

Supplement to “Nonparametric Regression under Location Shifts”

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THIS APPENDIX PROVIDES PROOFS FOR TECHNICAL LEMMAS IN THE ABOVE PAPER.

Lemma 5.1 *Let $f_u(\cdot)$ be the p.d.f. of $\{u_i\}$. Suppose A2-A3 hold. Then*

$$\int u \left[L_n (2m)^{-1} \sum_{\alpha=-m}^m f(u, x - \mu_\alpha) - f_u(u) \right] du = O(L_n^2/m^2 + L_n^{-v}).$$

Proof. The trapezoidal rule approximates $\int_a^b g(p) dp$ by $\frac{1}{2}(b-a)(g(a) + g(b))$ and the approximation error is given by the Newton-Cotes formula (e.g., Stoer and Bulirsch (1993, p. 162))

$$\frac{1}{2}(b-a)(g(a) + g(b)) - \int_a^b g(p) dp = \frac{1}{12}(b-a)^3 g''(c), \quad (5.1)$$

where the second order derivative $g''(\cdot)$ of $g(\cdot)$ is continuous and $c \in (a, b)$. Let $\beta_n = 2m/L_n$. Write

$$\begin{aligned} & \frac{L_n}{2m} \sum_{\alpha=-m}^m f(u, x - \mu_\alpha) - f_u(u) \\ = & \left\{ \frac{1}{\beta_n} \sum_{\alpha=-m}^m f(u, x - \mu_\alpha) - \int_{-L_n/2}^{L_n/2} f(u, x - p) dp \right\} \\ & + \left\{ \int_{-L_n/2}^{L_n/2} f(u, x - p) dp - \int_{-\infty}^{\infty} f(u, x - p) dp \right\} \\ = & \left\{ \sum_{\alpha=-m}^m \frac{1}{2} \left[f\left(u, x - \frac{\alpha-1}{\beta_n}\right) + f\left(u, x - \frac{\alpha}{\beta_n}\right) \right] \frac{1}{\beta_n} - \int_{-L_n/2}^{L_n/2} f(u, x - p) dp \right\} \\ & + \left\{ \int_{-L_n/2}^{L_n/2} f(u, x - p) dp - \int_{-\infty}^{\infty} f(u, x - p) dp \right\} \\ & + \frac{1}{2\beta_n} \left\{ f\left(u, x - \frac{L_n}{2}\right) - f\left(u, x + \frac{(m+1)L_n}{2m}\right) \right\} \equiv I_{1m}(u) + I_{2m}(u) + I_{3m}(u). \end{aligned}$$

By (5.1) and Assumption A3(i)

$$\begin{aligned}\beta_n^2 \int_{-\infty}^{\infty} u I_{1m}(u) du &= \int_{-\infty}^{\infty} \frac{u}{12\beta_n} \sum_{\alpha=-m}^m f_2''(u, x - p_\alpha) du \\ &\rightarrow \frac{1}{12} \int_{-\infty}^{\infty} u \int_{-\infty}^{\infty} f_2''(u, x - p) dp du = \frac{1}{12} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u f_2''(u, p) dp du,\end{aligned}$$

where p_α lies between $(\alpha - 1)/\beta_n$ and α/β_n . By Assumptions A2 and A3 (ii)-(iii)

$$\begin{aligned}\left| \int_{-\infty}^{\infty} u I_{2m}(u) du \right| &\leq \int_{-\infty}^{\infty} |u| \left| \int_{-L_n/2}^{L_n/2} f(u, x - p) dp - \int_{-\infty}^{\infty} f(u, x - p) dp \right| du \\ &\leq (L_n/2)^{-v} \int_{-\infty}^{\infty} |u| C_f(u) du = O(L_n^{-v}),\end{aligned}$$

and

$$\begin{aligned}\left| \int_{-\infty}^{\infty} u I_{3m}(u) du \right| &\leq \frac{1}{2\beta_n} \left\{ \int_{-\infty}^{\infty} |u| f\left(u|x - \frac{L_n}{2}\right) du f_{u_x}\left(x - \frac{L_n}{2}\right) \right. \\ &\quad \left. + \int_{-\infty}^{\infty} |u| f\left(u|x + \frac{(m+1)L_n}{2m}\right) du f_{u_x}\left(x + \frac{(m+1)L_n}{2m}\right) \right\} \\ &= O(L_n^{-v-1}/\beta_n) = O(L_n^{-v}m^{-1}),\end{aligned}$$

where $f(\cdot)$ is the conditional p.d.f. of u_i given u_{xi} . This completes the proof. ■

Lemma 5.2 *Let $\Theta_{jn} \equiv \frac{L_n}{n} \sum_{i=1}^n (X_i - x)^j K_h(X_i - x)$ for $j = 0, 1, 2$. Suppose A1-A6 hold. Then (i) $E\Theta_{0n} = 1 + o(1)$, (ii) $E\Theta_{1n} = o(h^2)$, (iii) $E\Theta_{2n} = h^2\mu_2(K) + o(h^2)$, and (iv) $\text{Var}(\Theta_{jn}) = (h^{2j-1}L_n/n) \int z^{2j} K^2(z) dz + o(h^{2j-1}L_n/n)$ for $j = 0, 1, 2$.*

Proof. Note that

$$\begin{aligned}E\Theta_{jn} &= \frac{L_n}{2m+1} \sum_{\alpha=-m}^m E \left[(\mu_\alpha + u_{xi} - x)^j K_h(\mu_\alpha + u_{xi} - x) \right] \\ &= \frac{L_n h^j}{2m+1} \sum_{\alpha=-m}^m \int z^j K(z) f_{u_x}(x - \mu_\alpha + hz) dz.\end{aligned}$$

If $j = 0$, we have

$$E\Theta_{0n} = \frac{L_n}{2m+1} \sum_{\alpha=-m}^m f_{u_x}(x - \mu_\alpha) + \frac{L_n h^2 \mu_2(K)}{2m+1} \sum_{\alpha=-m}^m f_{u_x}''(x - \mu_\alpha) + R_{0n},$$

where $R_{0n} = \frac{L_n}{2(2m+1)} h^2 \int z^2 K(z) \int_0^1 \sum_{\alpha=-m}^m [f_{u_x}''(x - \mu_\alpha + whz) - f_{u_x}''(x - \mu_\alpha)] (1-w) dw dz$ is the remainder term. Noting

$$\begin{aligned}\frac{L_n}{2m+1} \sum_{\alpha=-m}^m f_{u_x}(x - \mu_\alpha) &\approx \int_{-L_n/2}^{L_n/2} f_{u_x}(x - p) dp \rightarrow \int_{-\infty}^{\infty} f_{u_x}(x - p) dp = 1, \\ \frac{L_n}{2m+1} \sum_{\alpha=-m}^m |f_{u_x}''(x - \mu_\alpha)| &\approx \int_{-L_n/2}^{L_n/2} |f_{u_x}''(x - p)| dp \rightarrow \int_{-\infty}^{\infty} |f_{u_x}''(x - p)| dp < \infty,\end{aligned}$$

it follows that $R_{0n} = o(h^2)$ by the dominated convergence theorem. Thus, $E\Theta_{0n} = 1 + o(1)$. Similarly, for $j = 1, 2$, we have

$$E\Theta_{1n} = \frac{L_n h^2 \mu_2(K)}{2m+1} \sum_{\alpha=-m}^m f'_{u_x}(x - \mu_\alpha) + o(h^2) = h^2 \mu_2(K) \int_{-\infty}^{\infty} f'_{u_x}(x-p) dp + o(h^2) = o(h^2),$$

since $\int_{-\infty}^{\infty} f'_{u_x}(x-p) dp = 0$, and

$$E\Theta_{2n} = \frac{L_n h^2 \mu_2(K)}{2m+1} \sum_{\alpha=-m}^m f_{u_x}(x - \mu_\alpha) + o(h^4) = h^2 \mu_2(K) + o(h^2).$$

Now, by Jensen's inequality, a change of variables, and since (u_i, u_{xi}) is *iid*, we have

$$\begin{aligned} \text{Var}(\Theta_{jn}) &= \frac{L_n^2}{M(2m+1)^2} \sum_{\alpha=-m}^m \text{Var}\left((\mu_\alpha + u_x - x)^j K_h(\mu_\alpha + u_x - x)\right) \\ &\leq \frac{L_n^2 h^{2j}}{M(2m+1)^2 h} \sum_{\alpha=-m}^m \int z^{2j} K^2(z) f_{u_x}(x - \mu_\alpha + hz) dz \\ &= \frac{L_n h^{2j-1} \int z^{2j} K^2(z) dz}{M(2m+1)} \frac{L_n}{2m+1} \sum_{\alpha=-m}^m f_{u_x}(x - \mu_\alpha) + O\left(\frac{L_n h^{2j+1}}{M(2m+1)}\right) \\ &= \frac{L_n h^{2j-1} \int z^{2j} K^2(z) dz}{n} + o\left(\frac{L_n h^{2j-1}}{n}\right). \end{aligned}$$

This completes the proof of the lemma. ■

Recall $b_n = c_n b$, $a_n = c_n a$, and in the special case where $L_n = L$ is fixed, $c_n = 1$.

Lemma 5.3 *Let $\Phi_{jn} \equiv \int_{a_n}^{b_n} x^j \hat{f}(x) dx$ for $j = 0, 1, 2$. Suppose the conditions in Theorem 3.3 hold. Then for $j = 0, 1, 2$, (i) $E\Phi_{jn} = (b_n^{j+1} - a_n^{j+1}) / (j+1) + O(h^2 c_n^{j+1})$, (ii) $\text{Var}(\Phi_{jn}) = O(L_n c_n^{2j+1} / n)$, and (iii) $c_n^{-(j+1)} \Phi_{jn} = (b^{j+1} - a^{j+1}) / (j+1) + o(1)$.*

Proof. Recall $N = n/L_n = (2m+1)M/L_n$. By the Fubini theorem and Taylor expansion,

$$\begin{aligned} E\Phi_{jn} &= \frac{L_n}{2m+1} \sum_{\alpha=-m}^m \int_{a_n}^{b_n} \int x^j K(z) f_{u_x}(x - \mu_\alpha + hz) dz dx \\ &= \frac{L_n}{2m+1} \sum_{\alpha=-m}^m \int_{a_n}^{b_n} x^j f_{u_x}(x - \mu_\alpha) dx \\ &\quad + \frac{h^2 \mu_2(K) L_n}{2m+1} \sum_{\alpha=-m}^m \int_{a_n}^{b_n} x^j f''_{u_x}(x - \mu_\alpha) dx + R_{jn}. \end{aligned} \tag{5.2}$$

where $R_{jn} = \frac{h^2 L_n}{2(2m+1)} \sum_{\alpha=-m}^m \int_{a_n}^{b_n} \int \int x^j z^2 K(z) [f''_{u_x}(x - \mu_\alpha + whz) - f''_{u_x}(x - \mu_\alpha)] (1-w) dw dz dx$ is the remainder term. Noting

$$\begin{aligned} \frac{L_n}{2m+1} \sum_{\alpha=-m}^m f_{u_x}(x - \mu_\alpha) &\approx \int_{L_n/2}^{L_n/2} f_{u_x}(x-p) dp \rightarrow \int_{-\infty}^{\infty} f_{u_x}(x-p) dp = 1, \\ \frac{L_n}{2m+1} \sum_{\alpha=-m}^m f''_{u_x}(x - \mu_\alpha) &\approx \int_{L_n/2}^{L_n/2} f''_{u_x}(x-p) dp \rightarrow \int_{-\infty}^{\infty} f''_{u_x}(x-p) dp < \infty, \end{aligned}$$

it follows that the first term in (5.2) is approximately

$$\int_{a_n}^{b_n} x^j dx = \frac{b_n^{j+1} - a_n^{j+1}}{j+1}, \quad (5.3)$$

the second term in (5.2) is approximately

$$h^2 \mu_2(K) \int_{-\infty}^{\infty} f''_{u_x}(p) dp \int_{a_n}^{b_n} x^j dx = O(h^2 c_n^{j+1}),$$

and $R_{jn} = o(h^2 c_n^{j+1})$ by dominated convergence.

Now, by Jensen's inequality, the Fubini theorem, a change of variables, and since (u_i, u_{xi}) is *iid*, we have

$$\begin{aligned} & \text{Var}(\Phi_{jn}) \\ &= \frac{L_n^2}{(2m+1)^2 M^2} \sum_{i=1}^n \text{Var} \left(\int_{a_n}^{b_n} x^j K_h(X_i - x) dx \right) \\ &= \frac{L_n^2}{(2m+1)^2 M} \sum_{\alpha=-m}^m \text{Var} \left(\int_{a_n}^{b_n} x^j K_h(\mu_\alpha + u_{x1} - x) dx \right) \\ &\leq \frac{L_n^2}{(2m+1)^2 M} \sum_{\alpha=-m}^m \int_{a_n}^{b_n} \int_{a_n}^{b_n} x^j \tilde{x}^j E[K_h(\mu_\alpha + u_{x1} - x) K_h(\mu_\alpha + u_{x1} - \tilde{x})] dx d\tilde{x} \\ &= \frac{L_n^2}{(2m+1)^2 M} \sum_{\alpha=-m}^m \int_{a_n}^{b_n} \int_{a_n}^{b_n} x^j \tilde{x}^j \int K_h(\mu_\alpha + u_{x1} - x) K_h(\mu_\alpha + u_{x1} - \tilde{x}) f_{u_x}(u_{x1}) du_{x1} dx d\tilde{x} \\ &= \frac{L_n^2}{h(2m+1)^2 M} \sum_{\alpha=-m}^m \int_{a_n}^{b_n} \int_{a_n}^{b_n} x^j \tilde{x}^j \int K(z) K\left(z + \frac{x - \tilde{x}}{h}\right) f_{u_x}(x - \mu_\alpha + hz) dz dx d\tilde{x} \\ &\approx \frac{L_n}{nh} \int_{a_n}^{b_n} \int_{a_n}^{b_n} x^j \tilde{x}^j \int K(z) K\left(z + \frac{x - \tilde{x}}{h}\right) \frac{L_n}{2m+1} \sum_{\alpha=-m}^m f_{u_x}(x - \mu_\alpha) dz dx d\tilde{x} \\ &= \frac{L_n}{nh} \int_{a_n}^{b_n} \int_{a_n}^{b_n} x^j \tilde{x}^j \int K(z) K\left(z + \frac{x - \tilde{x}}{h}\right) dz dx d\tilde{x} \{1 + o(1)\} \\ &= \frac{L_n}{n} \int_{a_n}^{b_n} \int_{(a_n-x)/h}^{(b_n-x)/h} x^j (x + hv)^j \int K(z) K(z - v) dz dv dx \{1 + o(1)\} \\ &= O\left(\frac{L_n c_n^{2j+1}}{n}\right). \end{aligned}$$

(iii) follows from (i) and (ii) and the Chebyshev inequality. ■

Lemma 5.4 *Let $\Xi_{jn} \equiv \int_{a_n}^{b_n} N^{-1/2} \sum_{i=1}^n x^j (X_i - x) K_h(X_i - x) dx$ for $j = 0, 1$. Then $\Xi_{jn} = O_P(h^2 \sqrt{n/L_n} c_n^{j+1} + h c_n^{j+1/2})$.*

Proof. By the Fubini theorem and Taylor expansion,

$$\begin{aligned}
E(\Xi_{jn}) &= \sqrt{\frac{ML_n}{2m+1}} \sum_{\alpha=-m}^m \int_{a_n}^{b_n} E[x^j (\mu_\alpha + u_{x1} - x) K_h(\mu_\alpha + u_{x1} - x)] dx \\
&= \sqrt{\frac{ML_n}{2m+1}} \sum_{\alpha=-m}^m \int_{a_n}^{b_n} \int x^j z K(z) f_{u_x}(x - \mu_\alpha + hz) dz dx \\
&= h^2 \sqrt{\frac{n}{L_n}} \mu_2(K) \left\{ \int_{a_n}^{b_n} x^j \frac{L_n}{2m+1} \sum_{\alpha=-m}^m f''_{u_x}(x - \mu_\alpha) dx \{1 + o(1)\} \right\} \\
&= h^2 \sqrt{n/L_n} \mu_2(K) \int_{a_n}^{b_n} x^j dx \int f''_{u_x}(p) dp \{1 + o(1)\} = O\left(h^2 \sqrt{n/L_n} c_n^{j+1}\right).
\end{aligned}$$

Analogous to the proof of Lemma 5.3(ii), we can show that $\text{Var}(\Xi_{jn}) = O(h^2 c_n^{2j+1})$. The result follows from the Chebyshev inequality. ■