_i Q_i^- \bar{p}^K(x) = o_p(1)$ and $\zeta_{2inT} \equiv T^{-1/2} \bar{p}^K(x)' [(p'_i m_h p_i / T)^- - Q_i^-] p'_i m_h \varepsilon_i = o_p(1)$, given the fact that $T V_{inT} / \|p^K(x)\|^2 = O_p(1)$. By the triangle inequality, Assumptions 1(vii) and 3(iv), and Lemma A.1,

$$\begin{aligned} |\zeta_{1inT}| &\leq \left| \bar{p}^K(x)' \left[(p'_i m_h p_i / T)^- - Q_i^- \right] p'_i m_h \Psi_{i,T} m_h p_i (p'_i m_h p_i)^- \bar{p}^K(x) \right| \\ &\quad + \left| T^{-1} \bar{p}^K(x)' Q_i^- p'_i m_h \Psi_{i,T} m_h p_i \left[(p'_i m_h p_i / T)^- - Q_i^- \right] \bar{p}^K(x) \right| \\ &\leq \left\| \left[(p'_i m_h p_i / T)^- - Q_i^- \right] p'_i m_h \Psi_{i,T} m_h p_i (p'_i m_h p_i)^- \right\| \\ &\quad + \left\| T^{-1} Q_i^- p'_i m_h \Psi_{i,T} m_h p_i \left[(p'_i m_h p_i / T)^- - Q_i^- \right] \right\| \\ &\leq \lambda_{\max}(\Psi_{i,T}) \left\| (p'_i m_h p_i / T)^- - Q_i^- \right\| \\ &\quad + \lambda_{\max}(\Psi_{i,T}) \left\| Q_i^- (p'_i m_h p_i / T) \left[(p'_i m_h p_i / T)^- - Q_i^- \right] \right\| \\ &\leq O(1) O_p(K/\sqrt{T}) + O(1) [\lambda_{\min}(Q_i)]^{-1} \lambda_{\max}(p'_i m_h p_i / T) \left\| (p'_i m_h p_i / T)^- - Q_i^- \right\| \\ &= O_p(K/\sqrt{T}) + O(1) O(1) O_p(1) O_p(K/\sqrt{T}) = o_p(1). \end{aligned}$$

By Lemmas A.1 and A.4 and Assumption 4(ii)

$$\begin{aligned} |\zeta_{2inT}| &\leq T^{1/2} \left\| (p'_i m_h p_i / T)^- - Q_i^- \right\| \left\| T^{-1} p'_i m_h \varepsilon_i \right\| \\ &= T^{1/2} O_p(K/\sqrt{T}) O_p(\sqrt{K/T}) = O_p(\sqrt{K^3/T}) = o_p(1). \end{aligned}$$

Thus (A.13) holds.

Similarly, we prove (A.14) by showing that $\zeta_{1inT}^* \equiv T^{-1} \bar{p}^K(x)' Q_i^- (p'_i m_h \Psi_{i,T} m_h p_i - p'_i m_b \Psi_{i,T} m_b p_i) Q_i^- \bar{p}^K(x) = o_p(1)$ and $\zeta_{2inT}^* \equiv T^{-1/2} \bar{p}^K(x)' Q_i^- p'_i (m_h - m_b) \varepsilon_i = o_p(1)$. The first part is analogous to the proof that $\zeta_{1inT} = o_p(1)$ and Lemma A.5(v). For the second part, we apply Lemma A.5(vi) to obtain

$$\begin{aligned} |\zeta_{2inT}^*|^2 &= T^{-1} \text{tr} \left(\bar{p}^K(x)' Q_i^- p'_i (m_h - m_b) \varepsilon_i \varepsilon_i' (m_h - m_b) p'_i Q_i^- \bar{p}^K(x) \right) \\ &\leq [\lambda_{\min}(Q_i)]^{-2} T^{-1} \text{tr} \left(p'_i (m_h - m_b) \varepsilon_i \varepsilon_i' (m_h - m_b) p'_i \right) \\ &= T [\lambda_{\min}(Q_i)]^{-2} \left\| T^{-1} p'_i (m_h - m_b) \varepsilon_i \right\|^2 = O_p(K/n) = o_p(1). \end{aligned}$$

Now we prove (A.15). Let $\bar{\sigma}_{inT}^2 = T^{-1} \bar{p}^K(x)' Q_i^- p'_i m_b \Psi_{i,T} m_b p_i Q_i^- \bar{p}^K(x)$, and $c'_{inT} = \bar{p}^K(x)' Q_i^- p'_i m_b$. Then $T_{inT}^* = T^{-1/2} \sum_{t=1}^T c_{inT,t} \varepsilon_{it} / \bar{\sigma}_{inT}$, where $c_{inT,t}$ denotes the t th element of c_{inT} . Let ι_t denote a $T \times 1$ vector with 1 in the t th place and 0 elsewhere. Then $c_{inT,t} = \bar{p}^K(x)' Q_i^- p'_i m_b \iota_t$. By construction, $E_{\mathcal{D}}(T_{inT}^*) = 0$ and $E_{\mathcal{D}}(T_{inT}^{*2}) = 1$. We now prove the lemma under different conditions specified in Assumption 4(i).

Case 1: $\{\varepsilon_{it}, \mathcal{F}_{it}\}$ is a martingale difference sequence (MDS). By White (2001, Theorem 5.24), we prove that conditional on \mathcal{D} , $T_{inT}^* \xrightarrow{d} N(0, 1)$ by showing that: (i) For every $\varepsilon > 0$, $\bar{\sigma}_{inT}^{-2} T^{-1} \sum_{t=1}^T E_{\mathcal{D}}[c_{inT,t}^2 \varepsilon_{it}^2 \mathbf{1}(c_{inT,t}^2 \varepsilon_{it}^2 > T \varepsilon \bar{\sigma}_{inT}^2)] \xrightarrow{p} 0$; (ii) $T^{-1} \sum_{t=1}^T c_{inT,t}^2 \varepsilon_{it}^2 - \bar{\sigma}_{inT}^2 = o_p(1)$.

First, by Assumptions 1(vii) and 3(iv),

$$\begin{aligned} \bar{\sigma}_{inT}^2 &= T^{-1} \bar{p}^K(x)' Q_i^- p'_i m_b \Psi_{i,T} p'_i m_b p_i Q_i^- \bar{p}^K(x) \\ &\geq \lambda_{\min}(\Psi_{i,T}) \bar{p}^K(x)' Q_i^- (p'_i m_b p_i / T) Q_i^- \bar{p}^K(x) \\ &\geq \lambda_{\min}(\Psi_{i,T}) [\lambda_{\max}(Q_i)]^{-2} \lambda_{\min}(p'_i m_b p_i / T) > C > 0 \text{ for large } (n, T), \end{aligned}$$

where we have used the fact that $\|T^{-1}(p'_i m_b p_i - p'_i m_h p_i)\| = o_p(1)$ and Lemma A.1(v). By the Cauchy-Schwarz and Chebyshev's inequalities, Assumptions 1-2 and 3(iv), and repeated use of the fact that $\text{tr}(AB) \leq \lambda_{\max}(A)\text{tr}(B)$ for symmetric A and p.s.d. B ,

$$\begin{aligned} &\bar{\sigma}_{inT}^{-2} T^{-1} \sum_{t=1}^T E_{\mathcal{D}} [c_{inT,t}^2 \varepsilon_{it}^2 \mathbf{1}(c_{inT,t}^2 \varepsilon_{it}^2 > T \varepsilon \bar{\sigma}_{inT}^2)] \\ &\leq \frac{1}{T \bar{\sigma}_{inT}^2} \sum_{t=1}^T \{E_{\mathcal{D}} [c_{inT,t}^4 \varepsilon_{it}^4]\}^{1/2} \{P(c_{inT,t}^2 \varepsilon_{it}^2 > T \varepsilon \bar{\sigma}_{inT}^2 | \mathcal{D})\}^{1/2} \leq \frac{1}{T^2 \varepsilon \bar{\sigma}_{inT}^4} \sum_{t=1}^T E_{\mathcal{D}} [c_{inT,t}^4 \varepsilon_{it}^4] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{E(\varepsilon_{i1}^4)}{T^2 \varepsilon_{inT}^4} \sum_{t=1}^T \text{tr}(\bar{p}^K(x)' Q_i^- p_i' m_b \iota_t \iota_t' m_b p_i Q_i^- \bar{p}^K(x) \bar{p}^K(x)' Q_i^- p_i' m_b \iota_t \iota_t' m_b p_i Q_i^- \bar{p}^K(x)) \\
&\leq \frac{E(\varepsilon_{i1}^4) [\lambda_{\min}(Q_i)]^{-4}}{T^2 \varepsilon_{inT}^4} \sum_{t=1}^T \text{tr}(p_i' m_b \iota_t \iota_t' m_b p_i p_i' m_b \iota_t \iota_t' m_b p_i) \\
&\leq \frac{E(\varepsilon_{i1}^4) [\lambda_{\min}(Q_i)]^{-4}}{T^2 \varepsilon_{inT}^4} \sum_{t=1}^T \text{tr}(\iota_t p_i p_i' \iota_t \iota_t' p_i p_i' \iota_t) \\
&= \frac{E(\varepsilon_{i1}^4) [\lambda_{\min}(Q_i)]^{-4}}{T^2 \varepsilon_{inT}^4} \sum_{t=1}^T \|p_{it}\|^4 = O(K^2/T) = o(1). \tag{A.16}
\end{aligned}$$

This completes the proof of (i).

Now, let $\varsigma_{inT} = T^{-1} \sum_{t=1}^T c_{inT,t}^2 \varepsilon_{it}^2 - \bar{\sigma}_{inT}^2$. Then $E_{\mathcal{D}}(\varsigma_{inT}) = 0$ and

$$\begin{aligned}
E_{\mathcal{D}}(\varsigma_{inT}^2) &= \frac{1}{T^2} \sum_{t=1}^T \text{Var}_{\mathcal{D}}(c_{inT,t}^2 \varepsilon_{it}^2) + \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1, s \neq t}^T \text{Cov}_{\mathcal{D}}(c_{inT,t}^2 \varepsilon_{it}^2, c_{inT,s}^2 \varepsilon_{is}^2) \\
&= \frac{\text{Var}(\varepsilon_{i1}^2)}{T^2} \sum_{t=1}^T c_{inT,t}^4 + \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1, s \neq t}^T c_{inT,t}^2 c_{inT,s}^2 \text{Cov}(\varepsilon_{it}^2, \varepsilon_{is}^2). \tag{A.17}
\end{aligned}$$

As in the proof of (A.16), the first term of (A.17) is bounded above by $\frac{C}{T^2} \sum_{t=1}^T \|p_{it}\|^4 = O(K^2/T) = o(1)$. By the Davydov inequality, the second term of (A.17) is bounded above by

$$\frac{1}{T^2} \sum_{t=1}^T c_{inT,t}^4 \sum_{s=1, s \neq t}^T \alpha_i (|t-s|)^{\eta/(4+\eta)} \{E|\varepsilon_{it}|^{4+\eta}\}^{4/(4+\eta)} \leq \frac{C}{T^2} \sum_{t=1}^T c_{inT,t}^4 = O(K^2/T) = o(1).$$

Consequently, $\varsigma_{inT} = T^{-1} \sum_{t=1}^T c_{inT,t}^2 \varepsilon_{it}^2 - \bar{\sigma}_{inT}^2 = o_p(1)$ by the conditional Chebyshev inequality.

Case 2: ϕ -mixing with $\phi(j) = O(j^{-1-\epsilon})$ for some $\epsilon > 0$. By White (2001, Theorem 5.20), it suffices to show that for large (n, T) : (i) $E_{\mathcal{D}}[c_{inT,t}^2 \varepsilon_{it}^2]$ is stochastically bounded for all t ; (ii) $\bar{\sigma}_{inT}^2 \equiv \text{Var}_{\mathcal{D}}(T^{-1/2} \sum_{t=1}^T c_{inT,t} \varepsilon_{it}) > c$ for some $c > 0$. (ii) follows from the bound of $\bar{\sigma}_{inT}^2$ in Case 1 where we need not use the MDS property of $\{\varepsilon_{it}, \mathcal{F}_{it}\}$. For (i), noting that $p_i' m_b \iota_t \iota_t' m_b p_i \leq p_{it} p_{it}'$, by Assumption 3(iv) we have

$$\begin{aligned}
E[c_{inT,t}^2 \varepsilon_{it}^2] &= \sigma_i^2 E[\bar{p}^K(x)' Q_i^- p_i' m_b \iota_t \iota_t' m_b p_i Q_i^- \bar{p}^K(x)] \\
&\leq \sigma_i^2 \text{tr}\{\bar{p}^K(x)' Q_i^- E[p_{it} p_{it}'] Q_i^- \bar{p}^K(x)\} \\
&\leq \sigma_i^2 \lambda_{\max}(Q_{ipp}) [\lambda_{\min}(Q_i)]^{-2} \leq C < \infty
\end{aligned}$$

where $\sigma_i^2 = E(\varepsilon_{i1}^2)$.

Case 3: α -mixing with $\lambda_{\max}(E(p_{it} p_{it}' p_{it} p_{it}')) \leq C < \infty$. For a scalar random variable ς , let $\|\varsigma\|_p \equiv \{E|\varsigma|^p\}^{1/p}$. Note that $\sum_{j=1}^{\infty} \alpha(j)^{\frac{2+\eta/2}{(2+\eta/2)-2}} = \sum_{j=1}^{\infty} \alpha(j)^{\frac{4+\eta}{\eta}} < \infty$ by Assumption 1.

By White (2001, Theorem 5.20), it suffices to show that for large (n, T) : (i) $\|c_{inT,t}\varepsilon_{it}\|_{2+\eta/2} \leq C < \infty$ for all t ; (ii) $\bar{\sigma}_{inT}^2 \equiv \text{Var}_{\mathcal{D}} \left(T^{-1/2} \sum_{t=1}^T c_{inT,t}\varepsilon_{it} \right) > c$ for some $c > 0$. The proof of (ii) is identical to that of (ii) in Case 1. So we only prove (i). Noting that $E|\varepsilon_{it}|^{2+\eta/2} = E|\varepsilon_{i1}|^{2+\eta/2} < \infty$ and ε_{it} is independent of $c_{inT,t}$, (i) holds provided $\|c_{inT,t}\|_{2+\eta/2} \leq C < \infty$ for all t and large (n, T) . By the Liapounov's inequality, repeated use of the facts that $\lambda_{\max}(m_b) \leq 1$ and $\text{tr}(AB) \leq \lambda_{\max}(A)\text{tr}(B)$ for symmetric A and p.s.d. B , and Assumption 3(iv),

$$\begin{aligned} & \left(\|c_{inT,t}\|_{2+\eta/2} \right)^4 \leq \left(\|c_{inT,t}\|_4 \right)^4 \\ & \leq E \left\{ \text{tr} \left(Q_i^- p_i' m_b \iota_t \iota_t' m_b p_i Q_i^- Q_i^- p_i' m_b \iota_t \iota_t' m_b p_i Q_i^- \right) \right\} \\ & = [\lambda_{\min}(Q_i)]^{-2} \text{tr} \left\{ E \left(p_i' \iota_t \iota_t' p_i p_i' \iota_t \iota_t' p_i \right) Q_i^- Q_i^- \right\} \\ & \leq [\lambda_{\min}(Q_i)]^{-4} \lambda_{\max} \left\{ E \left(p_{it} p_{it}' p_{it} p_{it}' \right) \right\} \leq C^4 < \infty \text{ for all } t. \end{aligned}$$

That is, $\|c_{inT,t}\|_{2+\eta/2} \leq C < \infty$. This completes the proof. ■

Now, recall $\hat{e}_i = m_h(y_i - \hat{\mathbf{g}}_i)$ and \hat{e}_{it} denotes the t th element of \hat{e}_i . Write

$$\hat{e}_i - \varepsilon_i = m_h(y_i - \hat{\mathbf{g}}_i) - \varepsilon_i = m_h(\mathbf{g}_i - \hat{\mathbf{g}}_i) + m_h f_2 \gamma_{2i} + (m_h \varepsilon_i - \varepsilon_i). \quad (\text{A.18})$$

The following lemma is used in the proof of Theorem 4.3.

Lemma A.8 *Suppose Assumptions 1-4 hold, then*

- (i) $T^{-1} \|\hat{e}_i - \varepsilon_i\|^2 = O_p(K/T + K^{-2\lambda_i/d} + K/n)$;
- (ii) $T^{-1} \|\mathbf{h}(\hat{e}_i - \varepsilon_i)\|^2 = T^{-1} \sum_{t=1}^T \|h_t\|^2 (\hat{e}_{it} - \varepsilon_{it})^2 = O_p(K(K/T + K^{-2\lambda_i/d} + K/n))$;
- (iii) $T^{-1} \|\mathbf{p}_i(\hat{e}_i - \varepsilon_i)\|^2 = T^{-1} \sum_{t=1}^T \|p_{it}\|^2 (\hat{e}_{it} - \varepsilon_{it})^2 = O_p(K^2(K/T + K^{-2\lambda_i/d} + K/n))$.

Proof. (i) By Theorem 4.1 and the Chebyshev inequality, $T^{-1} \|\mathbf{g}_i - \hat{\mathbf{g}}_i\|^2 = O_p(K/T + K^{-2\lambda_i/d} + K/n)$. By Lemmas A.5(iv) and A.6(ii), $T^{-1} \|m_h f_2\|^2 = O_p(1/n)$. Similar to the proof of Lemma A.4, we can show that $T^{-1} \|h(h'h)^{-1} h' \varepsilon_i\|^2 = O_p(1/T + 1/n)$. By (A.18), the triangle and Cauchy-Schwarz inequalities and the fact that $\lambda_{\max}(m_h) = 1$,

$$\begin{aligned} T^{-1} \|\hat{e}_i - \varepsilon_i\|^2 & \leq 3T^{-1} \|m_h(\mathbf{g}_i - \hat{\mathbf{g}}_i)\|^2 + 3T^{-1} \|m_h f_2 \gamma_{2i}\|^2 + 3T^{-1} \|m_h \varepsilon_i - \varepsilon_i\|^2 \\ & \leq 3T^{-1} \|\mathbf{g}_i - \hat{\mathbf{g}}_i\|^2 + 3T^{-1} \|m_h f_2 \gamma_{2i}\|^2 + 3T^{-1} \left\| h(h'h)^{-1} h' \varepsilon_i \right\|^2 \\ & = O_p(K/T + K^{-2\lambda_i/d} + K/n) + O_p(1/n) + O_p(1/T + 1/n) \\ & = O_p(K/T + K^{-2\lambda_i/d} + K/n). \end{aligned} \quad (\text{A.19})$$

(ii) By (A.18) and the triangle and Cauchy-Schwarz inequalities,

$$\begin{aligned} & T^{-1} \|\mathbf{h}(\hat{e}_i - \varepsilon_i)\|^2 \\ & \leq 3T^{-1} \|\mathbf{h} m_h(\mathbf{g}_i - \hat{\mathbf{g}}_i)\|^2 + 3T^{-1} \|\mathbf{h} m_h f_2 \gamma_{2i}\|^2 + 3T^{-1} \|\mathbf{h}(m_h \varepsilon_i - \varepsilon_i)\|^2 \\ & \equiv 3\tau_{1T} + 3\tau_{2T} + 3\tau_{3T}. \end{aligned} \quad (\text{A.20})$$

Let $m_{h,ts}$ denote the (t, s) element of m_h . Then

$$\begin{aligned} \sum_{s=1}^T |m_{h,ts}| &= \sum_{s=1}^T \left| \mathbf{1}(s=t) - h'_s (h'h)^{-1} h_t \right| \leq 1 + T^{-1} \sum_{s=1}^T \left| h'_s (h'h/T)^{-1} h_t \right| \\ &\leq 1 + \left\| (h'h/T)^{-1} \right\| T^{-1} \sum_{s=1}^T \|h_s\| \|h_t\| = 1 + \alpha_{1T} \|h_t\|, \end{aligned} \quad (\text{A.21})$$

where $\mathbf{1}(\cdot)$ is the usual indicator function and

$$\alpha_{1T} \equiv \left\| (h'h/T)^{-1} \right\| T^{-1} \sum_{s=1}^T \|h_s\| = O_p(1). \quad (\text{A.22})$$

Similarly,

$$\begin{aligned} \sum_{s=1}^T \|p_{is} m_{h,ts}\| &= \sum_{s=1}^T \left\| p_{is} \left(\mathbf{1}(s=t) - h'_t (h'h)^{-1} h_s \right) \right\| \\ &\leq \|p_{it}\| + T^{-1} \sum_{s=1}^T \left\| h'_t (h'h/T)^{-1} h_s p'_{is} \right\| \\ &\leq \|p_{it}\| + \left\| (h'h/T)^{-1} \right\| \left\{ T^{-1} \sum_{s=1}^T \|p_{is} h'_s\| \right\} \|h_t\| \\ &= \|p_{it}\| + \alpha_{2T} \|h_t\|, \end{aligned} \quad (\text{A.23})$$

where

$$\alpha_{2T} = \left\| (h'h/T)^{-1} \right\| T^{-1} \sum_{s=1}^T \|p_{is} h'_s\| = O_p(\sqrt{K}). \quad (\text{A.24})$$

Noting that uniformly in s , $\widehat{g}_i(x_{is}) - g_i(x_{is}) = (\widehat{\alpha}_{g_i} - \alpha_{g_i})' p_{is} + (\alpha'_{g_i} p_{is} - g_i(x_{is})) = (\widehat{\alpha}_{g_i} - \alpha_{g_i})' p_{is} + O_p(K^{-\lambda_i/d})$, it follows from Theorem 4.1(i) and the triangle and Cauchy-Schwarz inequalities that

$$\begin{aligned} \tau_{1T} &= T^{-1} \sum_{t=1}^T \|h_t\|^2 \left[\sum_{s=1}^T m_{h,ts} (\widehat{g}_i(x_{is}) - g_i(x_{is})) \right]^2 \\ &\leq T^{-1} \sum_{t=1}^T \|h_t\|^2 \left[\sum_{s=1}^T |m_{h,ts}| \left\{ O_p(K^{-\lambda_i/d}) + (\widehat{\alpha}_{g_i} - \alpha_{g_i})' p_{is} \right\} \right]^2 \\ &\leq O_p(K^{-2\lambda_i/d}) \left\{ T^{-1} \sum_{t=1}^T \|h_t\|^2 \left[\sum_{s=1}^T |m_{h,ts}| \right]^2 \right\} \\ &\quad + 2 \|\widehat{\alpha}_{g_i} - \alpha_{g_i}\|^2 \left\{ T^{-1} \sum_{t=1}^T \|h_t\|^2 \left[\sum_{s=1}^T \|p_{is} m_{h,ts}\| \right]^2 \right\} \\ &= O_p(K^{-2\lambda_i/d}) O_p(1) + O_p(K/T + K^{-2\lambda_i/d} + K/n) O_p(K) \\ &= O_p(K(K/T + K^{-2\lambda_i/d} + K/n)), \end{aligned} \quad (\text{A.25})$$

where the next to last equality follows because by (A.21) and (A.23) and the Markov inequality, one can show

$$\begin{aligned} T^{-1} \sum_{t=1}^T \|h_t\|^2 \left[\sum_{s=1}^T |m_{h,ts}| \right]^2 &= T^{-1} \sum_{t=1}^T \|h_t\|^2 [1 + \alpha_{1T} \|h_t\|]^2 = O_p(1), \text{ and} \\ T^{-1} \sum_{t=1}^T \|h_t\|^2 \left[\sum_{s=1}^T \|p_{is} m_{h,ts}\| \right]^2 &= T^{-1} \sum_{t=1}^T \|h_t\|^2 [\|p_{it}\| + \alpha_{2T} \|h_t\|]^2 = O_p(K). \end{aligned}$$

Next, by Lemmas A.5(iv) and A.6(ii)

$$\begin{aligned} \tau_{2T} &= T^{-1} \|\mathbf{h} m_h f_2 \gamma_{2i}\|^2 = T \|T^{-1} \mathbf{h} (m_h f_2 - m_b f_2)\|^2 \|\gamma_{2i}\|^2 \\ &= O_p(T) O_p(1/(nT)) O_p(1) = O_p(1/n). \end{aligned} \tag{A.26}$$

Now,

$$\begin{aligned} \tau_{3T} &= T^{-1} \|\mathbf{h} (m_h \varepsilon_i - \varepsilon_i)\|^2 = T^{-1} \sum_{t=1}^T \|h_t\|^2 h_t' (h'h)^{-1} h' \varepsilon_i \varepsilon_i' h (h'h)^{-1} h_t \\ &= T^{-1} \sum_{t=1}^T \|h_t\|^2 \left\{ h_t' (T^{-1} h'h)^{-1} \left(T^{-1} \sum_{s=1}^T h_s \varepsilon_{is} \right) \right\}^2 \\ &\leq \left\| T^{-1} \sum_{t=1}^T \|h_t\|^2 h_t' \right\| \left\| (T^{-1} h'h)^{-1} \right\| \left\| T^{-1} \sum_{s=1}^T h_s \varepsilon_{is} \right\|^2 \\ &= O_p(1/T + 1/n). \end{aligned} \tag{A.27}$$

Combining (A.20), (A.25), (A.26), and (A.27) yields

$$T^{-1} \|\mathbf{h} (\hat{e}_i - \varepsilon_i)\|^2 = O_p \left(K(K/T + K^{-2\lambda_i/d} + K/n) \right). \tag{A.28}$$

(iii) The proof is analogous to that of (ii) and is thus omitted. ■

B Lemmas used in the proofs for section 5

Here we prove several lemmas that are used in the proof of Theorems 5.1-5.3.

Lemma B.1 *Suppose the conditions in Lemma A.1 and Assumption 5 hold, then*

- (i) $\|(nT)^{-1} P' M_h P - \bar{Q}_n\| = o_p(K^{-1/2})$;
- (ii) $\lambda_{\max}(P' M_h P / (nT)) = \lambda_{\max}(\bar{Q}_n) + o_p(K^{-1/2}) = O_p(1)$;
- (iii) $\lambda_{\min}(P' M_h P / (nT)) \geq \lambda_{\min}(\bar{Q}_n) / 2$ w.p.a.1.

Proof. We first prove (i). Noting that M_h is block diagonal, we have

$$\begin{aligned} \|(nT)^{-1} P' M_h P - \bar{Q}_n\| &= n^{-1} \left\| \sum_{i=1}^n (T^{-1} p_i' m_h p_i - Q_i) \right\| \\ &\leq n^{-1} \sum_{i=1}^n \|T^{-1} p_i' m_h p_i - Q_i\| = \sum_{s=3}^6 A_s, \end{aligned}$$

where $A_s = n^{-1} \sum_{i=1}^n A_{is}$ and A_{is} , $s = 3, 4, 5, 6$, are defined in the proof of Lemma A.1(iii). For A_3 , by the proof of Lemma A.1(i) and Assumption 5(iv),

$$E[A_3] = n^{-1} \sum_{i=1}^n E[A_{i3}] \leq \max_{1 \leq i \leq n} E \|T^{-1} p'_i p_i - Q_{ipp}\| = O(K/\sqrt{T}).$$

For A_4 , we have

$$A_4 = n^{-1} \sum_{i=1}^n \left\| (T^{-1} p'_i h - Q_{iph}) (h'h)^{-1} h' p_i \right\| \leq \left\| (T^{-1} h'h)^{-1} \right\| \bar{A}_4$$

where $\bar{A}_4 \equiv n^{-1} \sum_{i=1}^n \left\| (T^{-1} p'_i h - Q_{iph}) \right\| \left\| T^{-1} p'_i h \right\|$. By the Cauchy-Schwarz inequality, the proof of Lemma A.1(ii), Assumptions 1(vi) and 5(iv),

$$\begin{aligned} E[\bar{A}_4] &\leq n^{-1} \sum_{i=1}^n \left\{ E \left\| (T^{-1} p'_i h - Q_{iph}) \right\|^2 E \left\| T^{-1} p'_i h \right\|^2 \right\}^{1/2} \\ &\leq \left\{ \max_{1 \leq i \leq n} E \left\| (T^{-1} p'_i h - Q_{iph}) \right\|^2 \right\}^{1/2} \left\{ \max_{1 \leq i \leq n} E \left\| T^{-1} p'_i h \right\|^2 \right\}^{1/2} \\ &= O(\sqrt{K/T}) O(\sqrt{K}) = O(K/\sqrt{T}). \end{aligned}$$

It follows that $A_4 = O_p(K/\sqrt{T})$ by the Markov inequality and the fact that $\left\| (T^{-1} h'h)^{-1} \right\| = O_p(1)$. Next, $A_5 \leq \left\| (T^{-1} h'h)^{-1} - Q_{hh}^{-1} \right\| \max_{1 \leq i \leq n} \|Q_{iph}\| \bar{A}_5$, where $\bar{A}_5 \equiv n^{-1} \sum_{i=1}^n \left\| T^{-1} p'_i h \right\|$. $E[\bar{A}_5] \leq \max_{1 \leq i \leq n} E \left\| T^{-1} p'_i h \right\| = O(\sqrt{K})$, implying that $\bar{A}_5 = O_p(\sqrt{K})$. So $A_5 = O_p(T^{-1/2}) O(\sqrt{K}) O_p(\sqrt{K}) = O_p(K/\sqrt{T})$. Similarly, we can show that $A_6 = O_p(K/\sqrt{T})$. It follows that $n^{-1} \sum_{i=1}^n \left\| T^{-1} p'_i m_h p_i - Q_i \right\| = O_p(K/\sqrt{T}) = o_p(K^{-1/2})$ provided $K^3/T \rightarrow 0$ as $T \rightarrow \infty$. Thus we have proved (i) for the case where $T \rightarrow \infty$.

If T is fixed, we follow the proof of Lemma A.1(iii) and prove (i) by showing that

$$\left\| n^{-1} \sum_{i=1}^n (T^{-1} p'_i p_i - Q_{ipp}) \right\| = o_p(K^{-1/2}) \text{ and } \left\| n^{-1} \sum_{i=1}^n (T^{-1} p'_i h - Q_{iph}) \right\| = o_p(K^{-1/2}).$$

The first part is ensured by Assumption 5(iii). For the second part, we have

$$\begin{aligned} &\left\| n^{-1} \sum_{i=1}^n (T^{-1} p'_i h - Q_{iph}) \right\| \\ &= \left\| (nT)^{-1} \sum_{i=1}^n [p'_i b - E(p'_i b)] + (nT)^{-1} \sum_{i=1}^n [(p'_i h - p'_i b) - E(p'_i h - p'_i b)] \right\| \\ &\leq \left\| (nT)^{-1} \sum_{i=1}^n [p'_i b - E(p'_i b)] \right\| + \left\| (nT)^{-1} \sum_{i=1}^n [p'_i \bar{v}^* - E(p'_i \bar{v}^*)] \right\| \end{aligned}$$

The first term on the right hand side of the last expression is $o_p(K^{-1/2})$ by Assumption 5(iii). The expectation of the second term is no bigger than $2n^{-1} \sum_{i=1}^n E |T^{-1} p'_i \bar{v}^*|$. So $\left\| n^{-1} \sum_{i=1}^n (T^{-1} p'_i h - Q_{iph}) \right\| = o_p(K^{-1/2})$ if we can show $\max_{1 \leq i \leq n} E |T^{-1} p'_i \bar{v}^*| = o(K^{-1/2})$.

This follows from the proof of Lemma A.5(i), Assumptions 1(vi) and 5(iv), and the fact that $K^2/n = o(1)$. It follows that $\|n^{-1} \sum_{i=1}^n (T^{-1}p'_i h - Q_{iph})\| = o_p(K^{-1/2})$.

The proof of (ii) and (iii) is analogous to that of Lemmas A.1(iv)-(v), now using (i) instead. ■

Lemma B.2 *Suppose Assumptions 3(i)-(iii) hold with $g_i(\cdot)$ replaced by $g(\cdot)$, then $(nT)^{-1} E\|\mathbf{g} - Pa_g\|^2 = O(K^{-2\lambda/d})$ where $\mathbf{g} = (\mathbf{g}'_1, \dots, \mathbf{g}'_n)'$, $\mathbf{g}_i = (g(x_{i1}), \dots, g(x_{iT}))'$, and $\lambda = \min_{1 \leq i \leq n} \lambda_i$.*

Proof. The proof is analogous to that of Lemma A.2 and thus is omitted. ■

Lemma B.3 *Suppose Assumptions 1(iii), (iv) and (vii), 2(i)-(ii), and 3(iv) hold, then $E\|(NT)^{-1} P'M_h \varepsilon\|^2 = O(K/(nT))$.*

Proof. The proof is analogous to that of Lemma A.4 and thus is omitted. The main difference is that we now require that $\lambda_{\max}(\Psi_{nT})$ be $O(1)$. The latter condition is satisfied because Ψ_{nT} is block diagonal by Assumption 1(v), and $\max_{1 \leq i \leq n} \lambda_{\max}(\Psi_{i,T}) \leq \bar{c}_\Psi < \infty$ by Assumption 5(iv). ■

Lemma B.4 *Let $M_b \equiv I_n \otimes m_b$. Suppose Assumptions 1, 2, and 3(iv) hold and $\lambda_{\max}(E(\gamma_2 \gamma_2')) = O(r_n)$, then $\|(P'M_h F_2 - P'M_b F_2) \gamma_2\| / \sqrt{nT} = O_p(\sqrt{r_n K/n} + \sqrt{KT/n})$.*

Proof. By the proof of Lemma A.5(v),

$$\begin{aligned} \frac{1}{\sqrt{nT}} \|(P'M_h F_2 - P'M_b F_2) \gamma_2\| &= \sqrt{\frac{T}{n}} \left\| \sum_{i=1}^n T^{-1} (p'_i m_h f_2 - p'_i m_b f_2) \gamma_{2i} \right\| \\ &= \|B_{3a} + B_{3b} + B_{3c} + B_{5a} - B_{5b}\|, \end{aligned}$$

where $B_{3l} = \sqrt{T/n} \sum_{i=1}^n B_{i3l}$ for $l = a, b, c$, $B_{5l} = \sqrt{T/n} \sum_{i=1}^n B_{i5l}$ for $l = a, b$, and B_{i3a} , B_{i3b} , B_{i3c} , B_{i5a} , and B_{i5b} are defined in the proof of Lemma A.5(v).

Define the block diagonal matrices $\bar{V}^* \equiv I_n \otimes \bar{v}^*$ and $\bar{\Gamma}_2 \equiv I_n \otimes \Gamma_2^{a'}$. Then

$$\begin{aligned} E_{\mathcal{D}} \|B_{3a}\|^2 &= \frac{1}{nT} E_{\mathcal{D}} \left\| \sum_{i=1}^n p'_i m_b \bar{v}^* \Gamma_2^{a'} \gamma_{2i} \right\|^2 \\ &= \frac{1}{nT} E_{\mathcal{D}} \left\| P'M_b \bar{V}^* \bar{\Gamma}_2 \gamma_2 \right\|^2 = \frac{1}{nT} \text{tr} \left(P'M_b \bar{V}^* \bar{\Gamma}_2 E(\gamma_2 \gamma_2') \bar{\Gamma}_2' \bar{V}^{*'} M_b P \right) \\ &\leq \frac{1}{nT} \text{tr}(P'M_b \bar{V}^* \bar{V}^{*'} M_b P) \lambda_{\max}(E(\gamma_2 \gamma_2')) \lambda_{\max}(\bar{\Gamma}_2 \bar{\Gamma}_2') \\ &= \frac{1}{nT} O_p(KT) O(r_n) O(1) = O_p(r_n K/n). \end{aligned}$$

It follows that $\|B_{3a}\| = O_p(\sqrt{r_n K/n})$. For B_{3b} , we have

$$\|B_{3b}\| \leq \sqrt{\frac{T}{n}} \sum_{i=1}^n \|A_{i3b}\| \leq \left\| \left(\frac{h'h}{T} \right)^{-1} - \left(\frac{b'b}{T} \right)^{-1} \right\| \bar{B}_{3b}$$

where $\overline{B}_{3b} = \sqrt{T/n} \sum_{i=1}^n \|T^{-1}p'_i \overline{v}^*\| \|T^{-1}b' f_2 \gamma_{2i}\|$. Noting that $\|T^{-1}h'h - T^{-1}b'b\| = O_p(1/\sqrt{n})$, and $E|\overline{B}_{3b}| \leq \sqrt{nT} \max_{1 \leq i \leq n} \{E\|T^{-1}p'_i \overline{v}^*\|^2\}^{1/2} \{E\|T^{-1}b' f_2 \gamma_{2i}\|^2\}^{1/2} = \sqrt{nT} O(\sqrt{K/n}) O(1) = O(\sqrt{KT})$, we have $\|B_{3b}\| = O_p(\sqrt{KT/n})$. Similarly, we can show that $\|B_{3c}\| = O_p(\sqrt{KT/n})$.

For B_{5a} , we have

$$\|B_{5a}\| \leq [\lambda_{\min}(T^{-1}h'h)]^{-1} \overline{B}_{5a}$$

where $\overline{B}_{5a} = \sqrt{T/n} \sum_{i=1}^n \|T^{-2}p'_i \overline{v}^* \overline{v}^{*'} f_2 \gamma_{2i}\|$. Noting that

$$\begin{aligned} E\|\overline{B}_{5a}\| &\leq \sqrt{nT} \max_{1 \leq i \leq n} \left\{ E\|T^{-1}p'_i \overline{v}^*\|^2 \right\}^{1/2} \left\{ E\|T^{-1}\overline{v}^{*'} f_2 \gamma_{2i}\|^2 \right\}^{1/2} \\ &= \sqrt{nT} O(\sqrt{K/n}) O(1/\sqrt{n}) = O(\sqrt{KT/n}), \end{aligned}$$

we have $B_{5a} = O_p(\sqrt{KT/n})$. By the same token, $B_{5b} = O_p(\sqrt{KT/n})$. Consequently $\|(P'M_h F_2 - P'M_b F_2)\gamma_2\|/\sqrt{nT} = O_p(\sqrt{r_n K/n} + \sqrt{KT/n})$. ■

Lemma B.5 *Suppose the conditions in Theorem 5.2 hold, then $T_{nT} \equiv A_{nT} p^K(x)' (P'M_h P)^- P'M_h \varepsilon \xrightarrow{d} N(0, 1)$.*

Proof. Recall $A_{nT} = V_{nT}^{-1/2}$ where $V_{nT} = p^K(x)' (P'M_h P)^- P'M_h \Psi_{nT} M_h P (P'M_h P)^- p^K(x)$ and $\Psi_{nT} = E(\varepsilon \varepsilon')$. Let $\overline{p}^K(x) = p^K(x) / \|p^K(x)\|$. Then $T_{nT} = (nT)^{1/2} \overline{p}^K(x)' (P'M_h P)^- P'M_h \varepsilon / \sqrt{nT V_{nT} / \|p^K(x)\|^2}$. Define

$$\begin{aligned} \overline{T}_{nT} &\equiv \frac{(nT)^{-1/2} \overline{p}^K(x)' \overline{Q}_n^{-1} P'M_h \varepsilon}{\sqrt{(nT)^{-1} \overline{p}^K(x)' \overline{Q}_n^{-1} P'M_h \Psi_{nT} M_h P \overline{Q}_n^{-1} \overline{p}^K(x)}}, \text{ and} \\ T_{nT}^* &\equiv \frac{(nT)^{-1/2} \overline{p}^K(x)' \overline{Q}_n^{-1} P'M_b \varepsilon}{\sqrt{(nT)^{-1} \overline{p}^K(x)' \overline{Q}_n^{-1} P'M_b \Psi_{nT} M_b P \overline{Q}_n^{-1} \overline{p}^K(x)}}. \end{aligned}$$

By Assumption 5(iv) and Lemma B.1,

$$\begin{aligned} nT V_{nT} / \|p^K(x)\|^2 &= nT p^K(x)' (P'M_h P)^- P'M_h \Psi_{nT} M_h P (P'M_h P)^- p^K(x) \\ &\geq \lambda_{\min}(\Psi_{nT}) \overline{p}^K(x)' (P'M_h P / (nT))^- \overline{p}^K(x) \\ &\geq \lambda_{\min}(\Psi_{nT}) [\lambda_{\max}(P'M_h P / (nT))]^{-1} \geq C > 0 \text{ w.p.a.1.} \end{aligned}$$

The proof is complete if we can show that $T_{nT} - \overline{T}_{nT} = o_p(1)$, $\overline{T}_{nT} - T_{nT}^* = o_p(1)$, and $T_{nT}^* \xrightarrow{d} N(0, 1)$. The proof of the first two parts are analogous to that of (A.13) and (A.14). So we only prove the third part by showing that conditional on \mathcal{D} , $T_{nT}^* \xrightarrow{d} N(0, 1)$.

Let $A_{nT}^* = ((nT)^{-1} \overline{p}^K(x)' \overline{Q}_n^{-1} P'M_b \Psi_{nT} M_b P \overline{Q}_n^{-1} \overline{p}^K(x))^{-1/2}$, $c'_i = (nT)^{-1/2} A_{nT}^* p^K(x)' \overline{Q}_n^{-1} p'_i m_b$, and $\xi_{ni} = c'_i \varepsilon_i$, where we have suppressed the dependence of c_i on (n, T) for notational simplicity. Then

$$T_{nT}^* = (nT)^{-1/2} A_{nT}^* p^K(x)' \overline{Q}_n^{-1} P'M_b \varepsilon = \sum_{i=1}^n c'_i \varepsilon_i = \sum_{i=1}^n \xi_{ni}.$$

Clearly $E_{\mathcal{D}}(\xi_{ni}) = 0$, $\sigma_{ni}^2 \equiv \text{Var}_{\mathcal{D}}(\xi_{ni}) = c_i' \text{Var}(\varepsilon_i) c_i$, and

$$\begin{aligned} \sum_{i=1}^n \sigma_{ni}^2 &= (nT)^{-1} A_{nT}^{*2} p^K(x)' \overline{Q}_n^{-1} \sum_{i=1}^n p_i' m_b \text{Var}(\varepsilon_i) m_b p_i' \overline{Q}_n^{-1} p^K(x) \\ &= (nT)^{-1} A_{nT}^{*2} p^K(x)' \overline{Q}_n^{-1} P' M_b \Psi_{nT} M_b P' \overline{Q}_n^{-1} p^K(x) = 1. \end{aligned}$$

To check the Liapounov condition, let c_{it} denote the t th element in c_i . Then

$$\begin{aligned} E_{\mathcal{D}}[\xi_{ni}^4] &= \sum_{t_1=1}^T \sum_{t_2=1}^T \sum_{t_3=1}^T \sum_{t_4=1}^T c_{it_1} c_{it_2} c_{it_3} c_{it_4} E[\varepsilon_{it_1} \varepsilon_{it_2} \varepsilon_{it_3} \varepsilon_{it_4}] \\ &= \sum_{t=1}^T c_{it}^4 E[\varepsilon_{it}^4] + 3 \sum_{t_2 \neq t_1} c_{it_1}^2 c_{it_2}^2 E[\varepsilon_{it_1}^2 \varepsilon_{it_2}^2] \\ &\quad + 4 \sum_{t_2 \neq t_1} c_{it_1}^3 c_{it_2} E[\varepsilon_{it_1}^3 \varepsilon_{it_2}] + 6 \sum_{t_3 \neq t_2 \neq t_1} c_{it_1}^2 c_{it_2} c_{it_3} E[\varepsilon_{it_1}^2 \varepsilon_{it_2} \varepsilon_{it_3}] \\ &\quad + \sum_{t_4 \neq t_3 \neq t_2 \neq t_1} c_{it_1} c_{it_2} c_{it_3} c_{it_4} E[\varepsilon_{it_1} \varepsilon_{it_2} \varepsilon_{it_3} \varepsilon_{it_4}] \\ &\equiv \xi_{ni1} + 3\xi_{ni2} + 4\xi_{ni3} + 6\xi_{ni4} + \xi_{ni5}, \end{aligned} \tag{B.1}$$

where, for example, $\sum_{t_4 \neq t_3 \neq t_2 \neq t_1} \sum_{t_1=1}^T \sum_{t_2 \neq t_1}^T \sum_{t_3 \neq t_1, t_3 \neq t_2}^T \sum_{t_4 \neq t_1, t_4 \neq t_2, t_4 \neq t_3}^T$

Let $m_{b,st}$ denote the (s, t) element of m_b . Then

$$|c_{it}| = \left| (nT)^{-1/2} A_{nT}^* p^K(x)' \overline{Q}_n^{-1} \sum_{s=1}^T p_{is} m_{b,st} \right| \leq \{A_1\}^{1/2} \{A_{2it}\}^{1/2}, \tag{B.2}$$

where $A_1 \equiv A_{nT}^{*2} p^K(x)' \overline{Q}_n^{-1} p^K(x)$ and $A_{2it} \equiv (nT)^{-1} \sum_{s=1}^T p_{is}' m_{b,st} \overline{Q}_n^{-1} \sum_{r=1}^T p_{ir} m_{b,rt}$. By the idempotence of M_b and Assumptions 1(v) and (vii),

$$\begin{aligned} A_1 &= A_{nT}^{*2} p^K(x)' \overline{Q}_n^{-1} \overline{Q}_n \overline{Q}_n^{-1} p^K(x) = A_{nT}^{*2} p^K(x)' \overline{Q}_n^{-1} (P' M_b P / (nT)) \overline{Q}_n^{-1} p^K(x) + o_p(1) \\ &\leq A_{nT}^{*2} p^K(x)' \overline{Q}_n^{-1} (P' M_b \Psi_{nT} M_b P / (nT)) \overline{Q}_n^{-1} p^K(x) / \lambda_{\min}(\Psi_{nT}) + o_p(1) \\ &= 1 / \lambda_{\min}(\Psi_{nT}) + o_p(1) = O_p(1). \end{aligned} \tag{B.3}$$

Next by the symmetry of m_b and arguments similar to (A.23),

$$\begin{aligned} A_{2it} &\leq (nT)^{-1} [\lambda_{\min}(\overline{Q}_n)]^{-1} \left[\sum_{s=1}^T \|m_{b,ts} p_{is}\| \right]^2 \\ &\leq (nT)^{-1} [\lambda_{\min}(\overline{Q}_n)]^{-1} [\|p_{it}\| + \beta_{2T} \|b_t\|]^2. \end{aligned}$$

where

$$\beta_{2T} = \| (b'b/T)^{-1} \| T^{-1} \sum_{s=1}^T \| p_{is} b_s' \| = O_p(\sqrt{K}) \tag{B.4}$$

(c.f. (A.24)). Hence $A_{2it} \leq \alpha_n^2 (\|p_{it}\| + \beta_{2T} \|b_t\|)^2$, where by Assumption 5 and Lemma B.1(iii),

$$\alpha_n \equiv (nT)^{-1/2} [\lambda_{\min}(\bar{Q}_n)]^{-1/2} = O_p((nT)^{-1/2}). \quad (\text{B.5})$$

Combining this result with (B.2) and (B.3) yields

$$c_{it} \leq \alpha_n (\|p_{it}\| + \beta_{2T} \|b_t\|). \quad (\text{B.6})$$

It follows from Assumption 3(iv), the remarks after Assumption 2, (B.5), and (B.4) that

$$\begin{aligned} \sum_{i=1}^n \sum_{t=1}^T c_{it}^4 &= \alpha_n^4 \sum_{i=1}^n \sum_{t=1}^T (\|p_{it}\| + \alpha_{2T} \|b_t\|)^4 \leq 8\alpha_n^4 \sum_{i=1}^n \sum_{t=1}^T (\|p_{it}\|^4 + \beta_{2T}^4 \|b_t\|^4) \\ &= O_p(n^{-2}T^{-2}) O_p(nTK^2) = O_p(n^{-1}T^{-1}K^2). \end{aligned} \quad (\text{B.7})$$

Similarly,

$$\sum_{i=1}^n \sum_{t=1}^T c_{it}^2 = \alpha_n^2 \sum_{i=1}^n \sum_{t=1}^T (\|p_{it}\| + \beta_{2T} \|b_t\|)^2 = O_p(n^{-1}T^{-1}) O_p(nTK) = O_p(K). \quad (\text{B.8})$$

By (B.7),

$$\sum_{i=1}^n \xi_{ni1} \leq \max_{1 \leq i \leq n} E(\varepsilon_{i1}^4) \sum_{i=1}^n \sum_{t=1}^T c_{it}^4 = O_p(n^{-1}T^{-1}K^2). \quad (\text{B.9})$$

By the Cauchy-Schwarz and Jensen inequalities and (B.7),

$$\sum_{i=1}^n \left[\sum_{t=1}^T c_{it}^2 \right]^2 = T^2 \sum_{i=1}^n \left[\frac{1}{T} \sum_{t=1}^T c_{it}^2 \right]^2 \leq T \sum_{i=1}^n \sum_{t=1}^T c_{it}^4$$

$$\sum_{i=1}^n \xi_{ni2} \leq \max_{1 \leq i \leq n} E(\varepsilon_{i1}^4) \sum_{i=1}^n \left[\sum_{t=1}^T c_{it}^2 \right]^2 \leq \max_{1 \leq i \leq n} E(\varepsilon_{i1}^4) T \sum_{i=1}^n \sum_{t=1}^T c_{it}^4 = O_p(n^{-1}K^2). \quad (\text{B.10})$$

Similarly, we can show that

$$\sum_{i=1}^n \xi_{ni3} = O_p(n^{-1}K^2). \quad (\text{B.11})$$

Next, noting that $c_{it_1}^2 c_{it_2} c_{it_3} \leq \frac{1}{2} c_{it_1}^2 (c_{it_2}^2 + c_{it_3}^2) \leq \frac{1}{4} (2c_{it_1}^4 + c_{it_2}^4 + c_{it_3}^4)$, we have

$$\begin{aligned} \xi_{ni4} &= \sum_{t_3 \neq t_2 \neq t_1} \sum_{i=1}^n \sum_{t=1}^T c_{it_1}^2 c_{it_2} c_{it_3} E[\varepsilon_{it_1}^2 \varepsilon_{it_2} \varepsilon_{it_3}] \\ &\leq \frac{1}{4} \sum_{t_3 \neq t_2 \neq t_1} \sum_{i=1}^n \sum_{t=1}^T (2c_{it_1}^4 + c_{it_2}^4 + c_{it_3}^4) |E[\varepsilon_{it_1}^2 \varepsilon_{it_2} \varepsilon_{it_3}]| \\ &= \frac{1}{2} \sum_{t_3 \neq t_2 \neq t_1} \sum_{i=1}^n \sum_{t=1}^T c_{it_1}^4 |E[\varepsilon_{it_1}^2 \varepsilon_{it_2} \varepsilon_{it_3}]| + \frac{1}{2} \sum_{t_3 \neq t_2 \neq t_1} \sum_{i=1}^n \sum_{t=1}^T c_{it_2}^4 |E[\varepsilon_{it_1}^2 \varepsilon_{it_2} \varepsilon_{it_3}]| \\ &\equiv \frac{1}{2} \xi_{ni41} + \frac{1}{2} \xi_{ni42}. \end{aligned}$$

For the first term in the last expression, we have

$$\begin{aligned}\xi_{ni41} &= 2 \sum_{1 \leq t_1 < t_2 < t_3 \leq T} \sum_{i=1}^n c_{it_1}^4 |E[\varepsilon_{it_1}^2 \varepsilon_{it_2} \varepsilon_{it_3}]| + 2 \sum_{1 \leq t_2 < t_1 < t_3 \leq T} \sum_{i=1}^n c_{it_1}^4 |E[\varepsilon_{it_1}^2 \varepsilon_{it_2} \varepsilon_{it_3}]| \\ &+ 2 \sum_{1 \leq t_2 < t_3 < t_1 \leq T} \sum_{i=1}^n c_{it_1}^4 |E[\varepsilon_{it_1}^2 \varepsilon_{it_2} \varepsilon_{it_3}]| \equiv 2\xi_{ni41a} + 2\xi_{ni41b} + 2\xi_{ni41c}.\end{aligned}$$

By the Davydov inequality,

$$\begin{aligned}\sum_{i=1}^n \xi_{ni41a} &= \sum_{i=1}^n \sum_{t_1=1}^{T-2} c_{it_1}^4 \sum_{t_2=t_1+1}^{T-1} \sum_{t_3=t_2+1}^T |\text{Cov}(\varepsilon_{it_1}^2 \varepsilon_{it_2}, \varepsilon_{it_3})| \\ &\leq C \sum_{i=1}^n \sum_{t_1=1}^{T-2} c_{it_1}^4 \sum_{t_2=t_1+1}^{T-1} \sum_{t_3=t_2+1}^T \alpha_i (t_3 - t_2)^{\eta/(4+\eta)} \left\{ E|\varepsilon_{i1}|^{4+\eta} \right\}^{4/(4+\eta)} \\ &\leq CT \left\{ \sum_{i=1}^n \sum_{t=1}^T c_{it}^4 \right\} \left\{ \sum_{\tau=1}^{\infty} \alpha(\tau)^{\eta/(4+\eta)} \right\} = O(T) O_p(n^{-1} T^{-1} K^2) O(1) = O_p(n^{-1} K^2).\end{aligned}$$

Similarly, one can show that $\sum_{i=1}^n \xi_{ni41b}$ and $\sum_{i=1}^n \xi_{ni41c}$ are of the same order as $\sum_{i=1}^n \xi_{ni4a}$ and hence $\sum_{i=1}^n \xi_{ni41} = O_p(n^{-1} K^2)$. Analogous arguments show that $\sum_{i=1}^n \xi_{ni42} = O_p(n^{-1} K^2)$. It follows that

$$\sum_{i=1}^n \xi_{ni4} = O_p(n^{-1} K^2). \quad (\text{B.12})$$

To analyze ξ_{ni5} , noting that $|c_{it_1} c_{it_2} c_{it_3} c_{it_4}| \leq \frac{1}{4}(c_{it_1}^2 + c_{it_2}^2)(c_{it_3}^2 + c_{it_4}^2) \leq \frac{1}{4}(c_{it_1}^4 + c_{it_2}^4 + c_{it_3}^4 + c_{it_4}^4)$, we have

$$\begin{aligned}\xi_{ni5} &= \sum_{t_4 \neq t_3 \neq t_2 \neq t_1} \sum_{i=1}^n c_{it_1} c_{it_2} c_{it_3} c_{it_4} E[\varepsilon_{it_1} \varepsilon_{it_2} \varepsilon_{it_3} \varepsilon_{it_4}] \\ &\leq \frac{1}{4} \sum_{t_4 \neq t_3 \neq t_2 \neq t_1} \sum_{i=1}^n (c_{it_1}^4 + c_{it_2}^4 + c_{it_3}^4 + c_{it_4}^4) |E[\varepsilon_{it_1} \varepsilon_{it_2} \varepsilon_{it_3} \varepsilon_{it_4}]| \\ &\simeq \sum_{1 \leq t_1 < t_2 < t_3 < t_4 \leq T} \sum_{i=1}^n c_{it_1}^4 |E[\varepsilon_{it_1} \varepsilon_{it_2} \varepsilon_{it_3} \varepsilon_{it_4}]|,\end{aligned} \quad (\text{B.13})$$

where $A \simeq B$ denotes that A and B are of the same probability order. Let $1 \leq t_1 < t_2 < t_3 < t_4 \leq T$. Let d_c denote the c -th largest difference among $t_{j+1} - t_j$, $j = 1, 2, 3$. Define

$$\xi_i(t_1, \dots, t_4) = \varepsilon_{it_1} \varepsilon_{it_2} \varepsilon_{it_3} \varepsilon_{it_4}.$$

For any $1 \leq j \leq 3$, put $P_0^{(4)}(E^{(4)}) = P((\varepsilon_{it_1}, \dots, \varepsilon_{it_4}) \in E^{(4)})$, and

$$P_j^{(4)}(E^{(j)} \times E^{(4-j)}) = P((\varepsilon_{it_1}, \dots, \varepsilon_{it_j}) \in E^{(j)}) P((\varepsilon_{i,t_{j+1}}, \dots, \varepsilon_{it_4}) \in E^{(4-j)}),$$

where $E^{(j)}$ is a Borel set in \mathbb{R}^j and P is the probability law associated with $\varepsilon_{it_1}, \dots, \varepsilon_{it_4}$. (Note we suppress the dependence of P on i .) By Assumption 1 and the Jensen inequality, for any $0 \leq j \leq 3$,

$$\int |\xi_i(t_1, \dots, t_4)|^{1+\eta/4} dP_j^{(4)} \leq E|\varepsilon_{it}|^{4+\eta} \leq \bar{\mu}_{4+\eta}.$$

By Lemma B.6 below with $\tilde{\eta} = \eta/4$,

$$|E[\xi_i(t_1, \dots, t_4)]| \leq \begin{cases} c_{1\eta} \alpha_i(t_2 - t_1)^{\eta/(4+\eta)} & \text{if } t_2 - t_1 = d_1 \\ c_{1\eta} \alpha_i(t_4 - t_3)^{\eta/(4+\eta)} & \text{if } t_4 - t_3 = d_1 \end{cases}$$

where $c_{1\eta} \equiv 4(\bar{\mu}_{4+\eta})^{4/(4+\eta)}$. If $t_3 - t_2 = d_1$, using Lemma B.6 three times yields

$$\begin{aligned} |E[\xi_i(t_1, \dots, t_4)]| &\leq |E(\varepsilon_{it_1} \varepsilon_{it_2}) E(\varepsilon_{it_4} \varepsilon_{it_3})| + c_{1\eta} \alpha_i(t_3 - t_2)^{\eta/(4+\eta)} \\ &\leq c_{2\eta}^2 \alpha_i(t_2 - t_1)^{\eta/(4+\eta)} \alpha_i(t_4 - t_3)^{\eta/(4+\eta)} + c_{1\eta} \alpha_i(t_3 - t_2)^{\eta/(4+\eta)} \end{aligned} \quad (\text{B.14})$$

where $c_{2\eta} \equiv 4(\bar{\mu}_{2+\eta/2})^{4/(4+\eta)}$.

Therefore

$$\begin{aligned} &\sum_{i=1}^n \sum_{\substack{1 \leq t_1 < \dots < t_4 \leq T \\ t_2 - t_1 = d_1}} c_{it_1}^4 |E[\xi(t_1, \dots, t_4)]| \\ &\leq c_{1\eta} \sum_{i=1}^n \sum_{t_1=1}^{T-3} \sum_{t_2=t_1+\max_{j \geq 3} \{t_j - t_{j-1}\}}^{T-2} \sum_{t_3=t_2+1}^{T-1} \sum_{t_4=t_3+1}^T c_{it_1}^4 \alpha_i(t_2 - t_1)^{\eta/(4+\eta)} \\ &\leq c_{1\eta} \sum_{i=1}^n \sum_{t_1=1}^{T-3} c_{it_1}^4 \sum_{t_2=t_1}^{T-2} (t_2 - t_1)^2 \alpha_i^{\frac{\eta}{4+\eta}}(t_2 - t_1) \\ &\leq c_{1\eta} \left\{ \sum_{i=1}^n \sum_{t_1=1}^T c_{it_1}^4 \right\} \left\{ \sum_{\tau=1}^{\infty} j^2 \alpha^{\frac{\eta}{4+\eta}}(\tau) \right\} = O_p(n^{-1} T^{-1} K^2) O(1) = O_p(n^{-1} T^{-1} K^2). \end{aligned} \quad (\text{B.15})$$

Similarly,

$$\begin{aligned} &\sum_{i=1}^n \sum_{\substack{1 \leq t_1 < \dots < t_4 \leq T \\ t_4 - t_3 = d_1}} c_{it_1}^4 |E[\xi(t_1, \dots, t_4)]| \\ &\leq C \sum_{i=1}^n \sum_{t_1=1}^{T-3} c_{it_1}^4 \sum_{t_2=t_1+1}^{T-2} \sum_{t_3=t_2+1}^T \sum_{t_4=t_3+\max_{j \geq 3} \{t_j - t_{j-1}\}}^{T-1} \alpha_i(t_4 - t_3)^{\eta/(4+\eta)} \\ &\leq C \sum_{i=1}^n \sum_{t_1=1}^{T-3} c_{it_1}^4 \sum_{t_4=t_3+1}^T (t_4 - t_3)^2 \alpha_i(t_4 - t_3)^{\eta/(4+\eta)} = O_p(n^{-1} T^{-1} K^2). \end{aligned} \quad (\text{B.16})$$

By (B.14),

$$\begin{aligned} &\sum_{i=1}^n \sum_{\substack{1 \leq t_1 < \dots < t_4 \leq T \\ t_3 - t_2 = d_1}} c_{it_1}^4 |E[\xi(t_1, \dots, t_4)]| \\ &\leq c_{1\eta} \sum_{i=1}^n \sum_{t_1=1}^{T-3} c_{it_1}^4 \sum_{t_2=t_1+1}^{T-2} \sum_{t_3=t_2+\max\{t_2-t_1, t_4-t_3\}}^{T-1} \sum_{t_4=t_3+1}^T \alpha_i(t_3 - t_2)^{\eta/(4+\eta)} \\ &\quad + c_{2\eta}^2 \sum_{i=1}^n \sum_{t_1=1}^{T-3} c_{it_1}^4 \sum_{t_2=t_1+1}^{T-2} \sum_{t_3=t_2+\max\{t_2-t_1, t_4-t_3\}}^{T-1} \sum_{t_4=t_3+1}^T \alpha_i(t_2 - t_1)^{\eta/(4+\eta)} \alpha_i(t_4 - t_3)^{\eta/(4+\eta)} \\ &\equiv E_1 + E_2, \text{ say.} \end{aligned}$$

By the analysis similar to that of the cases where $t_2 - t_1 = d_1$ or $t_4 - t_3 = d_1$, the first term E_1 in the last expression is $O_p(n^{-1}T^{-1}K^2)$. For the second term, without loss of generality (w.l.o.g.) we assume that $t_2 - t_1 = d_2$ (which implies that $t_4 - t_3 \leq d_2$ because $t_3 - t_2 = d_1$). It follows that

$$\begin{aligned}
E_2 &= c_{2\eta}^2 \sum_{i=1}^n \sum_{t_1=1}^{T-3} c_{it_1}^4 \sum_{t_2=t_1+1}^{T-2} \sum_{t_3=t_2+\max\{t_2-t_1, t_4-t_3\}}^{T-1} \sum_{t_4=t_3+1}^T \alpha_i(t_2-t_1)^{\eta/(4+\eta)} \alpha_i(t_4-t_3)^{\eta/(4+\eta)} \\
&\leq c_{2\eta}^2 \sum_{i=1}^n \sum_{t_1=1}^{T-3} c_{it_1}^4 \sum_{t_2=t_1+1}^{T-2} \alpha(t_2-t_1)^{\eta/(4+\eta)} \sum_{t_3=1}^{T-1} \sum_{t_4=t_3+1}^T \alpha(t_4-t_3)^{\eta/(4+\eta)} \\
&\leq c_{2\eta}^2 \left\{ T \sum_{i=1}^n \sum_{t=1}^T c_{it}^4 \right\} \left\{ \sum_{\tau=1}^{\infty} \alpha(\tau)^{\eta/(4+\eta)} \right\}^2 = O_p(n^{-1}K^2)O(1) = O_p(n^{-1}K^2).
\end{aligned}$$

Hence

$$\sum_{i=1}^n \sum_{\substack{1 \leq t_1 < \dots < t_4 \leq T \\ t_3 - t_2 = d_1}} c_{it_1}^4 |E[\xi_i(t_1, \dots, t_4)]| = O_p(n^{-1}K^2). \quad (\text{B.17})$$

Combining (B.13), (B.15), (B.16) and (B.17) yields

$$\sum_{i=1}^n \xi_{ni5} = O_p(n^{-1}K^2). \quad (\text{B.18})$$

Then by (B.1), (B.9), (B.10), (B.11), (B.12) and (B.18) we have

$$\sum_{i=1}^n E_{\mathcal{D}}[\xi_{ni}^4] = O_p(n^{-1}K^2) = o_p(1). \quad (\text{B.19})$$

This verifies the Liapounov condition. Then by the Liapounov CLT, we have $T_{nT} \xrightarrow{d} N(0, 1)$.

■

Lemma B.6 *Let $\{W_t, t \geq 1\}$ be an α -mixing process with mixing coefficients $\alpha(j)$. For any integer $q > 1$ and integers (t_1, \dots, t_q) such that $1 \leq t_1 < t_2 < \dots < t_q$, let b be a Borel measurable function such that $\max\{\int |b(w_1, \dots, w_q)|^{1+\tilde{\eta}} dF(w_1, \dots, w_q), \int |b(w_1, \dots, w_q)|^{1+\tilde{\eta}} dF^{(1)}(w_1, \dots, w_j) dF^{(2)}(w_{j+1}, \dots, w_q)\} \leq M$ for some $\tilde{\eta} > 0$ and $M > 0$, where F , $F^{(1)}$ and $F^{(2)}$ are the distribution functions of $(W_{t_1}, \dots, W_{t_q})$, $(W_{t_1}, \dots, W_{t_j})$ and $(W_{t_{j+1}}, \dots, W_{t_q})$, respectively. Then $|\int b(w_1, \dots, w_q) dF(w_1, \dots, w_q) - \int b(w_1, \dots, w_q) dF^{(1)}(w_1, \dots, w_j) dF^{(2)}(w_{j+1}, \dots, w_q)| \leq 4M^{1/(1+\tilde{\eta})} \times \alpha(t_{j+1} - t_j)^{\tilde{\eta}/(1+\tilde{\eta})}$.*

Proof. See Lemma 2.1 of Sun and Chiang (1997). ■

C Proof of equation (4.5) in the text

Recall $\tilde{V}_{inT} = \phi^{K'}(p'_i m_h p_i)^- p'_i m_h \text{Var}(\varepsilon_i) m_h p_i (p'_i m_h p_i)^- \phi^K$ and $\tilde{A}_{inT} = \tilde{V}_{inT}^{-1/2}$. Since $\lambda_{\min}(\text{Var}(\varepsilon_i)) > c_1 > 0$ for some c_1 by Assumption 1(vii), $\text{Var}(\varepsilon_i) \geq c_1 I_T$. By Assumption

3(iv) and Lemma A.1, $(p'_i m_h p_i / T)^- \geq c_2 I_K$ for large (n, T) . Then by the assumption that $\lambda_{\min}(\phi^{K'} \phi^K) > c_0 > 0$ for some c_0 , we have

$$\begin{aligned} \lambda_{\min}(T \tilde{V}_{inT}) &= \lambda_{\min}\left(T \phi^{K'} (p'_i m_h p_i)^- p'_i m_h \text{Var}(\varepsilon_i) m_h p_i (p'_i m_h p_i)^- \phi^K\right) \\ &\geq c_1 \lambda_{\min}\left(\phi^K (p'_i m_h p_i / T)^- \phi^K\right) \\ &\geq c_1 c_2 \lambda_{\min}(\phi^K \phi^K) \geq c_1 c_2 c_0 > 0 \text{ for large } (n, T). \end{aligned} \quad (\text{C.1})$$

Consider any $a \in \mathbb{R}^d$ with $\|a\| = 1$. Let $g_{iK}(\cdot) \equiv p^K(\cdot)' \alpha_{g_i}$ and $g_{iK}^r(\cdot)$ be the remainder function from approximating g_i by $g_{iK} : g_{iK}^r = g_i - g_{iK}$. Noting that $\Phi(g_i) = \Phi(g_{iK}) + \Phi(g_{iK}^r)$, we can write

$$\begin{aligned} &a' \tilde{A}_{inT} [\Phi(\hat{g}_i) - \Phi(g_i)] \\ &= a' \tilde{A}_{inT} \phi^{K'} (\hat{\alpha}_{g_i} - \alpha_{g_i}) - a' \tilde{A}_{inT} \Phi(g_{iK}^r) \\ &= a' \tilde{A}_{inT} \phi^{K'} (p'_i m_h p_i)^- p'_i m_h \varepsilon_i + a' \tilde{A}_{inT} \phi^{K'} (p'_i m_h p_i)^- p'_i m_h f_2 \gamma_{2i} \\ &\quad + a' \tilde{A}_{inT} \phi^{K'} (p'_i m_h p_i)^- p'_i m_h (\mathbf{g}_i - p_i \alpha_{g_i}) + a' \tilde{A}_{inT} \Phi(g_{iK}^r) \\ &\equiv \tilde{E}_{inT,1} + \tilde{E}_{inT,2} + \tilde{E}_{inT,3} + \tilde{E}_{inT,4}. \end{aligned} \quad (\text{C.2})$$

It suffices to show that $\tilde{E}_{inT,1} \xrightarrow{d} N(0, 1)$ and $\tilde{E}_{inT,s} = o_p(1)$ for $s = 2, 3, 4$.

We first show $\tilde{E}_{inT,2} \xrightarrow{p} 0$ as $(n, T) \rightarrow \infty$. By the Cauchy-Schwarz inequality, Assumptions 1(vii), 3(iv) and 4(ii), Lemmas A.1(v) and A.5(iv)-(v),

$$\begin{aligned} \left| \tilde{E}_{inT,2} \right| &= |a' \tilde{A}_{inT} \phi^{K'} (p'_i m_h p_i)^- p'_i m_h f_2 \gamma_{2i}| \\ &\leq \left[a' \tilde{A}_{inT} \phi^{K'} (p'_i m_h p_i)^- \phi^K \tilde{A}_{inT} a \right]^{1/2} \left[\gamma'_{2i} f'_2 m_h p_i (p'_i m_h p_i)^- p'_i m_h f_2 \gamma_{2i} \right]^{1/2} \\ &\leq \left[a' \tilde{A}_{inT} \phi^{K'} (p'_i m_h p_i)^- p'_i m_h \Psi_{i,T} m_h p_i (p'_i m_h p_i)^- \phi^K \tilde{A}_{inT} a / \lambda_{\min}(\Psi_{i,T}) \right]^{1/2} \\ &\quad \times \left[\gamma'_{2i} f'_2 m_h p_i (p'_i m_h p_i)^- p'_i m_h f_2 \gamma_{2i} \right]^{1/2} \\ &\leq [1 / \lambda_{\min}(\Psi_{i,T})]^{1/2} [\lambda_{\min}(p'_i m_h p_i / T)]^{-1/2} \left\{ \|p'_i m_h f_2 \gamma_{2i}\| / \sqrt{T} \right\} \\ &= O(1) O_p(1) O_p(\sqrt{KT/n}) = o_p(1). \end{aligned}$$

Similarly, by the Cauchy-Schwarz inequality, Assumptions 1(vii), 3(iv) and 4(ii), Lemmas A.1(v) and A.2,

$$\begin{aligned} \left| \tilde{E}_{inT,3} \right| &= |a' \tilde{A}_{inT} \phi^{K'} (p'_i m_h p_i)^- p'_i m_h (\mathbf{g}_i - p_i \alpha_{g_i})| \\ &\leq \left[a' \tilde{A}_{inT} \phi^{K'} (p'_i m_h p_i)^- \phi^K \tilde{A}_{inT} a \right]^{1/2} [(\mathbf{g}_i - p_i \alpha_{g_i})' (\mathbf{g}_i - p_i \alpha_{g_i})]^{1/2} \\ &\leq [1 / \lambda_{\min}(\Psi_{i,T})]^{1/2} \|\mathbf{g}_i - p_i \alpha_{g_i}\| \\ &\leq O(1) O_p(\sqrt{T} K^{-\lambda_i/d}) = o_p(1). \end{aligned}$$

By Assumption D, $|\tilde{E}_{inT,4}| = a' \tilde{A}_{inT} \Phi(g_{iK}^r) \leq \lambda_{\max}(\tilde{A}_{inT}) \|\Phi(g_{iK}^r)\| \leq [\lambda_{\min}(T \tilde{V}_{inT})]^{-1/2} \tilde{C} \sqrt{T} \|g_{iK}^r\|_s = O_p(1) o(1) = o_p(1)$.

To prove $\tilde{E}_{inT,1} \xrightarrow{d} N(0, 1)$, let $\bar{\phi}_a^K \equiv \phi^K a / \|\phi^K a\|$ and $\tilde{A}_{inT,a} \equiv \tilde{V}_{inT,a}^{-1/2}$, where $\tilde{V}_{inT,a} = a' \tilde{V}_{inT} a / \|\phi^K a\|^2$. Then it is equivalent to showing that $\tilde{A}_{inT,a} \bar{\phi}_a^{K'} (p_i' m_h p_i)^{-1} p_i' m_h \varepsilon_i \xrightarrow{d} N(0, 1)$. The proof of Lemma A.7 continues to work when we replace $\bar{p}^K(x)$ and A_{inT} by $\bar{\phi}_a^K$ and $\tilde{A}_{inT,a}$, respectively. This completes the proof. ■

Additional References

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