

Semiparametric Estimator of Time Series Conditional Variance

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We propose a new combined semiparametric estimator, which incorporates the parametric and nonparametric estimators of the conditional variance in a multiplicative way. We derive the asymptotic bias, variance, and normality of the combined estimator under general conditions. We show that under correct parametric specification, our estimator can do as well as the parametric estimator in terms of convergence rates; whereas under parametric misspecification our estimator can still be consistent. It also improves over the nonparametric estimator of Ziegelman (2002) in terms of bias reduction. The superiority of our estimator is verified by Monte Carlo simulations and empirical data analysis.

KEY WORDS: Conditional variance; Nonparametric estimator; Semiparametric models.

1. INTRODUCTION

There are two main approaches to modeling volatility as the conditional variance σ_t^2 for a stochastic process y_t . The first is based upon the GARCH family of models, which was pioneered by Engle (1982) and soon spawned a plethora of complicated models to capture empirically stylized facts. The second is based upon the stochastic volatility model, which treats σ_t^2 as a latent variable and is expressed as a mixture of predictable and noise components. Both approaches provide parametric models of conditional variance σ_t^2 . It is well known that when the parametric model is correctly specified it gives a consistent estimator of σ_t^2 , whereas when the parametric model is incorrectly specified the resulting volatility estimator is usually inconsistent with σ_t^2 .

Nonparametric and semiparametric estimation of conditional variance can provide consistent estimation of σ_t^2 . Pagan and Schwert (1990) and Pagan and Hong (1990) are among the initial works in nonparametric ARCH literature. Härdle and Tsybakov (1997) and Härdle, Tsybakov, and Yang (1998) deal with univariate and multivariate local linear fit for conditional variance, respectively. Fan and Yao (1998) and Ziegelmann (2002) model squared residuals (from the conditional mean equation) nonparametrically to capture the volatility dynamics. Most of the above literature includes one lag in the conditional variance model. Additive and multiplicative models have tried to capture the impact of several period lags in the conditional variance equation. Yang, Härdle, and Nielsen (1999) propose a model where the conditional mean is additive and the conditional variance equation is given as $\sigma_t^2 = \gamma_0 \prod_{j=1}^d \sigma_j^2(y_{t-j})$, where γ_0 is an unknown parameter and $\sigma_j^2(\cdot)$, $j = 1, \dots, d$, are smooth but unknown functions. Linton and Mammen (2005) propose a semiparametric ARCH(∞) model where the conditional variance equation is given as $\sigma_t^2(\boldsymbol{y}, m) = \mu_t + \sum_{j=1}^{\infty} \psi_j(\boldsymbol{y})m(y_{t-j})$, where \boldsymbol{y} is a finite dimensional parameter, $m(\cdot)$ is a smooth but unknown function, and the functional forms of $\psi_j(\cdot)$, $j =$

$1, 2, \dots$, are known. Yang (2006) extends the GJR model (Glosten, Jagannathan, and Runkle 1993) to the semiparametric framework.

As explained in the next section, our paper takes a different approach where we combine a parametric estimation with a subsequent nonparametric estimation. We first model the parametric part of the conditional variance and then model the conditional variance of the standardized residual (nonparametric correction factor) nonparametrically capturing some features of σ_t^2 that the parametric model may fail to capture. Thus the combined heteroscedastic model is a multiplicative combination of the parametric model and the nonparametric model for the correction factor. The global parametric estimate of σ_t^2 can be obtained by using any parametric model based on the quasi maximum likelihood estimation (QMLE) principle, say. The estimate of the nonparametric correction factor can be obtained by various nonparametric methods, including the local linear or exponential estimation technique. The idea behind the combined estimation is that if the parametric estimator of σ_t^2 captures some information about the true shape of σ_t^2 , the standardized residual will be less volatile than σ_t^2 itself, which makes it easy to estimate the nonparametric correction factor.

It is worth mentioning that our semiparametric approach has a close analogy with the well-known prewhitening method in the time series literature. See Press and Tukey (1956), Andrews and Monahan (1992), among many others. The method has already been applied in the context of kernel density estimation by Hjort and Glad (1995) and in the context of conditional mean regression by Glad (1998). The seminonparametric estimators of Gallant (1981, 1987) and Gallant and Tauchen (1989)

employ a parametric component plus a flexible nonparametric component. Other combined estimators for density function or conditional mean function include Olkin and Spiegelman (1987) and Fan and Ullah (1999).

Built on aforementioned works, our paper provides asymptotic theory for the asymptotic bias, variance, and normality of the semiparametric estimator. It presents several potential improvements over both pure parametric and nonparametric estimators. First, in the case where the parametric model is misspecified so that the parametric estimator for the true volatility is usually inconsistent, our semiparametric estimator can still be consistent with the volatility. Second, in the case where the parametric model is correctly specified, as expected, our semiparametric estimator is generally less efficient than the parametric estimator but it can do as well as the parametric estimator in terms of convergence rates. Third, in comparison with the nonparametric estimator of Ziegelmann (2002), our estimator can result in bias reduction as long as the parametric model can capture some roughness feature of the true volatility function, whereas the two estimators have the same asymptotic variance. In case of correct specification in the first stage parametric modeling, our semiparametric estimator beats the Ziegelmann's estimator. Also, our estimator for the conditional volatility allows the conditioning variable to be estimated from the data and it incorporates the case where the information set is of infinite dimension but can be summarized by a finite dimensional conditioning variable (as in the typical GARCH framework), in sharp contrast with the setup in Ziegelmann (2002).

The structure of the paper is as follows. In Section 2 we introduce the semiparametric model and estimator, and study in detail the asymptotic properties of the semiparametric estimator with and without the assumption of correct parametric specification in the first stage. Section 3 provides simulations and empirical data analysis. Final remarks are contained in Section 4. All technical details are relegated to the Appendix.

2. THE SEMIPARAMETRIC ESTIMATION

2.1 The Model and Its Reformulation

In this paper we consider the model

$$y_t = g(U_t, \alpha^0) + \varepsilon_t, \quad t = 1, \dots, n, \quad (2.1)$$

where the error term ε_t satisfies $E(\varepsilon_t | \mathcal{F}_{t-1}) = 0$, $E(\varepsilon_t^2 | \mathcal{F}_{t-1}) = \sigma_t^2$, \mathcal{F}_{t-1} is the information set at time $t - 1$, U_t and α^0 are vectors of regressors and pseudo-true parameters, respectively. Since the seminal paper of Engle (1982), a vast literature has developed on the specification of σ_t^2 , among which the (G)ARCH family of models play a fundamental role. The majority of the GARCH family of models are specified parametrically, which is subject to the issue of misspecification. In case of misspecification, the conditional variance may not be estimated consistently even though the parameter estimator in the volatility model is still consistent for some pseudo-true parameter. Similar remarks also hold for the class of stochastic volatility models.

Here we propose a semiparametric estimator for the conditional variance based on a multiplicative combined model. The

point of divergence from the existing literature is that we represent the conditional variance in two parts: parametric and nonparametric. Analogous to Glad (1998), the idea builds on the simple identity

$$E(\varepsilon_t^2 | \mathcal{F}_{t-1}) = \sigma_{p,t}^2 E \left\{ \left(\frac{\varepsilon_t}{\sigma_{p,t}} \right)^2 \mid \mathcal{F}_{t-1} \right\}, \quad (2.2)$$

where $\sigma_{p,t}^2 \in \mathcal{F}_{t-1}$ is determined by a parametric specification of the conditional variance. Let $\sigma_{np,t}^2 = E\{(\varepsilon_t/\sigma_{p,t})^2 | \mathcal{F}_{t-1}\}$. We then have

$$\sigma_t^2 = \sigma_{p,t}^2 \sigma_{np,t}^2. \quad (2.3)$$

The key point is that if the parametric specification $\sigma_{p,t}^2$ captures some roughness features of σ_t^2 , the "nonparametric correction factor" $\sigma_{np,t}^2$ will be easier to estimate. In the extreme case when the parametric part $\sigma_{p,t}^2$ is correctly specified, we would hope $\sigma_{np,t}^2$ to be constant over time. To facilitate the presentation, we make the following assumption.

Assumption A0. $\sigma_{p,t}^2 = \sigma_p^2(X_{1,t}, \mathbf{y}^0)$ and $\sigma_{np,t}^2 = E\{(\varepsilon_t/\sigma_{p,t})^2 | \mathcal{F}_{t-1}\} = \sigma_{np}^2(X_{2,t})$, where \mathbf{y}^0 is the pseudo-true parameter, and $X_{1,t}$ and $X_{2,t}$ are a $d_1 \times 1$ and $d_2 \times 1$ vector, respectively.

Given Assumption A0, $\sigma_t^2 \equiv \sigma^2(X_t) = \sigma_p^2(X_{1,t}, \mathbf{y}^0) \times \sigma_{np}^2(X_{2,t})$, where X_t , a $d \times 1$ vector, is a disjoint union of $X_{1,t}$ and $X_{2,t}$. We are interested in estimating the volatility function $\sigma^2(\cdot)$ at an interior point, $x \in \mathbb{R}^d$. To see how restrictive Assumption A0 is, we make a definition:

Definition (Minimal reducible dimension d^*). If $E(\varepsilon_t^2 | \mathcal{F}_{t-1}) = E(\varepsilon_t^2 | Z_t)$ for some $Z_t \in \mathcal{F}_{t-1}$, a vector of finite dimension d^* , we say that the information set \mathcal{F}_{t-1} is reducible. If further $E(\varepsilon_t^2 | \mathcal{F}_{t-1}) \neq E(\varepsilon_t^2 | \tilde{Z}_t)$ for any subvector \tilde{Z}_t of Z_t , we say that the dimension d^* is minimal for estimating σ_t^2 and the set Z_t is a minimal reducible set.

Remark 1. Note that the minimal reducible set Z_t is not necessarily unique. For example, if the true data generating process (DGP) for ε_t is a GARCH(1, 1) process,

$$\varepsilon_t = \sqrt{\sigma_t^2} v_t, \quad \sigma_t^2 = \gamma_0^0 + \gamma_1^0 \sigma_{t-1}^2 + \gamma_2^0 \varepsilon_{t-1}^2, \quad (2.4)$$

where $\{v_t\}$ is a martingale difference sequence (m.d.s) with mean 0 and variance 1, then σ_t^2 depends on the entire past information set \mathcal{F}_{t-1} only through σ_{t-1}^2 and ε_{t-1}^2 . In this case, one can take $Z_t = (\sigma_{t-1}^2, \varepsilon_{t-1}^2)^T$ or $Z_t = (\sigma_{t-1}^2, \varepsilon_{t-1})^T$ in the above definition. In either case, $d^* = 2$ is minimal.

Remark 2. Clearly, Assumption A0 is a dimension reduction assumption, which is crucial for our asymptotic analysis. On the surface, both parts of the conditional variance, $\sigma_{p,t}^2$ and $\sigma_{np,t}^2$, may depend on a possibly infinite dimension of information in the information set \mathcal{F}_{t-1} . Assumption A0 requires that it is possible to summarize the information into a finite dimensional stochastic variable $X_{1,t}$ or $X_{2,t}$. This assumption looks restrictive but is not as restrictive as it appears. Suppose the true DGP for ε_t is a GARCH(1, 1) process in (2.4). If we correctly specify the parametric conditional variance model, then $\sigma_{p,t}^2 = \gamma_0^0 + \gamma_1^0 \sigma_{p,t-1}^2 + \gamma_2^0 \varepsilon_{t-1}^2$ and $\sigma_{np,t}^2 \equiv 1$. In this case we can simply set $X_{1,t} = (\sigma_{p,t-1}^2, \varepsilon_{t-1}^2)^T$ and $X_{2,t} = X_{1,t}$ or 1, say;

$\sigma_p^2(\cdot, \cdot)$ is affine in both $X_{1,t}$ and $\boldsymbol{\gamma}^0 \equiv (\gamma_0^0, \gamma_1^0, \gamma_2^0)^T$, and $\sigma_{np}^2(\cdot)$ is a constant function. On the other hand, if we specify the parametric conditional variance model as an ARCH(1) model, so that we can write $\sigma_{p,t}^2 = \gamma_0^* + \gamma_2^* \varepsilon_{t-1}^2$, where (γ_0^*, γ_2^*) is the pseudo-true parameter to which the ARCH(1) parametric estimate converges. In this latter case, we can easily identify $X_{1,t} = \varepsilon_{t-1}^2$ and $X_{2,t} = (\sigma_{p,t-1}^2, \varepsilon_{t-1}^2)^T$; $\sigma_p^2(\cdot, \cdot)$ is affine in both $X_{1,t}$ and $\boldsymbol{\gamma}^0 \equiv (\gamma_0^*, \gamma_2^*)^T$; and $\sigma_{np}^2(X_{2,t}) = (\gamma_0^0 + \gamma_1^0 \sigma_{p,t-1}^2 + \gamma_2^0 \varepsilon_{t-1}^2) / (\gamma_0^* + \gamma_2^* \varepsilon_{t-1}^2)$ is a highly nonlinear function.

Remark 3. X_t is not necessarily observable and it summarizes the entire past information set that has influence on the conditional variance. For example, if one specifies $\sigma_{p,t}^2$ as GARCH(1, 1) process: $\sigma_{p,t}^2 = \gamma_0^0 + \gamma_1^0 \sigma_{p,t-1}^2 + \gamma_2^0 \varepsilon_{t-1}^2$, then $\boldsymbol{\gamma}^0 = (\gamma_0^0, \gamma_1^0, \gamma_2^0)^T$, $X_{1,t} = (\sigma_{p,t-1}^2, \varepsilon_{t-1}^2)^T$. So we can rewrite the above process as $\sigma_{p,t}^2 = \sigma_p^2(X_{1,t}, \boldsymbol{\gamma}^0) = \gamma_0^0 + \gamma_2^0 \sigma_{p,t-1}^2 + \gamma_2^0 \varepsilon_{t-1}^2$. Given $X_{1,t}$ and $\boldsymbol{\gamma}^0$, we can recover $\sigma_{p,t}^2$ through the map defined by $(X_{1,t}, \boldsymbol{\gamma}^0) \rightarrow \sigma_p^2(X_{1,t}, \boldsymbol{\gamma}^0)$. This is sufficient for our purpose. If one believes that $X_{1,t}$ summarizes all the past information that affects the conditional variance at time t but has some doubt on the correct specification of the GARCH(1, 1) model, one can choose $X_{2,t} = X_{1,t}$. As a matter of fact, either $X_{1,t}$ or $X_{2,t}$ or both may depend on the unknown parameter

$$\boldsymbol{\theta}^0 = (\boldsymbol{\alpha}^{0T}, \boldsymbol{\gamma}^{0T})^T.$$

For this reason, we shall write $X_{1,t}$, $X_{2,t}$, and X_t explicitly as $X_{1,t}(\boldsymbol{\theta}^0)$, $X_{2,t}(\boldsymbol{\theta}^0)$, and $X_t(\boldsymbol{\theta}^0)$.

Remark 4. We do not need correct specification for modeling the parametric component. To be concrete, we focus on the case where the set of finite dimensional parameters are estimated by QMLE technique. Suppose the data $\{y_t, U_t\}$ are generated from the joint density $f(y|u)f(u)$, where $f(u)$ is the marginal density of U_t and $f(y|u)$ is the density of y_t given $U_t = u$. Let $f_{(\boldsymbol{\alpha}^0, \boldsymbol{\gamma}^0)}(y|u)f(u)$ be the density under the chosen parametric assumption. Whether the parametric model is true or not, under some regularity conditions for QMLE, the parameters, $\boldsymbol{\alpha}^0$ and $\boldsymbol{\gamma}^0$, can be estimated consistently at the regular \sqrt{n} rate (White 1994, chapter 6). One way is to use QMLE to estimate $\boldsymbol{\alpha}^0$ and $\boldsymbol{\gamma}^0$ jointly. The other is, under some orthogonality conditions, to estimate $\boldsymbol{\alpha}^0$ in (2.1) first, say by nonlinear least squares, and then use the residual series $\{\varepsilon_t\}$ to estimate $\boldsymbol{\gamma}^0$. In either case, we can generally establish \sqrt{n} -rate consistency for estimating the pseudo-true parameter $(\boldsymbol{\alpha}^{0T}, \boldsymbol{\gamma}^{0T})^T$, which is a minimizer of the Kullback–Leibler distance from the true density $f(y|u)f(u)$ to the suggested density $f_{(\boldsymbol{\alpha}^0, \boldsymbol{\gamma}^0)}(y|u)f(u)$.

2.2 The Semiparametric Estimator

To introduce our semiparametric estimator, we first define some notation.

Let $\widehat{\boldsymbol{\alpha}}$ be a consistent estimator of $\boldsymbol{\alpha}^0$ in the conditional mean model (2.1) and $\widehat{\boldsymbol{\gamma}}$ be a consistent estimator of $\boldsymbol{\gamma}^0$ in the parametric conditional variance model specified through the process $\{\sigma_p^2(X_{1,t}, \boldsymbol{\gamma}^0)\}$. Denote $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\alpha}}^T, \widehat{\boldsymbol{\gamma}}^T)^T$. Since the processes $\{X_{1,t}\}$, $\{X_{2,t}\}$, and $\{X_t\}$ may be unobservable and depend on the unknown finite dimensional parameter $\boldsymbol{\theta}^0$, we will write them as functions of $\boldsymbol{\theta}^0$, that is, $X_{1,t} \equiv X_{1,t}(\boldsymbol{\theta}^0)$, $X_{2,t} \equiv X_{2,t}(\boldsymbol{\theta}^0)$, and $X_t \equiv X_t(\boldsymbol{\theta}^0)$. Since $\boldsymbol{\theta}^0$ needs to be estimated, we use $\widehat{X}_{1,t}$,

$\widehat{X}_{2,t}$, and \widehat{X}_t to denote $X_{1,t}(\boldsymbol{\theta}^0)$, $X_{2,t}(\boldsymbol{\theta}^0)$, and $X_t(\boldsymbol{\theta}^0)$ with $\boldsymbol{\theta}^0$ being replaced by $\widehat{\boldsymbol{\theta}}$. Let $\varepsilon_t(\boldsymbol{\alpha}) \equiv y_t - g(U_t, \boldsymbol{\alpha})$ and $\widehat{\varepsilon}_t \equiv \varepsilon_t(\widehat{\boldsymbol{\alpha}})$. Then $\widehat{\varepsilon}_t = \varepsilon_t(\widehat{\boldsymbol{\alpha}})$. Define the “standardized” residual as $\widehat{z}_t \equiv \widehat{\varepsilon}_t / \sigma_p(\widehat{X}_{1,t}, \widehat{\boldsymbol{\gamma}})$. Let $\widehat{r}_t \equiv \sigma_p(x_{1,t}, \widehat{\boldsymbol{\gamma}}) \widehat{z}_t = \widehat{\varepsilon}_t \sigma_p(x_{1,t}, \widehat{\boldsymbol{\gamma}}) / \sigma_p(\widehat{X}_{1,t}, \widehat{\boldsymbol{\gamma}})$.

Let $r_t \equiv \varepsilon_t \sigma_p(x_{1,t}, \boldsymbol{\gamma}^0) / \sigma_p(X_{1,t}, \boldsymbol{\gamma}^0)$. Then by (2.2) and (2.3), Assumption A0, and the law of iterated expectations,

$$\begin{aligned} E(r_t^2 | X_{2,t} = x_2) &= \frac{\sigma_p^2(x_{1,t}, \boldsymbol{\gamma}^0)}{\sigma_p^2(X_{1,t}, \boldsymbol{\gamma}^0)} E[E(\varepsilon_t^2 | \mathcal{F}_{t-1}) | X_{2,t} = x_2] \\ &= \frac{\sigma_p^2(x_{1,t}, \boldsymbol{\gamma}^0)}{\sigma_p^2(X_{1,t}, \boldsymbol{\gamma}^0)} \sigma_p^2(X_{1,t}, \boldsymbol{\gamma}^0) \sigma_{np}^2(x_2) = \sigma^2(x). \end{aligned}$$

So in principle we can consider estimating $\sigma^2(x)$ by regressing r_t^2 nonparametrically on $X_{2,t}$, say, by the Nadaraya–Watson (NW) method or the local linear method. The NW method can ensure the nonnegativity of the estimate but it has the boundary bias problem. The local linear method does not have boundary bias problem but it cannot ensure the nonnegativity of the estimate. We now introduce a method that ensures the nonnegativity of the estimate whose bias on the boundary is of the same asymptotic order as that in the interior.

In the general setup, the local linear estimation of $m(x) \equiv E(Y_t | X_t = x)$ is based upon the approximation

$$m(X_t) \approx m(x) + \dot{m}(x)^T (X_t - x) \quad (2.5)$$

when X_t is close to x . Here $\dot{m}(x) \equiv \partial m(x) / \partial x$. When $m(X_t)$ is positive a.s. and X_t is close to x , we can approximate $m(X_t)$ as

$$\begin{aligned} m(X_t) &= \exp(\log(m(X_t))) \\ &\approx \exp(a(x) + b(x)^T (X_t - x)), \end{aligned} \quad (2.6)$$

where $a(x) \equiv \log(m(x))$, and $b(x) \equiv \dot{m}(x) / m(x)$ is the first derivative of $\log(m(x))$. In fact, we can replace the exponential function in (2.6) by any well behaved monotone function Ψ and the log function in (2.6) by the inverse function of Ψ . This motivates our estimator of $\sigma^2(x)$ below.

To proceed, we first fit the parametric model to obtain $\widehat{\boldsymbol{\theta}}$. Then we estimate $\sigma^2(x)$ by the local smoothing technique

$$\begin{aligned} \widehat{\beta} \equiv \arg \min_{\beta} n^{-1} \sum_{t=1}^n \left\{ \widehat{r}_t^2 - \Psi \left(\beta_0 + \sum_{j=1}^{d_2} \beta_j (\widehat{X}_{2,t,j} - x_{2,j}) \right) \right\}^2 \\ \times K_h(\widehat{X}_{2,t} - x_2), \end{aligned} \quad (2.7)$$

where $\boldsymbol{\beta} \equiv (\beta_0, \beta_1, \dots, \beta_{d_2})^T \in \mathbb{R}^{d_2+1}$, Ψ is a monotone function that has at least two continuous derivatives on its support, $h \equiv (h_1, \dots, h_{d_2})^T$ is a vector of bandwidth parameters, $K(\cdot)$ is a nonnegative product kernel of $k(\cdot)$, and $K_h(u) = \prod_{i=1}^{d_2} h_i^{-1} k(u_i/h_i)$. Note that we have suppressed the dependence of \widehat{r}_t and $\widehat{\boldsymbol{\beta}} \equiv (\widehat{\beta}_0, \widehat{\beta}_1, \dots, \widehat{\beta}_{d_2})^T$ on x , a disjoint union of x_1 and x_2 . We define the volatility estimator as $\widehat{\sigma}^2(x) = \Psi(\widehat{\beta}_0)$.

In Theorem 2.1 below, we show that $\Psi(\widehat{\beta}_0)$ is consistent for $\sigma^2(x)$. We will prove that this estimator, after being appropriately centered and scaled, is asymptotically normally distributed. Note that when $\Psi(u) \equiv u$, we have the local linear estimator for the conditional variance. When $\Psi(u) \equiv \exp(u)$, we have the local exponential estimator of the conditional variance introduced in Ziegelmann (2002) where the conditional variance is modeled fully nonparametrically as a function of a single observable. One obvious advantage of the local exponential

approach over the traditional local linear estimation is to ensure the nonnegativity of the estimator of the conditional variance. In contrast, our situation is different than Ziegelmann (2002) mainly in two aspects. First, our volatility function is specified semiparametrically as a product of a parametric component and a nonparametric component. Second, the arguments in our volatility function may depend on all the past information and have to be estimated from the data.

To simplify notation, let $L(x_2, \boldsymbol{\beta}) \equiv \Psi(\beta_0 + \sum_{j=1}^{d_2} \beta_j x_{2,j})$ and $\sigma_p^2(x_1) \equiv \sigma_p^2(x_1, \boldsymbol{\gamma}^0)$. Define

$$\begin{aligned} D_{\boldsymbol{\gamma}} \sigma_p^2(x_1, \boldsymbol{\gamma}) &\equiv \frac{\partial \sigma_p^2(x_1, \boldsymbol{\gamma})}{\partial \boldsymbol{\gamma}}, \\ D_{\boldsymbol{\gamma}\boldsymbol{\gamma}} \sigma_p^2(x_1, \boldsymbol{\gamma}) &\equiv \frac{\partial^2 \sigma_p^2(x_1, \boldsymbol{\gamma})}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\gamma}^T}, \\ \dot{\sigma}_p^2(x_1, \boldsymbol{\gamma}) &\equiv \frac{\partial \sigma_p^2(x_1, \boldsymbol{\gamma})}{\partial x_1}, & \dot{\sigma}_{np}^2(x_2) &\equiv \frac{\partial \sigma_{np}^2(x_2)}{\partial x_2}, \\ \ddot{\sigma}_{np}^2(x_2) &\equiv \frac{\partial^2 \sigma_{np}^2(x_2)}{\partial x_2 \partial x_2^T}, & \dot{L}(x_2, \boldsymbol{\beta}) &\equiv \frac{\partial L(x_2, \boldsymbol{\beta})}{\partial x_2}, \\ \ddot{L}(x_2, \boldsymbol{\beta}) &\equiv \frac{\partial^2 L(x_2, \boldsymbol{\beta})}{\partial x_2 \partial x_2^T}. \end{aligned}$$

For $i, j = 0, 1, 2$, let $\kappa_{ij} \equiv \int u^i k(u)^j du$. Define a $(d_2 + 1) \times (d_2 + 1)$ matrix:

$$S \equiv \begin{pmatrix} 1 & 0 \\ 0 & \kappa_{21} I_{d_2} \end{pmatrix}, \quad (2.8)$$

where I_{d_2} is a $d_2 \times d_2$ identity matrix.

2.3 Asymptotic Theory for the Semiparametric Estimator Under General Conditions

To introduce the asymptotic theory, we make the following set of assumptions.

Assumption A1. (i) $\boldsymbol{\alpha}^0$ lies in the interior of a compact set $A \subset \mathbb{R}^{k_1}$. The first derivative $D_{\boldsymbol{\alpha}} g(U_t, \boldsymbol{\alpha})$ of $g(U_t, \boldsymbol{\alpha})$ with respect to $\boldsymbol{\alpha}$ exists almost surely (a.s.). $D_{\boldsymbol{\alpha}} g(U_t, \boldsymbol{\alpha})$ is Lipschitz continuous in $\boldsymbol{\alpha}$ in the neighborhood of $\boldsymbol{\alpha}^0$.

(ii) $\boldsymbol{\gamma}^0$ lies in the interior of a compact set $\Gamma \subset \mathbb{R}^{k_2}$ such that the process $\{\sigma_p^2(X_{1,t}, \boldsymbol{\gamma}^0)\}_{t=1}^{\infty}$ is bounded below by a constant $\sigma_p^2 > 0$ and is strictly stationary and ergodic. $X_{2,t}$ has a continuous density $f(x_2)$ which is bounded away from zero at x_2 , an interior point of $f(\cdot)$.

(iii) $L(x_2, \boldsymbol{\beta})$ and $\dot{L}(x_2, \boldsymbol{\beta})$ are bounded uniformly when both x_2 and $\boldsymbol{\beta}$ are restricted in compact sets.

Assumption A2. (i) The process $\{U_t, X_t(\boldsymbol{\theta}^0)\}$ is stationary and α -mixing with a mixing coefficient $\alpha(j)$ satisfying $\alpha(j) = O(j^{-\gamma})$ with $\gamma > (2\nu - 2)/(\nu - 2)$ and $\nu > 2$.

(ii) Let $D_{\boldsymbol{\theta}} X_{1,t}(\boldsymbol{\theta}) \equiv (\partial/\partial \boldsymbol{\theta}^T) X_{1,t}(\boldsymbol{\theta})$ and $D_{\boldsymbol{\theta}} X_{2,t}(\boldsymbol{\theta}) \equiv (\partial/\partial \boldsymbol{\theta}^T) X_{2,t}(\boldsymbol{\theta})$. $D_{\boldsymbol{\theta}} X_{1,t}(\boldsymbol{\theta})$ and $D_{\boldsymbol{\theta}} X_{2,t}(\boldsymbol{\theta})$ exist and are Lipschitz continuous in the neighborhood of $\boldsymbol{\theta}^0$. $E\|D_{\boldsymbol{\theta}} X_{i,t}(\boldsymbol{\theta}^0)\|^{\nu} < \infty$ for some $\nu > 2$, $i = 1, 2$, where $\|\cdot\|$ denotes the Euclidean norm.

(iii) Write $\varepsilon_t = \sqrt{\sigma_t^2} v_t$. The process $\{v_t\}$ is a stationary mds such that $E(v_t|\mathcal{F}_{t-1}) = 0$, $E(v_t^2|\mathcal{F}_{t-1}) = 1$. $E(|\varepsilon_t|^{2\nu}) < \infty$ and $E\|X_t\|^{2\nu} < \infty$ for some $\nu > 2$.

Assumption A3. (i) $\sigma_p^2(X_{1,t}, \boldsymbol{\gamma})$ has two continuous derivatives in $\boldsymbol{\gamma}$ a.s. in the neighborhood of $\boldsymbol{\gamma}^0$. $\dot{\sigma}_p^2(x_1, \boldsymbol{\gamma})$ is Lipschitz continuous in x_1 for $\boldsymbol{\gamma}$ in the neighborhood of $\boldsymbol{\gamma}^0$. $\sigma_{np}^2(x_2)$ has two continuous derivatives in the neighborhood of x_2 . $\dot{\sigma}_{np}^2(x_2)$ is Lipschitz continuous in x_2 .

(ii) The gradient $(\mu(x_1, \boldsymbol{\gamma}))$ and Hessian matrix $(\nu(x_1, \boldsymbol{\gamma}))$ with respect to $\boldsymbol{\gamma}$ of $\log(\sigma_p^2(x_1, \boldsymbol{\gamma}))$ are Lipschitz continuous in the neighborhood of $\boldsymbol{\gamma}^0$, that is, for some $\epsilon > 0$ such that $\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^0\| \leq \epsilon$, we have $\|\mu(x_1, \boldsymbol{\gamma}) - \mu(\tilde{x}_1, \boldsymbol{\gamma})\| \leq C_1 \|x_1 - \tilde{x}_1\|$, and $\|\nu(x_1, \boldsymbol{\gamma}) - \nu(\tilde{x}_1, \boldsymbol{\gamma})\| \leq C_1 \|x_1 - \tilde{x}_1\|$ for some constant C_1 and all $x_1, \tilde{x}_1 \in \mathbb{R}^{d_1}$, where $\mu(x_1, \boldsymbol{\gamma}) \equiv D_{\boldsymbol{\gamma}} \sigma_p^2(x_1, \boldsymbol{\gamma})/\sigma_p^2(x_1, \boldsymbol{\gamma})$ and $\nu(x_1, \boldsymbol{\gamma}) \equiv \{\sigma_p^2(x_1, \boldsymbol{\gamma}) D_{\boldsymbol{\gamma}\boldsymbol{\gamma}} \sigma_p^2(x_1, \boldsymbol{\gamma}) - D_{\boldsymbol{\gamma}} \sigma_p^2(x_1, \boldsymbol{\gamma}) [D_{\boldsymbol{\gamma}} \sigma_p^2(x_1, \boldsymbol{\gamma})]^T\}/\sigma_p^2(x_1, \boldsymbol{\gamma})$. Denote $\mu_0(x_1) \equiv \mu(x_1, \boldsymbol{\gamma}^0)$.

Assumption A4. $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0) = n^{-1/2} \sum_{t=1}^n \varphi_t(\boldsymbol{\theta}^0) + o_p(1) \xrightarrow{d} N(0, V_{\boldsymbol{\theta}^0})$.

Assumption A5. (i) The kernel K is a product kernel of k that is a symmetric density with compact support on \mathbb{R} .

(ii) $|k(u) - k(v)| \leq C_2 |u - v|$ and $|\dot{k}(u) - \dot{k}(v)| \leq C_2 |u - v|$ for some finite constant C_2 and all u, v on the support of k , where $\dot{k}(u)$ is the first derivative of $k(u)$.

Assumption A6. As $n \rightarrow \infty$, (i) $h = (h_1, \dots, h_{d_2}) \rightarrow 0$, (ii) $n\|h\| \max(\prod_{i=1}^{d_2} h_i, \|h\|) \rightarrow \infty$, and (iii) $n(\prod_{i=1}^{d_2} h_i) \|h\|^4 \rightarrow C_3 \in [0, \infty)$.

Assumption A1 is standard in the literature. In particular, *Assumption A1(ii)* can be met for the GARCH family of processes as long as the intercept term is strictly positive, and *Assumption A1(iii)* is used to show the uniform convergence of some stochastic object. Note that the α -mixing condition in *Assumption A2(i)* is weaker than β -mixing condition in Hall, Wolf, and Yao (1999) and *Assumption A2(iii)* does not require v_t to be iid so that its higher-order moments may depend on $X_t \in \mathcal{F}_{t-1}$. *Assumption A3* imposes the smoothness properties of $\sigma_p^2(x_1, \boldsymbol{\gamma})$ and can easily be satisfied by a variety of GARCH-type models. *Assumption A4* follows from asymptotic normality results for QMLE estimators [e.g., Lee and Hansen 1994 and Lumsdaine 1996 for the GARCH(1, 1) case and Berkes, Horvath, and Koloszka 2003 and Berkes and Horvath 2003 for the GARCH(p, q) case]. *Assumptions A5* is similar to *assumption C2* in Hall, Wolf, and Yao (1999). As they remark, the requirement that K is compactly supported can be removed at the cost of much lengthier arguments used in the proofs, and in particular, Gaussian kernel is allowed. *Assumption A6(i)* is standard. *Assumption A6(ii)* is used in the proof of Theorem 2.1, and given *A6(i)* it is stronger than the usual requirement $n \prod_{i=1}^{d_2} h_i \rightarrow \infty$ because we use the Taylor expansion in the approximation of $K_h(\widehat{X}_{2,t} - x_2)$ by $K_h(X_{2,t} - x_2)$. *Assumption A6(iii)* is used toward the end of the proof of Theorem 2.2. Intuitively speaking, it is needed to ensure that the bias order $O(\|h\|^2)$ is of the order no bigger than $(n \prod_{i=1}^{d_2} h_i)^{-1/2}$.

Theorem 2.1 below shows that the estimator $\widehat{\boldsymbol{\beta}}$ defined in (2.7) converges to $\boldsymbol{\beta}^0$ in probability, where $\boldsymbol{\beta}^0$ is uniquely defined by $\sigma^2(x) = L(0, \boldsymbol{\beta}^0)$ and $\sigma_p^2(x_1) \dot{\sigma}_{np}^2(x_2) = \dot{L}(0, \boldsymbol{\beta}^0)$.

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Theorem 2.1. Under Assumptions A0–A6, we have

$$\widehat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}^0,$$

where $\boldsymbol{\beta}^0$ is uniquely defined by $\sigma^2(x) = L(0, \boldsymbol{\beta}^0)$ and $\sigma_p^2(x_1) \times \dot{\sigma}_{np}^2(x_2) = \dot{L}(0, \boldsymbol{\beta}^0)$.

For the proof of the above theorem, see Appendix A.1. Let $\dot{\Psi}(u)$ denote the first derivative of $\Psi(u)$ with respect to u . Write $\boldsymbol{\beta}^0 = (\beta_0^0, \boldsymbol{\beta}_1^{0T})^T$. Note that $L(0, \boldsymbol{\beta}^0) = \Psi(\boldsymbol{\beta}_0^0)$ and $\dot{L}(0, \boldsymbol{\beta}^0) = \dot{\Psi}(\boldsymbol{\beta}_0^0)\boldsymbol{\beta}_1^0$. Then Theorem 2.1 implies that

$$\beta_0^0 \equiv \Psi^{-1}(\sigma^2(x)) \quad \text{and} \quad \boldsymbol{\beta}_1^0 \equiv \frac{\sigma_p^2(x_1)\dot{\sigma}_{np}^2(x_2)}{\dot{\Psi}(\Psi^{-1}(\sigma^2(x)))},$$

where $\Psi^{-1}(\cdot)$ is the inverse function of $\Psi(\cdot)$.

The following theorem states the asymptotic normality of the estimator $\widehat{\sigma}^2(x)$.

Theorem 2.2. Under Assumptions A0–A6, we have

$$\begin{aligned} & \sqrt{n \prod_{i=1}^{d_2} h_i} \{ \widehat{\sigma}^2(x) - \sigma^2(x) \\ & \quad - \frac{\kappa_{21}}{2} \text{tr}\{D_h[\sigma_p^2(x_1)\ddot{\sigma}_{np}^2(x_2) - \ddot{L}(0, \boldsymbol{\beta}^0)]\} \\ & \quad \xrightarrow{d} N(0, \kappa_{02}^2(E(v_t^4|X_{2,t}=x_2) - 1)f^{-1}(x_2)\sigma^4(x)), \end{aligned} \quad (2.9)$$

where recall $\kappa_{ij} \equiv \int u^i k(u)^j du$ and $D_h \equiv \text{diag}(h_1^2, \dots, h_{d_2}^2)$.

Remark 5. In the case where $X_{1,t} = X_{2,t} \in \mathbb{R}^1$, we write the bandwidth $h = h_1$. Then the asymptotic variance of our estimator is $\kappa_{02}(E(v_t^4|X_t = x) - 1)f(x)^{-1}\sigma^4(x)/(nh)$ and the bias is $B(x) \equiv (\kappa_{21}h^2/2)[\sigma_p^2(x)\ddot{\sigma}_{np}^2(x) - \ddot{L}(0, \boldsymbol{\beta}^0)]$. We consider two choices for Ψ and compare our result to that in theorem 1 of Ziegelmann (2002) for the same bandwidth h and kernel K . For each case, we can see that the two estimators have exactly the same asymptotic variance and this result does not depend on the particular choice of Ψ .

(1) If we take $\Psi(u) \equiv \exp(u)$, then Theorem 2.1 implies that $\boldsymbol{\beta}^0 = (\beta_0^0, \boldsymbol{\beta}_1^0)^T = (\log(\sigma^2(x)), \dot{\sigma}_{np}^2(x)/\sigma_{np}^2(x))^T$. Hence

$$\begin{aligned} \dot{L}(0, \boldsymbol{\beta}^0) &= \left. \frac{\partial^2 \exp(\beta_0^0 + \boldsymbol{\beta}_1^0 x)}{\partial x^2} \right|_{x=0} = \exp(\beta_0^0)(\boldsymbol{\beta}_1^0)^2 \\ &= \frac{\sigma_p^2(x)(\dot{\sigma}_{np}^2(x))^2}{\sigma_{np}^2(x)}, \end{aligned}$$

so that $B(x) = \frac{h^2\kappa_{21}}{2} \frac{\partial^2 \log(\sigma_{np}^2(x))}{\partial x^2} \sigma^2(x)$ for our estimator. To compare with the bias for Ziegelmann's (2002) estimator, one can write the bias of his estimator as $\widetilde{B}(x) = \frac{h^2\kappa_{21}}{2} \frac{\partial^2 \log(\sigma^2(x))}{\partial x^2} \sigma^2(x)$. This means that our estimator will achieve bias reduction if one can choose $\sigma_{np}^2(x)$ in such a way that

$$\left| \frac{\partial^2 \log(\sigma_{np}^2(x))}{\partial x^2} \right| < \left| \frac{\partial^2 \log(\sigma^2(x))}{\partial x^2} \right|. \quad (2.10)$$

In other words, if $\sigma_{np}^2(x)$ can capture some of the shape features of $\sigma^2(x)$ in the neighborhood of x , $\log(\sigma_{np}^2(x))$ will be less rough than $\log(\sigma^2(x))$ itself so that (2.10) can be easily satisfied

and we achieve bias reduction. Also, in terms of global approximated weighted mean squared error, our estimator is better than that of Ziegelmann's (2002) if

$$\int \left\{ \frac{\partial^2 \log(\sigma_{np}^2(x))}{\partial x^2} \right\}^2 w(x) dx < \int \left\{ \frac{\partial^2 \log(\sigma^2(x))}{\partial x^2} \right\}^2 w(x) dx, \quad (2.11)$$

where $w(x)$ is a nonnegative weighting function.

(2) If we take $\Psi(u) \equiv u$, then $\dot{L}(0, \boldsymbol{\beta}^0) = 0$ so that the bias for our estimator is $B(x) = \frac{h^2\kappa_{21}}{2}\sigma_p^2(x)\ddot{\sigma}_{np}^2(x)$ and that for Ziegelmann's (2002) estimator is $\widetilde{B}(x) = \frac{h^2\kappa_{21}}{2}\ddot{\sigma}^2(x)$, where $\ddot{\sigma}^2(x)$ is the second derivative of $\sigma^2(x)$. We will achieve bias reduction provided that

$$|\sigma_p^2(x)\ddot{\sigma}_{np}^2(x)| < |\ddot{\sigma}^2(x)|. \quad (2.12)$$

Like in Glad (1998), if the initial parametric choice $\sigma_p^2(x)$ happens to be proportional to the true volatility $\sigma^2(x)$, the nonparametric correction factor $\sigma_{np}^2(x)$ is constant and hence the bias reduces to a negligible order for all potential values of C_3 in Assumption A6. If $\sigma_p^2(x)$ captures some of the shape features of $\sigma^2(x)$, $\sigma_{np}^2(x)$ will be less rough than $\sigma^2(x)$ itself so that (2.12) can be maintained.

For general function Ψ , bias reduction can be achieved with similar weak requirement. That is, the parametric component $\sigma_p^2(\cdot)$ bears some information on the shape of $\sigma^2(\cdot)$ in the neighborhood of x . In the case when the parametric component is correctly specified, that is, $\sigma^2(x) = \sigma_p^2(x_1, \boldsymbol{y}^0)$ so that $\sigma_{np}^2(x_2) \equiv 1$, our estimator is bias-free asymptotically ($\sqrt{nh^{d_2}}B(x) = 0$) while the asymptotic bias ($\sqrt{nh^{d_2}}\widetilde{B}(x)$) for the Ziegelmann's (2002) estimator does not vanish for the case where $C_3 > 0$ in Assumption A6. In the special case when $C_3 = 0$, the bias for both estimators are asymptotically negligible but different in higher order bias. In short, in case of correct specification for the parametric component, our semiparametric estimator always beats the Ziegelmann's estimator in terms of (integrated) mean squared error.

Remark 6. Using the notation and theories developed in Masry (1996a, 1996b), one can easily generalize our theory to allow for the use of higher-order polynomials in (2.7). Generally speaking, higher-order local polynomials will help reduce the bias but demand more data due to "sparsity."

Remark 7. Based on Theorem 2.2, we can develop a nonparametric test for the adequacy of the parametric conditional variance model. The null hypothesis is

$$H_0 : \sigma_{np}^2(X_{2,t}) = 1 \quad \text{a.s.},$$

and the alternative hypothesis H_1 is the negation of H_0 . Under H_0 , the parametric conditional variance model is correctly specified so that $\sigma_{np,t}^2 \equiv \sigma_{np}^2(X_{2,t}) = 1$ a.s. $\sigma_{np}^2(\cdot)$ is a nonunity function under the alternative. Let $u_t \equiv \varepsilon_t^2/\sigma_{p,t}^2 - 1$. Then the null hypothesis can be written as $H_0 : E(u_t|X_{2,t}) = 0$ a.s. We can construct consistent tests of H_0 versus H_1 using various distance measures. One convenient measure is

$$J \equiv E[u_t E(u_t|X_{2,t})f(X_{2,t})],$$

because $J = E\{[E(u_t|X_{2,t})]^2 f(X_{2,t})\} \geq 0$ and $J = 0$ if and only if H_0 holds. The sample analog of J is

$$J_n \equiv \frac{1}{n^2 \prod_{i=1}^{d_2} b_i} \sum_{t=1}^n \sum_{s \neq t}^n \widehat{u}_t \widehat{u}_s K_b(\widehat{X}_{2,t} - \widehat{X}_{2,s}),$$

where \widehat{u}_t and $\widehat{X}_{2,t}$ consistently estimate u_t and $X_{2,t}$ under H_0 and $b = (b_1, \dots, b_{d_2})$ is the bandwidth sequence. A statistic of this type has been recently used by Hsiao and Li (2001) to test for conditional heteroscedasticity where u_t can be estimated at \sqrt{n} -rate under the null. We conjecture that one can extend their analysis and show that after being suitably scaled, J_n converges to the standard normal distribution under the null and it diverges to infinity under the alternative. The detailed analysis is beyond the scope of this paper.

Remark 8. In the above analysis, we did not restrict the density function $f(\cdot)$ of $X_{2,t}$ to be compactly supported. In the case where $f(\cdot)$ is compactly supported, it is worthwhile to study the behavior of the estimator at the boundary of the support. Without loss of generality, assume that $d_2 = 1$ and the support of $f(\cdot)$ is $[0, 1]$. In this case, we denote the bandwidth simply as $h \equiv h(n)$ and consider the left boundary point $x_{n2} = ch$, where c is a positive constant. Also, following the literature, we assume that $f(0) \equiv \lim_{x_2 \downarrow 0} f(x_2)$ exists and is strictly positive. In this case, we can show that the asymptotic bias (Abias) and variance (Avar) of $\widehat{\sigma}^2(x)$ are given by

$$\begin{aligned} \text{Abias}(\widehat{\sigma}^2(x)) \\ = \frac{h^2}{2} \frac{\mu_{c2}^2 - \mu_{c1}\mu_{c3}}{\mu_{c0}\mu_{c2} - \mu_{c1}^2} [\sigma_p^2(x_1) \ddot{\sigma}_{np}^2(0) - \ddot{L}(0, \beta^0)] \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} \text{Avar}(\widehat{\sigma}^2(x)) \\ = \frac{\int_{-c}^{\infty} (\mu_{c2} - \mu_{c1}z)^2 k^2(z) dz}{nh(\mu_{c0}\mu_{c2} - \mu_{c1}^2)^2} \\ \times [E(v_t^4|X_{2,t} = x_2) - 1] f^{-1}(0) \sigma^4(x_1, 0), \end{aligned} \quad (2.14)$$

where $\mu_{cj} = \int_{-c}^{\infty} z^j k(z) dz$ for $j = 0, 1, 2, 3$. See Appendix A.3 for an outline of the proof.

2.4 Asymptotic Theory for the Semiparametric Estimator Under Correct Parametric Specification

In this subsection, we will derive the asymptotic properties of $\widehat{\beta}$ and $\widehat{\sigma}^2(x)$ under the additional assumption that the parametric conditional variance model is correctly specified, that is, $P(\sigma^2(X_t) = \sigma_p^2(X_{1t}, \gamma^0)) = 1$ for some $\gamma^0 \in \Gamma$. We will hold the bandwidth sequence $h \equiv (h_1, \dots, h_d)$ fixed and demonstrate that $\widehat{\beta}$ and $\widehat{\sigma}^2(x)$ converges to their population true values at the parametric \sqrt{n} -rate.

First, we state the consistency of $\widehat{\beta}$.

Theorem 2.3. Suppose $P(\sigma^2(X_t) = \sigma_p^2(X_{1t}, \gamma^0)) = 1$. Let $h \equiv (h_1, \dots, h_{d_2})$ be fixed. Under Assumptions A0–A5, we have

$$\widehat{\beta} \xrightarrow{p} \beta^0,$$

where $\beta^0 \equiv (\Psi^{-1}(\sigma_p^2(x_1)), 0_{1 \times d_2})^T$, and $\Psi^{-1}(\cdot)$ is the inverse function of $\Psi(\cdot)$.

Next, we study the asymptotic normality of the estimator $\widehat{\beta}$.

Theorem 2.4. Suppose $P(\sigma^2(X_t) = \sigma_p^2(X_{1t}, \gamma^0)) = 1$. Let $h \equiv (h_1, \dots, h_{d_2})$ be fixed. Under Assumptions A0–A5, we have

$$\sqrt{n}(\widehat{\beta} - \beta^0) \xrightarrow{d} N(0, [\dot{\Psi}(\beta^0)]^{-2} \Sigma_{\beta^0}^{-1} \Omega_{\beta^0} \Sigma_{\beta^0}^{-1}),$$

where

$$\Sigma_{\beta^0} \equiv E \left[\begin{pmatrix} 1 & (X_{2,t} - x_2)^T \\ (X_{2,t} - x_2) & (X_{2,t} - x_2)(X_{2,t} - x_2)^T \end{pmatrix} K_{ht} \right],$$

$$\begin{aligned} \Omega_{\beta^0} \equiv & \sigma_p^4(x_1) E \left\{ [E(v_t^4|X_{2,t} = x_2) - 1] \right. \\ & \times K_{ht}^2 \left(\begin{pmatrix} 1 & (X_{2,t} - x_2)^T \\ (X_{2,t} - x_2) & (X_{2,t} - x_2)(X_{2,t} - x_2)^T \end{pmatrix} \right) \\ & \left. + \Upsilon V_{\theta^0} \Upsilon^T, \right. \end{aligned}$$

$$\begin{aligned} \Upsilon = & E \left[\sigma_p^2(x_1) \begin{pmatrix} 1 \\ (X_{2,t} - x_2) \end{pmatrix} \right. \\ & \times K_{ht} \left\{ [0_{1 \times k_1}, (\mu_0(x_1) - \mu_0(X_{1,t}))^T] \right. \\ & \left. \left. - \frac{\dot{\sigma}_p^2(X_{1,t}, \gamma^0)^T}{\sigma_p^2(X_{1,t})} D_{\theta} X_{1,t}(\theta^0)^T \right\} \right], \end{aligned}$$

$K_{ht} \equiv K_h(X_{2,t} - x_2)$, V_{θ^0} is defined in Assumption A4, and recall $\dot{\Psi}(u)$ denotes the first derivative of $\Psi(u)$ with respect to u .

Note that when h is held fixed, Σ_{β^0} and Ω_{β^0} are generally nondiagonal matrices. By the delta method, we can prove the following corollary.

Corollary 2.5. Under the conditions of Theorem 2.4,

$$\sqrt{n}(\widehat{\sigma}^2(x) - \sigma^2(x)) \xrightarrow{d} N(0, e_1^T \Sigma_{\beta^0}^{-1} \Omega_{\beta^0} \Sigma_{\beta^0}^{-1} e_1),$$

where e_1 is a $(d_2 + 1) \times 1$ vector with 1 in the first entry and 0 elsewhere.

Remark 9. Corollary 2.5 states that in the case where the parametric conditional variance model is correctly specified, the combined estimator $\widehat{\sigma}^2(x)$ converges to $\sigma^2(x)$ at the parametric rate and is asymptotically normally distributed. A nice feature about $\sqrt{n}\widehat{\sigma}^2(x)$ in Corollary 2.5 is that it is asymptotically unbiased and its asymptotic variance does not depend on the particular tilting function Ψ in use. A second feature about $\sqrt{n}\widehat{\sigma}^2(x)$ is that its asymptotic variance is affected by the first stage estimation of the conditional mean and variance models. The impact of the first stage estimation is accounted for through the term $\Upsilon V_{\theta^0} \Upsilon^T$ in the definition of Ω_{β^0} .

Remark 10. To compare our estimator with the parametric estimator of conditional variance, first note that when the parametric component is correctly specified, as expected, our estimator is usually less efficient than the parametric one since our estimator has a slower convergence rate than the parametric estimator when $h \rightarrow 0$ as in Theorem 2.2, and generally has a larger asymptotic variance when h is kept fixed as in Corollary 2.5. To see the last point more clearly, we now explicitly calculate the asymptotic variance of the parametric conditional

1 variance estimator $\widehat{\sigma}^2(x_1, \widehat{\boldsymbol{y}})$. Under the correct specification of
 2 the parametric conditional variance model,

$$3 \sqrt{n}(\widehat{\sigma}^2(x_1, \widehat{\boldsymbol{y}}) - \sigma^2(x))$$

$$4 \xrightarrow{d} N(0, [D_{\boldsymbol{y}}\sigma_p^2(x_1, \boldsymbol{y}^0)]^T V_{\boldsymbol{y}_0} [D_{\boldsymbol{y}}\sigma_p^2(x_1, \boldsymbol{y}^0)]),$$

5 where $V_{\boldsymbol{y}_0}$ is the lower-right $k_2 \times k_2$ submatrix of $V_{\boldsymbol{\theta}_0}$. The dif-
 6 ference between the two asymptotic variances is given by

$$7 \text{Avar}(\sqrt{n}\widehat{\sigma}^2(x)) - \text{Avar}(\sqrt{n}\widehat{\sigma}^2(x_1, \widehat{\boldsymbol{y}}))$$

$$8 = e_1^T \Sigma_{\beta^0}^{-1} \Omega_{\beta^0} \Sigma_{\beta^0}^{-1} e_1 - [D_{\boldsymbol{y}}\sigma_p^2(x_1, \boldsymbol{y}^0)]^T V_{\boldsymbol{y}_0} [D_{\boldsymbol{y}}\sigma_p^2(x_1, \boldsymbol{y}^0)],$$

9 which is difficult to simplify unless both Ω_{β^0} and Υ are diag-
 10 onal. In this sense, we say that our estimator is as good as the
 11 parametric estimator in terms of convergence rates when h is
 12 kept fixed, which is consistent with Glad (1998) even though
 13 she did not explicitly point this fact out. In contrast, Fan and
 14 Ullah (1999) consider a combined estimator of the regression
 15 mean in the iid framework. Their combined estimator is a linear
 16 combination of a parametric estimator and a nonparametric es-
 17 timator with the weights automatically determined by the data.
 18 The parametric rate of convergence of their estimator in case of
 19 correct parametric specification can be achieved by letting the
 20 bandwidth approach zero.

21 *Remark 11.* Like Fan and Ullah (1999), in case of misspec-
 22 ification the parametric conditional variance estimator is usu-
 23 ally inconsistent (even though $\widehat{\boldsymbol{\theta}}$ is consistent for some pseudo-
 24 true parameter $\boldsymbol{\theta}^0$) while our semiparametric estimator is still
 25 consistent. Moreover, our semiparametric estimator can capture
 26 some shape structure of the conditional variance function that
 27 the parametric estimator fails to capture. Fan and Ullah (1999)
 28 also considered the case where the parametric model is approx-
 29 imately correct. In principle, we can extend their work to our
 30 framework and study the behavior of our combined estimator
 31 in the case where the parametric conditional variance model is
 32 approximately correct, that is,

$$33 \sigma^2(X_t) = \sigma_p^2(X_{1,t}, \boldsymbol{y}^0) + \delta_n \Delta(X_t) \quad \text{a.s.},$$

34 where $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, and $\Delta(x)$ is continuously differ-
 35 entiable with $|\Delta(x)| < \infty$. We conjecture that the rate of con-
 36 vergence of $\widehat{\sigma}^2(x)$ depends crucially on the magnitude of δ_n
 37 in relation to $n^{-1/2}$ and $(n \prod_{i=1}^{d_2} h_i)^{-1/2}$. For example, if $\delta_n =$
 38 $o(n^{-1/2})$, we can show that the result in Corollary 2.5 continues
 39 to hold by using fixed bandwidth; if $\delta_n \propto n^{-1/2}$, then $\widehat{\sigma}^2(x)$ con-
 40 tinues to converge to $\sigma^2(x)$ at the parametric rate by using fixed
 41 bandwidth and it has nonnegligible asymptotic bias determined
 42 by $\Delta(x)$; if $\delta_n n^{1/2} \rightarrow \infty$ and $\delta_n = o((n \prod_{i=1}^{d_2} h_i)^{-1/2})$, the result
 43 of Theorem 2.2 continues to hold for diminishing bandwidth; if
 44 $\delta_n (n \prod_{i=1}^{d_2} h_i) \rightarrow c \in (0, \infty]$, then $\delta_n \Delta(x)$ also contributes to the
 45 bias of $\widehat{\sigma}^2(x)$ in Theorem 2.2. For brevity, we omit the details.

52 2.5 Bandwidth Selection

53 As is the case for all nonparametric curve estimation, the
 54 bandwidth parameter plays an essential role in practice. It
 55 is desirable to have a reliable data-driven and yet easily-
 56 implementable bandwidth selection procedure.

57 One approach is to apply a ‘‘plug-in’’ method to obtain an es-
 58 timate of h as described for example in Fan and Gijbels (1996,

chapter 4.2 for the single regressor case). Without assuming
 the correct specification of the parametric conditional variance
 model, the asymptotic mean integrated squared error (AMISE)
 of $\widehat{\sigma}^2(x)$ is

$$64 \text{AMISE}(h) \equiv \int [B_h^2(x) + V_h(x)] w(x) dx, \quad (2.15)$$

65 where $B_h(x) = \frac{\kappa_{21}}{2} \text{tr}\{D_h[\sigma_p^2(x_1)\ddot{\sigma}_{np}^2(x_2) - \ddot{L}(0, \boldsymbol{\beta}^0)]\}$, $V_h(x) =$
 66 $\kappa_{02}^{d_2} (E(v_t^4 | X_{2,t} = x_2) - 1) f^{-1}(x_2) \sigma^4(x) / (n \prod_{i=1}^{d_2} h_i)$, and $w(x)$
 67 is a weighting function. In principle, one can choose the vector
 68 h to minimize $\text{AMISE}(h)$ but this may not result in an analytic
 69 solution. If we restrict that $h_1 = \dots = h_{d_2} = h_n$, then we only
 70 need to choose a scalar h_n to minimize $\text{AMISE}(h_n)$, the solution
 71 of which is given by

$$72 h_n^* = \left\{ \kappa_{02}^{d_2} \int (E(v_t^4 | X_{2,t} = x_2) - 1) f^{-1}(x_2) \sigma^4(x) w(x) dx \right.$$

$$73 \left. / \left(\kappa_{21}^2 \int \text{tr}\{[\sigma_p^2(x_1)\ddot{\sigma}_{np}^2(x_2) - \ddot{L}(0, \boldsymbol{\beta}^0)]\} \right. \right.$$

$$74 \left. \left. \times w(x) dx \right) \right\}^{1/(d_2+4)} n^{-1/(d_2+4)}. \quad (2.16)$$

75 Hence h_n^* converges to zero at the rate $n^{-1/(d_2+4)}$. Since h_n^*
 76 depends on several unknown quantities, to obtain an estimate
 77 for h_n^* , we need to estimate these unknown quantities first. This
 78 will generally require the choice of some pilot bandwidth. The
 79 performance of our estimate $\widehat{\sigma}^2(x)$ will be contingent upon the
 80 choice of such a pilot bandwidth and the estimates of these un-
 81 known quantities, which is a disadvantage of this approach. An-
 82 other drawback of this approach is that it can never be the op-
 83 timal bandwidth in the case of correct parametric specification.
 84 In this latter case, Corollary 2.5 suggests that we should hold
 85 the bandwidth fixed in order to achieve the parametric conver-
 86 gence rate of $\widehat{\sigma}^2(x)$. This parallels the case of a general local
 87 linear regression where underlying model may be linear or not.
 88 If the underlying model is indeed linear, it is well known that
 89 we should not let the bandwidth tend to zero. Instead, the band-
 90 width should be kept fixed.

91 So we propose to apply the leave-one-out least squares
 92 cross validation (LSCV) to obtain the data-driven bandwidth.
 93 Let $\widehat{\sigma}_{(-t)}^2(\widehat{X}_t)$ be the leave-one-out analog of $\widehat{\sigma}^2(\widehat{X}_t)$ without
 94 using the t th observation in the estimation. We choose $h =$
 95 (h_1, \dots, h_{d_2}) to minimize the following LSCV criterion func-
 96 tion:

$$97 \text{CV}(h) = \frac{1}{n} \sum_{t=1}^n (\widehat{r}_t^2 - \widehat{\sigma}_{(-t)}^2(\widehat{X}_t))^2 w(\widehat{X}_t), \quad (2.17)$$

98 where $w(\cdot)$ is a nonnegative weight function, for example,
 99 $w(\widehat{X}_t) = \prod_{i=1}^d 1(|\widehat{X}_{t,i} - \bar{x}_i| \leq 2s_i)$ with \bar{x}_i and s_i being the sample
 100 mean and standard deviation of $\widehat{X}_{t,i}$, respectively. Let \widehat{h} denote
 101 the minimizer of $\text{CV}(h)$. We conjecture \widehat{h} converges to the min-
 102 imizer of $\text{AMISE}(h)$ in (2.15) in case of misspecification of the
 103 parametric model and is not convergent to zero in case of cor-
 104 rect parametric specification. A formal study of the theoretical
 105 property of \widehat{h} is beyond the scope of this paper.

3. SIMULATION AND EMPIRICAL ANALYSIS

3.1 Simulation

To consider the data generating processes (DGPs), we focus on the case where $y_t = \varepsilon_t$, $\varepsilon_t = \sigma_t v_t$, $v_t \sim \text{iidN}(0, 1)$, $E(\varepsilon_t | \mathcal{F}_{t-1}) = 0$, and $E(\varepsilon_t^2 | \mathcal{F}_{t-1}) = \sigma_t^2$. Note that the parameter values for the DGPs are chosen such that the unconditional variance and persistence are similar across DGPs. The dynamics considered for σ_t^2 are specified as follows.

DGP 1: The ARCH(1) model of Engle (1982): $\sigma_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2$, where $\gamma_0 = 0.6$ and $\gamma_1 = 0.4$.

DGP 2: The GARCH(1, 1) model of Bollerslev (1986): $\sigma_t^2 = \gamma_0 + \gamma_1 \sigma_{t-1}^2 + \gamma_2 \varepsilon_{t-1}^2$, where $\gamma_0 = 0.03$, $\gamma_1 = 0.94$, and $\gamma_2 = 0.03$.

DGP 3: The threshold GARCH (GJR) model of Glosten, Jagannathan, and Runkle (1993):

$$\sigma_t^2 = \gamma_0 + \gamma_1 \sigma_{t-1}^2 + \gamma_2 \varepsilon_{t-1}^2 + \gamma_3 \varepsilon_{t-1}^2 \mathbf{1}(\varepsilon_{t-1} \leq 0),$$

where $\gamma_0 = 0.03$, $\gamma_1 = 0.93$, $\gamma_2 = 0.02$, and $\gamma_3 = 0.03$.

DGP 4: Combined GARCH (CGARCH): $\sigma_t^2 = (\gamma_0 + \gamma_1 \sigma_{t-1}^2 + \gamma_2 \varepsilon_{t-1}^2) \exp(\gamma_3 + \gamma_4 \sigma_{t-1}^2 + \gamma_5 |\varepsilon_{t-1}|)$, where $\gamma_0 = 0.03$, $\gamma_1 = 0.94$, and $\gamma_2 = 0.03$, $\gamma_3 = 0.56$, $\gamma_4 = -0.55$, and $\gamma_5 = 0.02$.

DGP 5: Combined ARCH (CARCH): $\sigma_t^2 = (\gamma_0 + \gamma_1 \varepsilon_{t-1}^2) \times \exp(\gamma_2 + \gamma_3 |\varepsilon_{t-1}|)$, where $\gamma_0 = 0.6$, $\gamma_1 = 0.4$, $\gamma_2 = 0.6$, and $\gamma_3 = -0.4$.

DGP 6: Stochastic Volatility (SV): $\log \sigma_t^2 = \gamma_0 + \gamma_1 \times \log \sigma_{t-1}^2 + \gamma_2 \zeta_t$, where $\gamma_0 = -0.04$, $\gamma_1 = 0.96$, and $\gamma_2 = 0.345$. We also assume that $(v_t, \zeta_t) \sim \text{iidN}(0, I_2)$, where I_2 is a 2×2 identity matrix.

For each DGP, we estimate the conditional variance by ARCH(1), GARCH(1, 1), the Nonparametric Local Exponential (NPLE) estimator of Ziegelmann (2002), and two versions of semiparametric estimators, SPGARCH and SPARCH, which correspond to the first stage GARCH(1, 1) and ARCH(1) parametric models, respectively. For both the SPARCH estimation and the NPLE estimation, we choose the conditioning variable $X_{2,t} = y_{t-1}$, while for the SPGARCH estimation, we choose $X_{2,t} = (y_{t-1}, \hat{\sigma}_{p,t-1}^2)^T$ where $\hat{\sigma}_{p,t-1}^2$ is the fitted parametric conditional variance in the first stage. In all cases we choose $\Psi(u) = \exp(u)$ for our semiparametric estimator to ensure the nonnegativity of the conditional variance estimator.

To obtain the NPLE estimator and our semiparametric estimator, we need to choose both the kernel and bandwidth. It is well known that the choice of kernel does not play any significant role in nonparametrics so that in all cases we use the normalized Epanechnikov kernel

$$k(u) = \frac{3}{4\sqrt{5}} \left(1 - \frac{1}{5}u^2\right) \mathbf{1}(|u| \leq \sqrt{5}).$$

In contrast, the choice of bandwidth is very important in nonparametric or semiparametric estimation. To avoid any ambiguity, we use the least squares cross-validation (LSCV) method for both estimators. The LSCV function for our second-stage nonparametric estimator is given in (2.17) and that for the NPLE estimator is similarly defined.

We use i, j , and t to denote the index of replications, models, and time, respectively. We draw replications of v_{it} independently (across both i and t) from the standard normal distribution, and use them to generate ε_{it} through the above specified conditional variance DGPs for $t = -n_0 + 1, -n_0 + 2, \dots, 1, \dots, n$. We throw away the first $n_0 = 500$ observations to avoid the starting-out effect and use sample sizes $n = 100, 200$, and 300 . The number of replications is $M = 200$ for each case. Let

$$B_t^j = \frac{1}{M} \sum_{i=1}^M [(\sigma_t^j)^2 - (\hat{\sigma}_{it}^j)^2], \quad t = 1, \dots, n, j = 1, \dots, 6,$$

$$S_t^j = \frac{1}{M} \sum_{i=1}^M \left[(\hat{\sigma}_{it}^j)^2 - \frac{1}{M} \sum_{i=1}^M (\hat{\sigma}_{it}^j)^2 \right]^2,$$

$$t = 1, \dots, n, j = 1, \dots, 6,$$

$B^j = [B_1^j, \dots, B_n^j]'$, and $S^j = [S_1^j, \dots, S_n^j]'$, where $(\sigma_t^j)^2$ is the true conditional variance for model j at time t , and $(\hat{\sigma}_{it}^j)^2$ is the estimated conditional variance for the i th replication and the j th model at time t . Here, B and S stand for bias and variance, respectively. Let $MSE_t^j = (B_t^j)^2 + S_t^j$ be the mean square error of estimates of $(\sigma_t^j)^2$. Thus we calculate the average MSE for model j by $MSE^j = n^{-1} \sum_{t=1}^n MSE_t^j$. We compare MSE^j for all the models and DGPs under study. For a given DGP, the lowest MSE value suggests the best fit of the model.

Table 1 provides the comparison of the MSEs. The MSEs for each DGP are presented across the columns [e.g., the first row corresponds to DGP ARCH(1)]. We have multiplied each MSE value by 10 for the convenience of presentation. The model corresponding to the minimum MSE for a given DGP is the best model. We find the following interesting points. (i) One of the main theoretical findings of the paper is that as long as the first-stage parametric conditional variance can capture some shape features of the true conditional variance function, the semiparametric estimators, namely SPGARCH and SPARCH, always dominate the NPLE estimator in terms of the MSEs. This is observed throughout our simulations. (ii) When the true DGP coincides with the fitted parametric model, then the parametric model is the dominant model. For example, ARCH(1) and GARCH(1, 1) are the best models for DGPs ARCH(1) and GARCH(1, 1), respectively. (iii) The parametric GARCH model outperforms SPARCH model for all DGPs except for the CARCH DGP. This can be explained by the fact that as the conditioning set of SPARCH model includes only y_{t-1} it cannot suitably capture the conditional variance of DGPs that have conditioning set $\{y_{-\infty}, \dots, y_{t-1}\}$. However, we find that SPGARCH model improves over GARCH model in all DGPs except for the GARCH(1, 1) and ARCH(1) DGPs. For example, when the sample size is 300 and the true DGP is GJR, the MSE for SPGARCH is 0.0021, and it is 0.0039 for GARCH(1, 1). (iv) When the true DGP is the stochastic volatility model, a model fundamentally different in structure compared to the GARCH class of models, we find that SPGARCH is the best model. (v) When the DGPs mimic the class of semiparametric model considered here (i.e., CGARCH and CARCH), we find that either the SPARCH or SPGARCH model dominate all other models. (vi) We note that whenever the first stage

Table 1. Comparison of mean square errors (MSEs) for various estimators of conditional variance

DGPs\Models	n	ARCH(1)	GARCH(1, 1)	NPLE	SPARCH	SPGARCH
ARCH(1)	100	0.084	0.096	0.254	0.098	0.228
	200	0.026	0.036	0.112	0.040	0.101
	300	0.010	0.024	0.076	0.026	0.069
GARCH(1, 1)	100	0.077	0.027	0.566	0.231	0.086
	200	0.029	0.013	0.302	0.086	0.044
	300	0.021	0.007	0.189	0.054	0.028
GJR	100	0.075	0.067	0.435	0.092	0.044
	200	0.064	0.042	0.420	0.087	0.040
	300	0.060	0.039	0.211	0.054	0.021
CGARCH	100	0.082	0.029	0.583	0.037	0.012
	200	0.076	0.009	0.324	0.031	0.006
	300	0.062	0.008	0.215	0.029	0.004
CARCH	100	0.131	0.121	0.973	0.112	0.119
	200	0.059	0.053	0.491	0.051	0.053
	300	0.029	0.042	0.296	0.026	0.040
SV	100	0.987	0.606	0.978	1.040	0.786
	200	0.415	0.300	0.638	0.606	0.102
	300	0.193	0.141	0.225	0.221	0.037

NOTE: The DGPs correspond to DGPs 1–6 in the text. We use five models to fit each DGP. The sample sizes are $n = 100, 200,$ and $300,$ and the number of simulations is $M = 200.$ The main entries are the mean square errors (MSEs) $\times 10$ associated with different DGPs. The comparison for a particular DGP in a given row is done across the columns, and given the DGP, the best model is the model with the lowest MSE value (boldfaced).

parametric model is misspecified SPGARCH model is the best model except for the CARCH case.

Another issue of relevance is the presence of residual nonlinearity. It is useful to analyze how close the second stage nonparametric estimation is to the true values. Since our second-stage estimator $\hat{\sigma}^2(X_t)$ estimates the conditional variance $\sigma_t^2 \equiv \sigma^2(X_t)$ directly, we can recover the estimator of $\sigma_{np}^2(X_{2,t})$ by

$\hat{\sigma}_{np}^2(X_{2,t}) \equiv \hat{\sigma}^2(X_t)/\sigma_p^2(X_{1,t}, \hat{\gamma}).$ If the first-stage parametric conditional variance model is correctly specified, one expects that $\hat{\sigma}_{np}^2(X_{2,t})$ vary in the neighborhood of unity. This is indeed observed in our simulations. In Table 2 we report the means and MSEs of the $\hat{\sigma}_{np}^2(X_{2,t})$ for the SPGARCH and SPARCH models under investigation. The MSE values are multiplied by 10^2 for the convenience of presentation. We observe that: (i) When the

Table 2. Mean and mean square error (MSE) for the estimator of the nonparametric component

DGPs\Models	n	SPARCH		SPGARCH	
		Mean	MSE $\times 10^2$	Mean	MSE $\times 10^2$
ARCH(1)	100	1.058	0.184	1.130	0.457
	200	1.044	0.083	1.111	0.185
	300	1.040	0.050	1.103	0.104
GARCH(1, 1)	100	1.363	0.978	1.086	0.147
	200	1.360	0.263	1.021	0.060
	300	1.395	0.185	1.019	0.038
GJR	100	0.848	0.524	0.921	0.158
	200	0.858	0.208	0.925	0.140
	300	0.862	0.065	0.932	0.036
CGARCH	100	0.860	1.221	1.094	0.057
	200	0.870	0.542	1.089	0.019
	300	0.880	0.316	1.090	0.012
CARCH	100	1.210	0.354	1.312	0.745
	200	1.201	0.188	1.310	0.400
	300	1.194	0.116	1.311	0.234
SV	100	1.782	6.500	1.630	1.840
	200	1.734	5.608	1.564	0.806
	300	1.742	4.308	1.597	0.563

NOTE: This table reports the mean and MSE for the estimator of the nonparametric component in the SPGARCH and SPARCH models. A mean value that is close to 1 and a small MSE value indicates a good first stage fit of the parametric conditional variance model to the true model.

1 first stage parametric conditional variance model is correctly
 2 specified, the mean of the estimator for the nonparametric
 3 component is close to 1 and the MSE of the estimator for the non-
 4 parametric component is lower than the case where the first
 5 stage model is misspecified. For example, when the DGP is
 6 GARCH(1, 1) and sample size $n = 300$, the MSE of $\hat{\sigma}_{np}^2(X_{2,t})$
 7 for the SPGARCH model is 0.00038, which is much lower than
 8 0.00185 for the SPARCH model. (ii) When the first stage para-
 9 metric model is misspecified, the mean of the estimator for the
 10 nonparametric component can be quite different from 1 and
 11 the performance of the SPGARCH and SPARCH estimators is
 12 mixed. For the CARCH process, we find that the MSE is lower
 13 for the SPARCH estimators, but for the GJR, CGARCH, and
 14 SV processes, we find that the SPGARCH estimator outper-
 15 forms SPARCH.

17 3.2 Empirical Data Analysis

18 In this subsection we fit the semiparametric model to the
 19 real data (collected from Datastream) and do some diagnostic
 20 checking to see whether the model is able to capture the time-
 21 varying conditional variance and leverage effect.

22 We consider the S&P 500 daily returns from January 3, 2002
 23 through January 3, 2007, a total of 1,258 observations. It
 24 is sometimes found that the conditional mean of stock return has
 25 AR or MA components. We therefore first carry out a diagnos-
 26 tic check for the return series to look for possible AR or MA
 27 components. The ACF and PACF of the series indicate no such
 28 memory structure. We also fit AR and MA models of lags of
 29 order 1 through 4 to check for the statistical significance of the
 30 parameter estimates. We also find that all the parameter esti-
 31 mates are statistically insignificant at the 5% level. It is possible
 32 to have nonlinear dynamics in the conditional mean equation,
 33 but we have not explored this avenue here. Consequently, we
 34 model the return series $\{y_t\}_{t=1}^n$ of a financial asset as $y_t = \alpha + \varepsilon_t$,
 35 where $E(\varepsilon_t | \mathcal{F}_{t-1}) = 0$ and $E(\varepsilon_t^2 | \mathcal{F}_{t-1}) = \sigma_t^2$. In addition to the
 36 two semiparametric models considered in the simulation, we
 37 consider a third semiparametric model where we fit the GJR
 38 model in the first-stage parametric estimation. It is useful to
 39 check whether the second stage estimation can capture some
 40 additional nonlinearity in the data.

41 In the first-stage parametric modeling, we estimate α by the
 42 sample average of y_t , and estimate the three parametric condi-
 43 tional variance models: ARCH(1), GARCH(1, 1), and GJR by
 44 using the Gaussian QMLE method. In the second-stage non-
 45 parametric estimation, we use y_{t-1} as the conditioning variable

for the ARCH(1) case and $(y_{t-1}, \hat{\sigma}_{p,t-1}^2)^T$ for the other two
 cases; and we denote the resulting semiparametric model as
 SPARCH, SPGARCH, and SPGJR, respectively. We find that
 estimate of α is statistically insignificant. Thus we do not pro-
 vide any result for this estimate. As in the simulations, we use
 the Epanechnikov kernel for all nonparametric estimation and
 use the least squares cross validation to choose the bandwidth
 for our second-stage nonparametric estimation.

Table 3 reports the estimates for the three parametric mod-
 els with corresponding standard errors in parentheses. All the
 parameter estimates appear significant at the conventional 5%
 significance level, which means the corresponding pseudo-true
 values are statistically different from 0. The last column in Ta-
 ble 3 indicates the percentage of variation of our combined
 volatility estimator that is captured by the first stage parametric
 estimator, that is, it is the sample variance of $\hat{\sigma}_p^2(\hat{X}_{1,t}, \hat{\mathcal{Y}})$ over
 that of $\hat{\sigma}^2(\hat{X}_t)$ multiplied by 100. The numbers indicate that the
 second-stage nonparametric estimation can capture 12%–19%
 of the variation of the conditional variance which the parametric
 model fails to capture.

Another issue of interest is the impact of y_{t-1} (innovation)
 on the current conditional variance when all other elements of
 the information set are kept constant. This is often referred to
 as news impact curve; see Engle and Ng (1993). Figure 1 plots
 the news impact curve for the ARCH, GARCH, and GJR mod-
 els, where we have normalized the conditional variance term to
 be zero when y_{t-1} is zero. For example, for the ARCH model
 this implies that it is a plot of $\hat{\sigma}_{p,t}^2 - \hat{\gamma}_0$ versus y_{t-1} , where
 $\hat{\sigma}_{p,t}^2 = \hat{\gamma}_0 + \hat{\gamma}_1 y_{t-1}^2$. All the plots have the expected shapes. To
 understand the role of second-stage estimation we also add the
 plot of $\hat{\sigma}_{np,t}^2$ versus y_{t-1} to the news impact curves for each para-
 metric model. We find that for all three cases, that is, SPARCH,
 SPGARCH, and SPGJR, the second-stage nonparametric fit is
 essentially downward sloping. Thus the multiplicative factor is
 higher for the negative shocks than that of the positive shocks.
 The second-stage asymmetry captures the leverage effect asso-
 ciated with negative news. Thus if any asymmetry is not ex-
 plained in the first stage then it is captured in the second-stage
 estimation (even in the case of SPGJR).

Figure 2 provides the pointwise 95% confidence intervals for
 the conditional variance estimates for all three semiparamet-
 ric models as a function of y_{t-1} (innovation). For the SPARCH

Table 3. Estimated parameters in the first stage estimation and decomposition of conditional variance

	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_3$	% variation explained by the parametric component
ARCH(1)	0.720 (0.037)	0.27 (0.043)			88.2
GARCH(1, 1)	0.004 (0.00025)	0.92 (0.0116)	0.062 (0.0105)		84.4
GJR	0.004 (0.00018)	0.94 (0.0136)	0.023 (0.0085)	0.040 (0.0061)	81.1

NOTE: The parameter interpretations are the same as given in the simulation. Numbers in parentheses are standard errors. The last column indicates the percentage of total variation of the volatility estimator explained by the first-stage parametric fit.

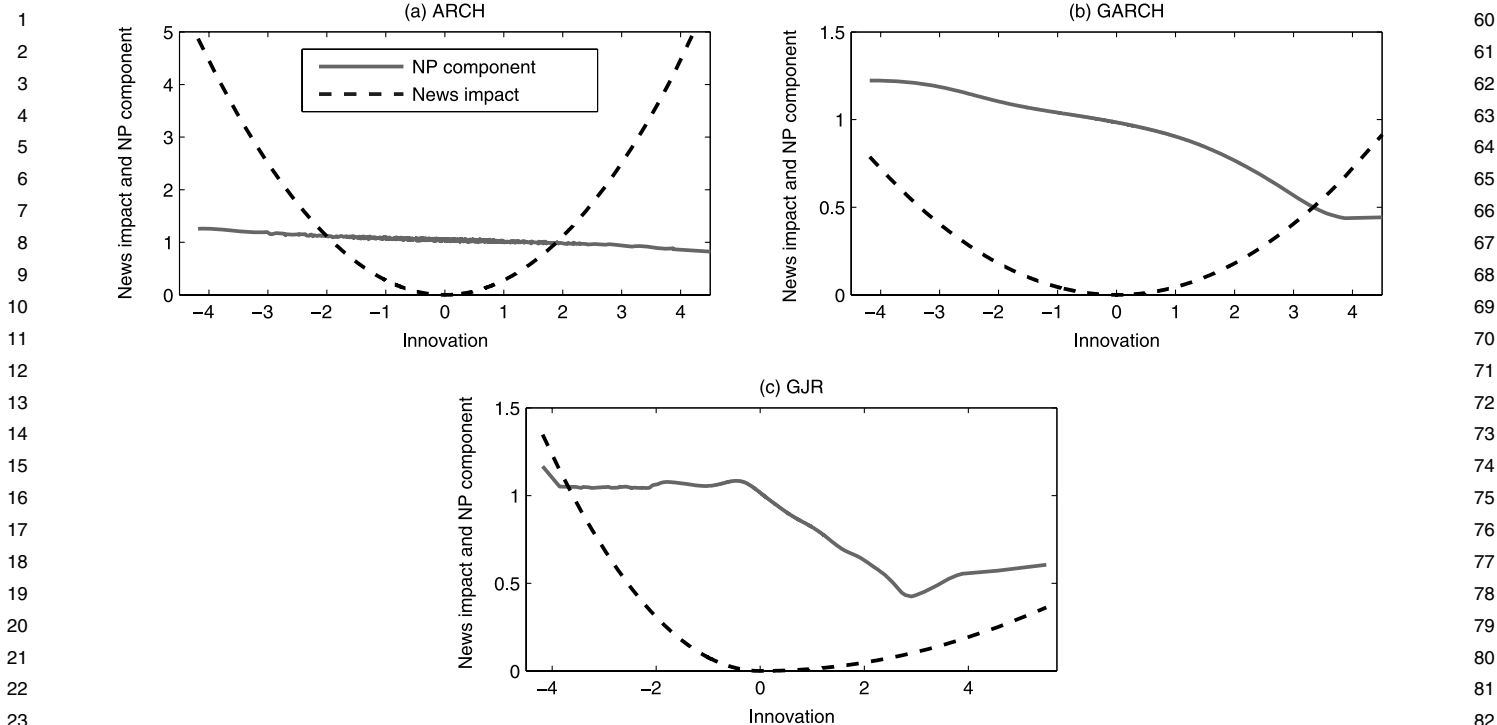


Figure 1. News impact curve and second-stage nonlinearity for S&P 500 daily returns.

case, there is only one conditioning variable, that is, y_{t-1} , in the construction of the semiparametric estimator. For the SP-GARCH and SPGJR cases, we have two conditioning variables

(y_{t-1} and $\hat{\sigma}_{p,t-1}^2$) to construct the semiparametric estimator and have held $\hat{\sigma}_{p,t-1}^2$ fixed at their sample mean level in order to obtain plots (b)–(c) in Figure 2. As we can tell from Figure 2,

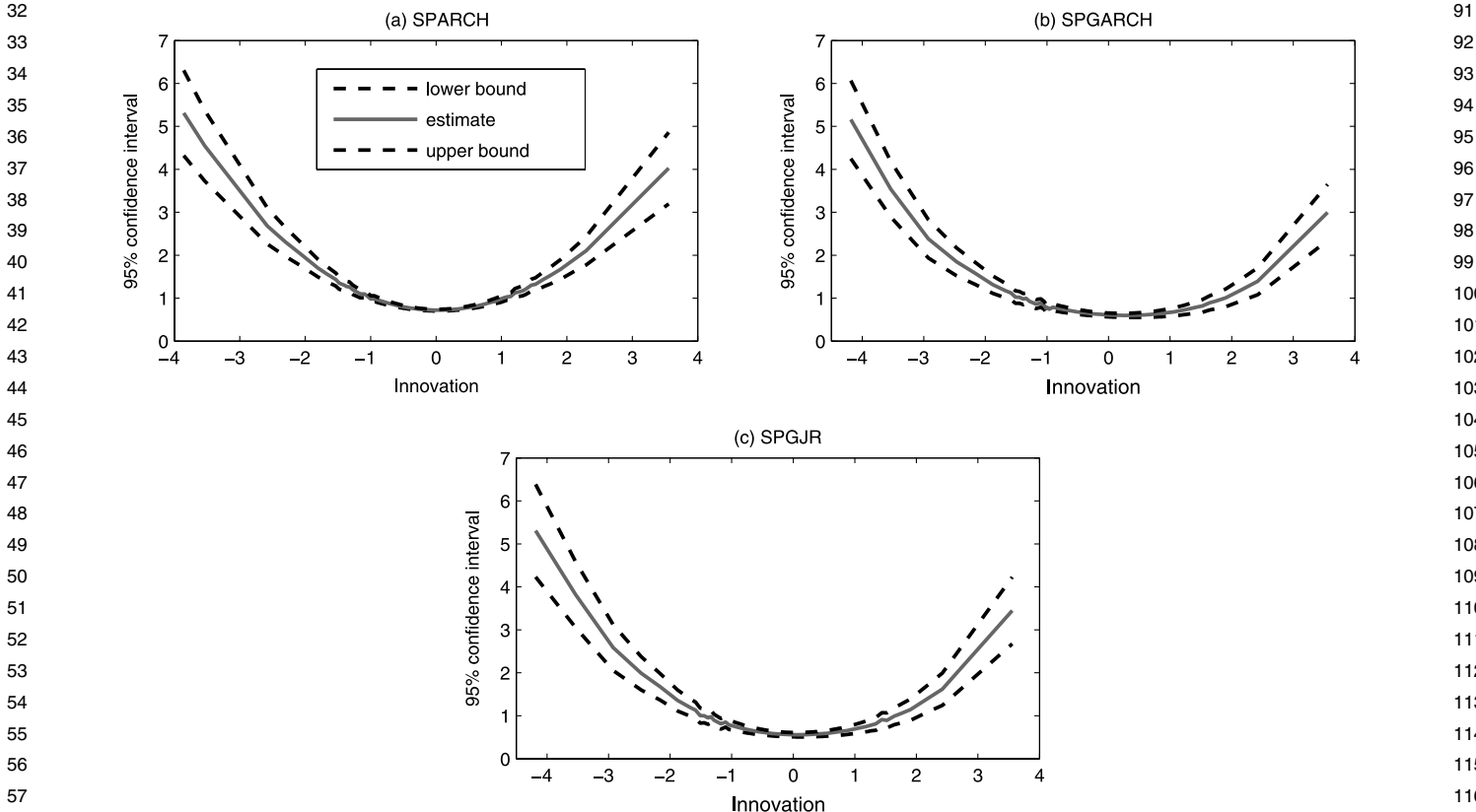


Figure 2. 95% confidence interval for the SPARCH, SPGARCH, and SPGJR estimates of conditional variance for S&P 500 daily returns.

the three semiparametric estimates share similar shapes despite the fact that different parametric models are fitted in the first stage. All of them can capture the leverage effects that are well documented in the finance literature.

Diagnostic check for the standardized residuals is an important aspect of modeling exercise. The basic diagnostic checks include visual inspection of the ACF and PACF plots of standardized residuals. It is also useful to check correlations of polynomials involving standardized residuals to detect nonlinear dependence. It is well known that the standard Box–Pierce statistic is not applicable in our framework due to the presence of conditional heteroscedasticity and nonparametric estimates. Li and Mak (1994) propose an alternative chi-square-based test to account for conditional heteroscedasticity. Nevertheless, their test statistic depends on the estimates of the Hessian matrix from a log-likelihood function (see lemma 3.3 of Ling and Li 1997 for details), which is not applicable in the presence of nonparametric estimation. Wong and Ling (2005) propose a modification to Li and Mak (1994) which allows for potential parametric misspecification. But the test statistic is posited in a one-step likelihood based framework and not applicable to our case either. Hence we eschew the Box–Pierce type of chi-square test and limit ourselves to the ACF and PACF tests indicated above.

We consider the ACF and PACF for the standardized residuals for models SPARCH, SPGARCH, and SPGJR. The lag length for our analysis is 10 and the 95% confidence bound for testing for white noise is ± 0.057 for our sample. Let ACF10 and PACF10 be the maximum of the absolute value of ACF and PACF up to lag 10, respectively. First, we find that the ACF and PACF value for the first lag for SPARCH model is 0.17 and 0.16, respectively, clearly rejecting the null hypothesis of white noise. However, for the SPGARCH model ACF10 and PACF10 values are 0.051 and 0.052, respectively, so all the ACF and PACF values till lag 10 remain within the 95% confidence bound. For the SPGJR model the corresponding ACF10 and PACF10 values are 0.049 and 0.050, respectively, thus, qualitatively, we draw the same conclusion as that of the SPGARCH model. To analyze higher-order dependence, we investigate the ACF and PACF values of squared and cubed standardized residuals for all three models. For the SPARCH model the first-order ACF and PACF for squared standardized residuals are 0.12 and 0.13, respectively, thus rejecting the null of white noise. The result is qualitatively similar for the cubed standardized residuals. For SPGARCH model the ACF10 and PACF10 for squared standardized residuals are given as 0.046 and 0.048, respectively. The corresponding ACF10 and PACF10 values for cubed standardized residuals are 0.050 and 0.053, respectively. Similarly, for SPGJR model ACF10 and PACF10 for squared standardized residuals are given as 0.049 and 0.047, respectively. The corresponding ACF10 and PACF10 values for cubed standardized residuals are 0.044 and 0.046, respectively. This provides support, albeit partial, for lack of dependence in standardized residuals obtained from SPGARCH and SPGJR model.

4. CONCLUSION

This paper proposes a new semiparametric estimator for time varying conditional variance. This is accomplished by combin-

ing the parametric and nonparametric estimators in a multiplicative way. We provide asymptotic theory for our semiparametric estimator and show that it can improve upon the pure parametric and nonparametric estimators in different scenarios. We also include simulations and empirical applications. We find from the simulations that our semiparametric estimators are superior to their parametric and nonparametric counterparts in terms of MSE criteria. From the empirical analysis we see that semiparametric models can capture the asymmetric effect in the data even beyond that captured by the parametric component in the model.

ACKNOWLEDGMENTS

The authors gratefully thank Arthur Lewbel, Serena Ng, the associate editor, and two anonymous referees for their many helpful comments and advice that have led to considerable improvements of the presentation. They are grateful to Torben G. Andersen, John Galbraith, Gloria González-Rivera, Qi Li, Essie Maasoumi, Nour Meddahi, and Victoria Zinde-Walsh for their discussions and comments on the subject matter of this paper. They are also thankful for the comments by the participants of the seminars at McGill University, Southern Methodist University, Syracuse University, Texas A&M University, Vanderbilt University, Université de Montréal, and University of Guelph. The second author gratefully acknowledges financial support from the NSFC under the grants 70501001 and 70601001. The third author acknowledges the support from the academic senate, UCR.

APPENDIX: PROOFS

We use $\|\cdot\|$ to denote the Euclidean norm of \cdot , C to signify a generic constant whose exact value may vary from case to case, and a^T to denote the transpose of a . For ease of presentation, we assume that $h_1 = \dots = h_{d_2}$ and with a little abuse of notation, we further denote $h_1 = \dots = h_{d_2} = h$. Recall $K_{ht} \equiv K_h(X_{2,t} - x_2)$ and $r_t \equiv \varepsilon_t \sigma_p(x_1) / \sigma_p(X_{1,t})$. Let $\xi_t \equiv r_t^2 - \sigma_p^2(x_1) \sigma_{np}^2(X_{2,t})$.

A.1. Proof of Theorem 2.1

Recall that $\hat{\beta}$ minimizes

$$\frac{1}{n} \sum_{t=1}^n \{\hat{r}_t^2 - L(\hat{X}_{2,t} - x_2, \beta)\}^2 K_h(\hat{X}_{2,t} - x_2).$$

It also minimizes the criterion function

$$\hat{G}_n(\beta) \equiv \frac{1}{n} \sum_{t=1}^n \{[\hat{r}_t^2 - L(\hat{X}_{2,t} - x_2, \beta)]^2 - [(\hat{r}_t^2)^2 - (r_t^2)^2]\} \times K_h(\hat{X}_{2,t} - x_2). \quad (\text{A.1})$$

By theorem 3.4 of White (1994), it suffices to show that

$$\hat{G}_n(\beta) - G_n(\beta) \xrightarrow{p} 0 \quad \text{uniformly in } \beta \text{ on a compact set } \mathcal{B}, \quad (\text{A.2})$$

and

$$\limsup_{n \rightarrow \infty} \max_{\beta \in N_\epsilon^c(\beta^0)} [G_n(\beta) - G_n(\beta^0)] > 0 \quad \text{for any } \epsilon > 0, \quad (\text{A.3})$$

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where

$$G_n(\boldsymbol{\beta}) \equiv E\{[\sigma_p^2(x_1)\sigma_{np}^2(X_{2,t}) - \varsigma_{1t}(\boldsymbol{\beta})]^2 K_{ht}\} + E\{\xi_t^2 K_{ht}\} \quad (\text{A.4})$$

$\varsigma_{1t}(\boldsymbol{\beta}) \equiv L(0, \boldsymbol{\beta}) - \dot{L}(0, \boldsymbol{\beta})^T(X_{2,t} - x_2)$, $N_\epsilon^c(\boldsymbol{\beta}^0)$ is the complement of an open neighborhood of $\boldsymbol{\beta}^0$ on \mathcal{B} of diameter ϵ , and $\boldsymbol{\beta}^0$ is uniquely defined by $\sigma^2(x) = L(0, \boldsymbol{\beta}^0)$ and $\sigma_p^2(x_1)\dot{\sigma}_{np}^2(x_2) = \dot{L}(0, \boldsymbol{\beta}^0)$. Write

$$\begin{aligned} \widehat{G}_n(\boldsymbol{\beta}) &= \frac{1}{n} \sum_{t=1}^n [r_t^2 - L(X_{2,t} - x_2, \boldsymbol{\beta})]^2 K_h(\widehat{X}_{2,t} - x_2) \\ &\quad + \frac{1}{n} \sum_{t=1}^n \{L^2(\widehat{X}_{2,t} - x_2, \boldsymbol{\beta}) - L^2(X_{2,t} - x_2, \boldsymbol{\beta})\} \\ &\quad \times K_h(\widehat{X}_{2,t} - x_2) \\ &\quad - \frac{2}{n} \sum_{t=1}^n \{r_t^2 L(\widehat{X}_{2,t} - x_2, \boldsymbol{\beta}) - r_t^2 L(X_{2,t} - x_2, \boldsymbol{\beta})\} \\ &\quad \times K_h(\widehat{X}_{2,t} - x_2) \\ &\equiv \widehat{G}_{n1}(\boldsymbol{\beta}) + \widehat{G}_{n2}(\boldsymbol{\beta}) - 2\widehat{G}_{n3}(\boldsymbol{\beta}). \end{aligned} \quad (\text{A.5})$$

We will show that

$$\widehat{G}_{n1}(\boldsymbol{\beta}) = G_n(\boldsymbol{\beta}) + o_p(1) \quad \text{uniformly in } \boldsymbol{\beta}, \quad (\text{A.6})$$

and

$$\widehat{G}_{nj}(\boldsymbol{\beta}) = o_p(1) \quad \text{uniformly in } \boldsymbol{\beta} \text{ for } j = 2, 3. \quad (\text{A.7})$$

By Assumptions A2 and A4–A6, and the Taylor expansion, it is easy to show that $\widehat{G}_{n1}(\boldsymbol{\beta}) = \widetilde{G}_{n1}(\boldsymbol{\beta}) + O_p(n^{-1/2}h^{-1} + n^{-1}h^{-(d_2+1)}) = \widetilde{G}_{n1}(\boldsymbol{\beta}) + o_p(1)$ uniformly in $\boldsymbol{\beta}$, where $\widetilde{G}_{n1}(\boldsymbol{\beta}) \equiv \frac{1}{n} \sum_{t=1}^n K_{ht} [r_t^2 - L(X_{2,t} - x_2, \boldsymbol{\beta})]^2$. By the ergodic theorem, $\widetilde{G}_{n1}(\boldsymbol{\beta}) \xrightarrow{p} E[q_{nt}(\boldsymbol{\beta})]$, where $q_{nt}(\boldsymbol{\beta}) \equiv [r_t^2 - L(X_{2,t} - x_2, \boldsymbol{\beta})]^2 \times K_{ht}$. To show that this convergence is also uniform in $\boldsymbol{\beta}$, it suffices to show the stochastic equicontinuity of $\widetilde{G}_{n1}(\boldsymbol{\beta})$. We do this by verifying the two conditions of lemma 3 in Andrews (1992). By the compactness of \mathcal{B} , the fact that $K(\cdot)$ is compactly supported (Assumption A5), and Assumption A1(iii), $q_{nt}(\boldsymbol{\beta}) \leq 2r_t^2 K_{ht} + 2CK_{ht} \equiv \bar{q}_{nt}$. Clearly, $E[\bar{q}_{nt}] < \infty$, which implies assumption DM of Andrews (1992). Similarly,

$$\begin{aligned} |q_{nt}(\boldsymbol{\beta}) - q_{nt}(\boldsymbol{\beta}')| &\leq 2r_t^2 |L(X_{2,t} - x_2, \boldsymbol{\beta}) - L(X_{2,t} - x_2, \boldsymbol{\beta}')| K_{ht} \\ &\quad + |L^2(X_{2,t} - x_2, \boldsymbol{\beta}) - L^2(X_{2,t} - x_2, \boldsymbol{\beta}')| K_{ht} \\ &\leq (r_t^2 \bar{C}_1 + \bar{C}_2) \|\boldsymbol{\beta} - \boldsymbol{\beta}'\| \end{aligned}$$

for some constants \bar{C}_1 and \bar{C}_2 , which implies assumption TSE of Andrews (1992) by the Markov inequality. Consequently, $\widetilde{G}_{n1}(\boldsymbol{\beta})$ is stochastic equicontinuous on \mathcal{B} and it converges to $E[q_{nt}(\boldsymbol{\beta})]$ uniformly in $\boldsymbol{\beta}$.

By the second-order Taylor expansion, we have

$$\begin{aligned} E[q_{nt}(\boldsymbol{\beta})] &= E\{[r_t^2 - \varsigma_{1t}(\boldsymbol{\beta}) - \varsigma_{2t}(\boldsymbol{\beta})]^2 K_{ht}\} \\ &= E\{[\xi_t + (\sigma_p^2(x_1)\sigma_{np}^2(X_{2,t}) \\ &\quad - \varsigma_{1t}(\boldsymbol{\beta})) - \varsigma_{2t}(\boldsymbol{\beta})]^2 K_{ht}\} \end{aligned}$$

$$\begin{aligned} &= G_n(\boldsymbol{\beta}) + E\{\varsigma_{2t}^2(\boldsymbol{\beta})K_{ht}\} \\ &\quad + 2E\{\xi_t[\sigma_p^2(x_1)\sigma_{np}^2(X_{2,t}) - \varsigma_{1t}(\boldsymbol{\beta})]K_{ht}\} \\ &\quad - 2E\{\xi_t\varsigma_{2t}(\boldsymbol{\beta})K_{ht}\} \\ &\quad - 2E\{[\sigma_p^2(x_1)\sigma_{np}^2(X_{2,t}) - \varsigma_{1t}(\boldsymbol{\beta})]\varsigma_{2t}(\boldsymbol{\beta})K_{ht}\} \\ &\equiv G_n(\boldsymbol{\beta}) + \bar{G}_{n1a}(\boldsymbol{\beta}) + \bar{G}_{n1b}(\boldsymbol{\beta}) \\ &\quad - 2\bar{G}_{n1c}(\boldsymbol{\beta}) - 2\bar{G}_{n1d}(\boldsymbol{\beta}), \end{aligned}$$

where $\varsigma_{2t}(\boldsymbol{\beta}) \equiv \frac{1}{2}(X_{2,t} - x_2)^T \ddot{L}(C_t(X_{2,t} - x_2), \boldsymbol{\beta})(X_{2,t} - x_2)$, and C_t is a $d_2 \times d_2$ matrix with elements that lie on the interval $[0, 1]$. We can easily verify that uniformly in $\boldsymbol{\beta}$, $\bar{G}_{n1a}(\boldsymbol{\beta}) = O(h^4)$, $\bar{G}_{n1b}(\boldsymbol{\beta}) = \bar{G}_{n1c}(\boldsymbol{\beta}) = 0$ because $E(\xi_t | \mathcal{F}_{t-1}) = 0$ and $X_t \in \mathcal{F}_{t-1}$, and $\bar{G}_{n1d}(\boldsymbol{\beta}) = O(h^2)$. Consequently,

$$E[q_{nt}(\boldsymbol{\beta})] = G_n(\boldsymbol{\beta}) + o(1) \quad \text{uniformly in } \boldsymbol{\beta}$$

and (A.6) follows. Now, by the Taylor expansion and Assumptions A1(iii)–A2 and A4–A6,

$$\begin{aligned} &\sup_{\boldsymbol{\beta} \in \mathcal{B}} |\widehat{G}_{n2}(\boldsymbol{\beta})| \\ &\leq \sup_{\boldsymbol{\beta} \in \mathcal{B}} \frac{1}{n} \sum_{t=1}^n |L^2(\widehat{X}_{2,t} - x_2, \boldsymbol{\beta}) - L^2(X_{2,t} - x_2, \boldsymbol{\beta})| \\ &\quad \times K_h(\widehat{X}_{2,t} - x_2) \\ &= \sup_{\boldsymbol{\beta} \in \mathcal{B}} \frac{2}{n} \sum_{t=1}^n |L(X_{2,t}^* - x_2, \boldsymbol{\beta}) \dot{L}(X_{2,t}^* - x_2, \boldsymbol{\beta})(\widehat{X}_{2,t} - X_{2,t})| \\ &\quad \times K_h(\widehat{X}_{2,t} - x_2) \\ &\leq \frac{2C}{n} \sum_{t=1}^n \|\widehat{X}_{2,t} - X_{2,t}\| K_h(\widehat{X}_{2,t} - x_2) \\ &= \frac{2C}{n} \sum_{t=1}^n \|D_{\boldsymbol{\theta}} X_{2,t}(\boldsymbol{\theta}_t)(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\| \\ &\quad \times |K_{ht} + [\dot{K}_h(X_{2,t}^{**} - x_2)]^T (\widehat{X}_{2,t} - X_{2,t})| \\ &\leq \frac{2C}{n} \sum_{t=1}^n (\|D_{\boldsymbol{\theta}} X_{2,t}(\boldsymbol{\theta}^0)\| \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0\| + O_p(n^{-1})) \\ &\quad \times |K_{ht} + [\dot{K}_h(X_{2,t} - x_2)]^T D_{\boldsymbol{\theta}} X_{2,t}(\boldsymbol{\theta}^0)(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0) \\ &\quad + O_p(n^{-1}h^{-d_2-1})| \\ &\leq \frac{2C}{n} \sum_{t=1}^n \|D_{\boldsymbol{\theta}} X_{2,t}(\boldsymbol{\theta}^0)\| K_{ht} \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0\| \\ &\quad + \frac{2C}{n} \sum_{t=1}^n \|D_{\boldsymbol{\theta}} X_{2,t}(\boldsymbol{\theta}^0)\|^2 \|\dot{K}_h(X_{2,t} - x_2)\| \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0\|^2 \\ &\quad + O_p(n^{-3/2}h^{-d_2-1}) \\ &= O_p(n^{-1/2}) + O_p(n^{-1}h^{-1}) + O_p(n^{-3/2}h^{-d_2-1}) = o_p(1), \end{aligned}$$

where $X_{2,t}^*$ and $X_{2,t}^{**}$ lie between $\widehat{X}_{2,t}$ and $X_{2,t}$, $\boldsymbol{\theta}_t$ lies between $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$ and $\dot{K}_h(u) \equiv (\partial/\partial u)K_h(u)$. Similarly, we can show that $\sup_{\boldsymbol{\beta} \in \mathcal{B}} |\widehat{G}_{n3}(\boldsymbol{\beta})| = o_p(1)$. Consequently,

$$\widehat{G}_n(\boldsymbol{\beta}) = G_n(\boldsymbol{\beta}) + o_p(1) \quad \text{uniformly in } \boldsymbol{\beta}.$$

Note that choosing β to minimize $G_n(\beta)$ is equivalent to choosing $L(0, \beta)$ and $\dot{L}(0, \beta)$ to minimize

$$G_n^*(\beta) \equiv E\{[\sigma_p^2(x_1)\sigma_{np}^2(X_{2,t}) - L(0, \beta) - \dot{L}(0, \beta)^T(X_{2,t} - x_2)]^2 K_{ht}\}.$$

It is easy to verify that when $L(0, \beta) = \sigma_p^2(x_1)\sigma_{np}^2(x_2)$ and $\dot{L}(0, \beta) = \sigma_p^2(x_1)\dot{\sigma}_{np}^2(x_2)$, $G_n^*(\beta) = O(h^4)$ is minimized. By the monotonicity of $\Psi(\cdot)$, β^0 is uniquely determined by $L(0, \beta^0) = \sigma_p^2(x_1)\sigma_{np}^2(x_2)$ and $\dot{L}(0, \beta^0) = \sigma_p^2(x_1)\dot{\sigma}_{np}^2(x_2)$. This implies (A.3)

A.2. Proof of Theorem 2.2

Let

$$R_n(x, \beta) \equiv \sum_{t=1}^n \{\hat{r}_t^2 - L(\hat{X}_{2,t} - x_2, \beta)\}^2 K_h(\hat{X}_{2,t} - x_2). \quad (\text{A.8})$$

A second-order Taylor expansion of $L(\hat{X}_{2,t} - x_2, \beta)$ around $L(0, \beta)$ yields

$$R_n(x, \beta) = \sum_{t=1}^n \left\{ \hat{r}_t^2 - L(0, \beta) - \dot{L}(0, \beta)^T(\hat{X}_{2,t} - x_2) - \frac{1}{2}(\hat{X}_{2,t} - x_2)^T \ddot{L}(C_t(\hat{X}_{2,t} - x_2), \beta)(\hat{X}_{2,t} - x_2) \right\}^2 \times K_h(\hat{X}_{2,t} - x_2),$$

where C_t is a $d_2 \times d_2$ matrix with elements that lie on the interval $[0, 1]$. Define $R_n^*(x, \beta)$ as $R_n(x, \beta)$ with β in $\ddot{L}(C_t(\hat{X}_{2,t} - x_2), \beta)$ replaced by $\hat{\beta}$. Let $\hat{\beta}^*$ denote the minimizer of $R_n^*(x, \beta)$ and put $\hat{\sigma}_*^2(x) = L(0, \hat{\beta}^*)$. It suffices to show that

$$\sqrt{nh^{d_2}}\{\hat{\sigma}_*^2(x) - \sigma^2(x) - B(x)\} \xrightarrow{d} N(0, \kappa_{02}^{d_2}(E(v_1^4|X_{2,t} = x_2) - 1)f(x_2)^{-1}\sigma^4(x)) \quad (\text{A.9})$$

and

$$\hat{\sigma}^2(x) = \hat{\sigma}_*^2(x) + o_p(n^{-1/2}h^{-d_2/2}), \quad (\text{A.10})$$

where $B(x) \equiv \frac{\kappa_{21}h^2}{2} \text{tr}\{\sigma_p^2(x_1)\ddot{\sigma}_{np}^2(x_2) - \ddot{L}(0, \beta^0)\}$.

By Ruppert and Wand (1994, p. 1348) or Fan and Gijbels (1996, p. 298),

$$\hat{\sigma}_*^2(x) = e_1^T (\hat{X}_x^T \hat{W}_x \hat{X}_x)^{-1} \hat{X}_x^T \hat{W}_x \hat{U}_x,$$

where e_1 is the $(d_2 + 1)$ vector having 1 in the first entry and all other entries 0, \hat{X}_x is an $n \times (d_2 + 1)$ matrix whose t th row is given by $\hat{X}_{x,t} \equiv (1, (\hat{X}_{2,t} - x_2)^T)$, $\hat{W}_x = \text{diag}\{K_h(\hat{X}_{2,1} - x_2), \dots, K_h(\hat{X}_{2,n} - x_2)\}$, and \hat{U}_x is an $n \times 1$ vector with typical element

$$\hat{U}_{x,t} \equiv \hat{r}_t^2 - \frac{1}{2}(\hat{X}_{2,t} - x_2)^T \ddot{L}(C_t(\hat{X}_{2,t} - x_2), \hat{\beta})(\hat{X}_{2,t} - x_2).$$

Define the $(d_2 + 1) \times (d_2 + 1)$ matrix

$$S_n(x_2) = \begin{pmatrix} n^{-1} \sum_{t=1}^n K_h(\hat{X}_{2,t} - x_2) & & \\ n^{-1} \sum_{t=1}^n K_h(\hat{X}_{2,t} - x_2) \frac{\hat{X}_{2,t} - x_2}{h} & & \\ & n^{-1} \sum_{t=1}^n K_h(\hat{X}_{2,t} - x_2) \frac{(\hat{X}_{2,t} - x_2)^T}{h} & \\ & & n^{-1} \sum_{t=1}^n K_h(\hat{X}_{2,t} - x_2) \frac{\hat{X}_{2,t} - x_2}{h} \frac{(\hat{X}_{2,t} - x_2)^T}{h} \end{pmatrix} \quad (\text{A.11})$$

and the equivalent kernel

$$K_n^*(u, x_2) = e_1^T S_n(x_2)^{-1} (1, (u - x_2)^T/h)^T K_h(u - x_2). \quad (\text{A.12})$$

Then by Ruppert and Wand (1994, p. 1351), $\hat{\sigma}_*^2(x) = \sum_{t=1}^n K_n^*(\hat{X}_{2,t}, x_2) \hat{U}_{x,t}$, $n^{-1} \sum_{t=1}^n K_n^*(\hat{X}_{2,t}, x_2) = 1$, and $n^{-1} \times \sum_{t=1}^n K_n^*(\hat{X}_{2,t}, x_2) (\hat{X}_{2,t} - x_2) = 0$. Hence

$$\begin{aligned} \hat{\sigma}_*^2(x) - \sigma^2(x) &= n^{-1} \sum_{t=1}^n K_n^*(\hat{X}_{2,t}, x_2) \\ &\times \left\{ \hat{r}_t^2 - \sigma^2(x) - \sigma_p^2(x_1) \dot{\sigma}_{np}^2(x_2)^T (\hat{X}_{2,t} - x_2) - \frac{1}{2} (\hat{X}_{2,t} - x_2)^T \ddot{L}(C_t(\hat{X}_{2,t} - x_2), \hat{\beta})(\hat{X}_{2,t} - x_2) \right\} \\ &= n^{-1} \sum_{t=1}^n K_n^*(\hat{X}_{2,t}, x_2) \left\{ [\hat{r}_t^2 - \sigma_p^2(x_1) \sigma_{np}^2(\hat{X}_{2,t})] + \frac{1}{2} \hat{A}_{nt} \right\} \\ &= n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) [\hat{r}_t^2 - \sigma_p^2(x_1) \sigma_{np}^2(\hat{X}_{2,t})] \\ &\quad + \frac{1}{2} n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) \hat{A}_{nt} \\ &\quad + n^{-1} \sum_{t=1}^n [K_n^*(\hat{X}_{2,t}, x_2) - K_n^*(X_{2,t}, x_2)] \\ &\quad \times [\hat{r}_t^2 - \sigma_p^2(x_1) \sigma_{np}^2(\hat{X}_{2,t})] \\ &\quad + \frac{1}{2} n^{-1} \sum_{t=1}^n [K_n^*(\hat{X}_{2,t}, x_2) - K_n^*(X_{2,t}, x_2)] \hat{A}_{nt} \\ &\equiv T_{n1} + T_{n2} + T_{n3} + T_{n4}, \end{aligned} \quad (\text{A.13})$$

where

$$\hat{A}_{nt} = (\hat{X}_{2,t} - x_2)^T [\sigma_p^2(x_1) \ddot{\sigma}_{np}^2(\tilde{C}_t(\hat{X}_{2,t} - x_2) + x_2) - \ddot{L}(C_t(\hat{X}_{2,t} - x_2), \hat{\beta})] (\hat{X}_{2,t} - x_2), \quad (\text{A.14})$$

and \tilde{C}_t is a $d_2 \times d_2$ matrix with elements that lie on the interval $[0, 1]$. T_{nj} , $j = 1, 2, 3, 4$, are analyzed in Lemmas A.1–A.4 below. Collecting the results in these lemmas, we obtain (A.9).

Next we show (A.10). By Theorem 2.1, under Assumptions A0–A6, $\beta^0 \equiv \beta^0(x)$ is uniquely defined by $\sigma^2(x) = L(0, \beta^0)$ and $\sigma_p^2(x_1) \dot{\sigma}_{np}^2(x_2) = \dot{L}(0, \beta^0)$, and $\|\hat{\beta} - \beta^0\| \xrightarrow{p} 0$. Further, following the arguments of Hall, Wolf, and Yao (1999, p. 163), one can show that $\hat{\beta}^* - \hat{\beta} = o_p(h^2)$. Consequently, $\hat{\sigma}^2(x) - \hat{\sigma}_*^2(x) = L(0, \hat{\beta}) - L(0, \hat{\beta}^*) = o_p(h^2) = o_p(n^{-1/2}h^{-d_2/2})$.

Let T_{nj} , $j = 1, 2, 3, 4$, be defined as in (A.13), we prove the following lemmas under the conditions of Theorem 2.2.

Lemma A.1. Recall $T_{n1} \equiv n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) [\hat{r}_t^2 - \sigma_p^2(x_1) \sigma_{np}^2(\hat{X}_{2,t})]$. Then $\sqrt{nh^{d_2}} T_{n1} \xrightarrow{d} N(0, \kappa_{02}^{d_2}(E(v_1^4|X_{2,t} = x_2) - 1)f(x_2)^{-1}\sigma^4(x))$.

Proof. Noting that $\widehat{\varepsilon}_t = \varepsilon_t + (g(U_t, \boldsymbol{\alpha}^0) - g(U_t, \widehat{\boldsymbol{\alpha}}))$, we have

$$\begin{aligned} \widehat{r}_t^2 &= \frac{\varepsilon_t^2 \sigma_p^2(x_1, \widehat{\boldsymbol{\gamma}})}{\sigma_p^2(X_{1,t}, \widehat{\boldsymbol{\gamma}})} - \frac{\varepsilon_t^2 \sigma_p^2(x_1, \widehat{\boldsymbol{\gamma}})}{\sigma_p^2(X_{1,t}, \widehat{\boldsymbol{\gamma}})} \left(1 - \frac{\sigma_p^2(X_{1,t}, \widehat{\boldsymbol{\gamma}})}{\sigma_p^2(\widehat{X}_{1,t}, \widehat{\boldsymbol{\gamma}})}\right) \\ &\quad + \frac{2\varepsilon_t [g(U_t, \boldsymbol{\alpha}^0) - g(U_t, \widehat{\boldsymbol{\alpha}})] \sigma_p^2(x_1, \widehat{\boldsymbol{\gamma}})}{\sigma_p^2(\widehat{X}_{1,t}, \widehat{\boldsymbol{\gamma}})} \\ &\quad + \frac{[g(U_t, \boldsymbol{\alpha}^0) - g(U_t, \widehat{\boldsymbol{\alpha}})]^2 \sigma_p^2(x_1, \widehat{\boldsymbol{\gamma}})}{\sigma_p^2(\widehat{X}_{1,t}, \widehat{\boldsymbol{\gamma}})} \\ &\equiv \xi_{n1t} - \xi_{n2t} + \xi_{n3t} + \xi_{n4t}. \end{aligned} \quad (\text{A.15})$$

Then

$$\begin{aligned} T_{n1} &= n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) [\xi_{n1t} - \sigma_p^2(x_1) \sigma_{np}^2(X_{2,t})] \\ &\quad - n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) \sigma_p^2(x_1) [\sigma_{np}^2(\widehat{X}_{2,t}) - \sigma_{np}^2(X_{2,t})] \\ &\quad - n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) \xi_{n2t} + n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) \xi_{n3t} \\ &\quad + n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) \xi_{n4t} \\ &\equiv T_{n11} - T_{n12} - T_{n13} + T_{n14} + T_{n15}. \end{aligned}$$

It suffices to show that

$$\sqrt{nh^{d_2}} T_{n11} \xrightarrow{d} \text{N}(0, \kappa_{02}^{d_2} (E(v_t^4 | X_{2,t} = x_2) - 1) f(x_2)^{-1} \sigma^4(x)) \quad (\text{A.16})$$

and

$$T_{n1j} = o_p(n^{-1/2} h^{-d_2/2}) \quad \text{for } j = 2, 3, 4, 5. \quad (\text{A.17})$$

First, by the second-order Taylor expansion,

$$\begin{aligned} \xi_{n1t} &= \frac{\varepsilon_t^2 \sigma_p^2(x_1, \widehat{\boldsymbol{\gamma}})}{\sigma_p^2(X_{1,t}, \widehat{\boldsymbol{\gamma}})} \\ &= r_t^2 \left\{ 1 + (\mu_0(x_1) - \mu_0(X_{1,t}))^T (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^0) \right. \\ &\quad \left. + \frac{1}{2} (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^0)^T G_t (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^0) \right\}, \end{aligned}$$

where recall $\mu_0(x_1) \equiv \mu(x_1, \boldsymbol{\gamma}^0)$, $G_t \equiv (v_*(x_1) - v_*(X_{1,t})) + (\mu_*(x_1) - \mu_*(X_{1,t}))^T (\mu_*(x_1) - \mu_*(X_{1,t}))$, and $\mu_*(x_1)$ and $v_*(x_1)$ are the gradient and the Hessian matrix with respect to $\boldsymbol{\gamma}$ of $\log \sigma_p^2(x_1, \boldsymbol{\gamma})$ evaluated at $\boldsymbol{\gamma}_t^*$, respectively. Here $\boldsymbol{\gamma}_t^*$ is the intermediate value that lies between $\boldsymbol{\gamma}^0$ and $\widehat{\boldsymbol{\gamma}}$. By Assumption A4, $\xi_{n1t} = r_t^2 \{1 + (\mu_0(x_1) - \mu_0(X_{1,t}))^T (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^0) + O_p(n^{-1}) \|G_t\|\}$, and

$$\begin{aligned} T_{n11} &= n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) \xi_t \\ &\quad + n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) r_t^2 (\mu_0(x_1) - \mu_0(X_{1,t}))^T (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^0) \end{aligned}$$

$$\begin{aligned} &+ \frac{O_p(n^{-1})}{2} n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) r_t^2 \|G_t\| \\ &\equiv T_{n11a} + T_{n11b} + T_{n11c}, \end{aligned} \quad (\text{A.18})$$

where recall $\xi_t \equiv r_t^2 - \sigma_p^2(x_1) \sigma_{np}^2(X_{2,t})$.

Following Masry (1996b), we can show that $S_n(x_2) = f(x_2) \times S(1 + O_p(h))$, where S is defined in (2.8). Then $\sqrt{nh^{d_2}} T_{n11a} = n^{-1/2} h^{d_2/2} f(x_2)^{-1} \sum_{t=1}^n K_{ht} \xi_t + o_p(1)$. Noting that $\{K_{ht} \xi_t, \mathcal{F}_t\}$ is an mds, $E[K_{ht} \xi_t] = 0$, and

$$\begin{aligned} \text{Var} \left(n^{-1/2} h^{d_2/2} f(x_2)^{-1} \sum_{t=1}^n K_{ht} \xi_t \right) \\ &= h^{d_2} f(x_2)^{-2} E\{K_{ht} \xi_t\}^2 \\ &\rightarrow \kappa_{02}^{d_2} (E(v_t^4 | X_{2,t} = x_2) - 1) f(x_2)^{-1} \sigma^4(x), \end{aligned}$$

by the dominated convergence theorem. By Assumptions A2 and A5–A6 and the arguments of Masry (1996a, theorem 3),

$$\begin{aligned} \sqrt{nh^{d_2}} T_{n11a} \\ \xrightarrow{d} \text{N}(0, \kappa_{02}^{d_2} (E(v_t^4 | X_{2,t} = x_2) - 1) f(x_2)^{-1} \sigma^4(x)). \end{aligned} \quad (\text{A.19})$$

By the ergodic theorem and Assumptions A2–A6,

$$\begin{aligned} T_{n11b} &= n^{-1} f(x_2)^{-1} \\ &\quad \times \sum_{t=1}^n K_h(X_{2,t} - x_2) r_t^2 (\mu_0(x_1) - \mu_0(X_{1,t}))^T (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^0) \\ &= O_p(1) O_p(n^{-1/2}) = o_p(n^{-1/2} h^{-d_2/2}). \end{aligned} \quad (\text{A.20})$$

Now by Assumptions A2–A6 and the weak law of large numbers, $\|G_t\| \leq C \sum_{i=1}^2 \|X_{1,t} - x_1\|^i$, and

$$\begin{aligned} T_{n11c} &= \frac{O_p(n^{-1})}{2} n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) r_t^2 \|G_t\| \\ &\leq \frac{O_p(n^{-1})}{2} C n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) r_t^2 \sum_{i=1}^2 \|X_{1,t} - x_1\|^i \\ &= \frac{O_p(n^{-1})}{2} C f(x_2)^{-1} \\ &\quad \times n^{-1} \sum_{t=1}^n K_{ht} r_t^2 \sum_{i=1}^2 \|X_{1,t} - x_1\|^i \{1 + O_p(h)\} \\ &= O_p(n^{-1}) O_p(1) = O_p(n^{-1}). \end{aligned} \quad (\text{A.21})$$

Hence (A.16) follows from (A.18)–(A.21).

By the Taylor expansion and Assumptions A2 and A4,

$$\begin{aligned} |T_{n12}| &= \left| n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) \sigma_p^2(x_1) \dot{\sigma}_{np}^2(X_{2,t}^*) (\widehat{X}_{2,t} - X_{2,t}) \right| \\ &= \sigma_p^2(x_1) n^{-1} f^{-1}(x_2) \\ &\quad \times \sum_{t=1}^n |K_{ht} \dot{\sigma}_{np}^2(X_{2,t}) D_{\boldsymbol{\theta}} X_{2,t}(\boldsymbol{\theta}^0) (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)| \\ &+ O_p(n^{-1} h^{-d_2}) \\ &= O_p(n^{-1/2}) + O_p(n^{-1} h^{-d_2}) = o_p(n^{-1/2} h^{-d_2/2}), \end{aligned}$$

where $X_{2,t}^*$ lies between $\widehat{X}_{2,t}$ and $X_{2,t}$. Next,

$$\begin{aligned} \xi_{n2t} &= \frac{\varepsilon_t^2 \sigma_p^2(x_1, \widehat{\mathcal{Y}})}{\sigma_p^2(X_{1,t}, \widehat{\mathcal{Y}})} \left(1 - \frac{\sigma_p^2(X_{1,t}, \widehat{\mathcal{Y}})}{\sigma_p^2(\widehat{X}_{1,t}, \widehat{\mathcal{Y}})} \right) \\ &= r_t^2 \{ 1 + (\mu_0(x_1) - \mu_0(X_{1,t}))^T (\widehat{\mathcal{Y}} - \mathcal{Y}^0) + O_p(n^{-1}) \|G_t\| \} \\ &\quad \times \frac{\dot{\sigma}_p^2(X_{1,t}, \mathcal{Y}_t^*)^T}{\sigma_p^2(X_{1,t}, \mathcal{Y}_t^*)} D_{\theta} X_{1,t}(\theta_t^*) (\widehat{\theta} - \theta^0), \end{aligned}$$

where θ_t^* lies between $\widehat{\theta}$ and θ^0 , and \mathcal{Y}_t^* lies between $\widehat{\mathcal{Y}}$ and \mathcal{Y}^0 .

$$\begin{aligned} T_{n13} &= n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) r_t^2 \frac{\dot{\sigma}_p^2(X_{1,t}, \mathcal{Y}_t^*)^T}{\sigma_p^2(X_{1,t}, \mathcal{Y}_t^*)} \\ &\quad \times D_{\theta} X_{1,t}(\theta_t^*) (\widehat{\theta} - \theta^0) + O_p(n^{-1}) \\ &= n^{-1} f(x_2)^{-1} \sum_{t=1}^n K_{ht} r_t^2 \frac{\dot{\sigma}_p^2(X_{1,t}, \mathcal{Y}^0)^T}{\sigma_p^2(X_{1,t})} \\ &\quad \times D_{\theta} X_{1,t}(\theta^0) (\widehat{\theta} - \theta^0) + O_p(n^{-1}) \\ &= O_p(1) O_p(n^{-1/2}) + O_p(n^{-1}) = o_p(n^{-1/2} h^{-d_2/2}). \end{aligned}$$

Similarly, we can show that $T_{n14} = o_p(n^{-1/2} h^{-d_2/2})$ and $T_{n15} = o_p(n^{-1/2} h^{-d_2/2})$. This completes the proof of the lemma.

Lemma A.2. $T_{n2} \equiv \frac{1}{2} n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) \widehat{A}_{nt} = B(x) + o_p(n^{-1/2} h^{-d_2/2})$.

Proof. Let A_{nt} be defined as \widehat{A}_{nt} in (A.14) but with $X_{2,t}$ replacing $\widehat{X}_{2,t}$. Decompose

$$\begin{aligned} T_{n2} &\equiv \frac{1}{2} n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) A_{nt} \\ &\quad + \frac{1}{2} n^{-1} \sum_{t=1}^n K_n^*(X_{2,t}, x_2) (\widehat{A}_{nt} - A_{nt}) \\ &\equiv T_{n21} + T_{n22}. \end{aligned}$$

By the ergodic theorem, the dominated convergence theorem, and Assumptions A2–A6,

$$\begin{aligned} T_{n21} &= \frac{1}{2} n^{-1} f(x_2)^{-1} \sum_{t=1}^n K_{ht} A_{nt} \{ 1 + O_p(h) \} \\ &= B(x) + o_p(h^2) \\ &= B(x) + o_p(n^{-1/2} h^{-d_2/2}). \end{aligned} \quad (\text{A.22})$$

Now, by Taylor expansions,

$$\begin{aligned} T_{n22} &= \frac{1}{2} n^{-1} f(x_2)^{-1} \sum_{t=1}^n K_{ht} (\widehat{A}_{nt} - A_{nt}) \{ 1 + O_p(h) \} \\ &= O_p(n^{-1/2}) = o_p(n^{-1/2} h^{-d_2/2}). \end{aligned}$$

Hence $T_{n2} = B(x) + o_p(n^{-1/2} h^{-d_2/2})$.

Lemma A.3. $T_{n3} \equiv n^{-1} \sum_{t=1}^n [K_n^*(\widehat{X}_{2,t}, x_2) - K_n^*(X_{2,t}, x_2)] \times [\widehat{r}_t^2 - \sigma_p^2(x_1) \sigma_{np}^2(\widehat{X}_{2,t})] = o_p(n^{-1/2} h^{-d_2/2})$.

Proof. Let $\eta_{nt} = f(x_2)^{-1} \{ K_h(\widehat{X}_{2,t} - x_2) - K_h(X_{2,t} - x_2) \}$. We can show $K_n^*(\widehat{X}_{2,t}, x_2) - K_n^*(X_{2,t}, x_2) = \eta_{nt} \{ 1 + O_p(h) \}$ where the order $O_p(h)$ does not depend on t . Then by the notation used in the proof of Lemma A.1 [Equation (A.15) in particular],

$$\begin{aligned} T_{n3} &= n^{-1} \sum_{t=1}^n \eta_{nt} [\xi_t + \sigma_p^2(x_1) (\sigma_{np}^2(X_{2,t}) - \sigma_{np}^2(\widehat{X}_{2,t})) \\ &\quad - \xi_{n2t} + \xi_{n3t} + \xi_{n4t}] \{ 1 + O_p(h) \} \\ &= \left\{ n^{-1} \sum_{t=1}^n \eta_{nt} \xi_t \right. \\ &\quad + n^{-1} \sum_{t=1}^n \eta_{nt} \sigma_p^2(x_1) (\sigma_{np}^2(X_{2,t}) - \sigma_{np}^2(\widehat{X}_{2,t})) \\ &\quad - n^{-1} \sum_{t=1}^n \eta_{nt} \xi_{n2t} + n^{-1} \sum_{t=1}^n \eta_{nt} \xi_{n3t} \\ &\quad \left. + n^{-1} \sum_{t=1}^n \eta_{nt} \xi_{n4t} \right\} \{ 1 + O_p(h) \} \\ &\equiv \{ T_{n31} + T_{n32} + T_{n33} + T_{n34} + T_{n35} \} \{ 1 + O_p(h) \}. \end{aligned}$$

It suffices to show that $T_{n3j} = o_p(n^{-1/2} h^{-d_2/2})$, $j = 1, 2, \dots, 5$. First,

$$\begin{aligned} T_{n31} &= n^{-1} f(x_2)^{-1} \sum_{t=1}^n \xi_t \dot{K}_h(X_{2,t} - x_2)^T D_{\theta} X_{2,t}(\theta^0) (\widehat{\theta} - \theta^0) \\ &\quad + O_p(n^{-1} h^{-d_2+2}) \\ &\equiv \widetilde{T}_{n31} (\widehat{\theta} - \theta^0) + O_p(n^{-1} h^{-d_2+2}), \end{aligned}$$

where $\dot{K}_h(u) \equiv (\partial/\partial u) K_h(u)$. It is easy to show that $E(\widetilde{T}_{n31}) = 0$ and $\| \text{Var}(\widetilde{T}_{n31}) \| = O(n^{-1} h^{-d_2-2})$. It follows from the Chebyshev inequality $\widetilde{T}_{n31} = O_p(n^{-1/2} h^{-d_2/2-1})$ and hence $T_{n31} = O_p(n^{-1} h^{-d_2/2-1} + n^{-1} h^{-d_2+2}) = o_p(n^{-1/2} h^{-d_2/2})$. Similarly, we can show that $T_{n3j} = O_p(n^{-1} h^{-1} + n^{-1} h^{-d_2+1}) = o_p(n^{-1/2} h^{-d_2/2})$, for $j = 2, 3, 4, 5$. This completes the proof.

Lemma A.4. $T_{n4} \equiv \frac{1}{2} n^{-1} \sum_{t=1}^n [K_n^*(\widehat{X}_{2,t}, x_2) - K_n^*(X_{2,t}, x_2)] \widehat{A}_{nt} = o_p(n^{-1/2} h^{-d_2/2})$.

Proof. Using the arguments and notation in the proof of Lemmas A.1–A.3, we have

$$\begin{aligned} T_{n4} &= \left\{ \frac{1}{2} n^{-1} \sum_{t=1}^n \eta_{nt} A_{nt} + \frac{1}{2} n^{-1} \sum_{t=1}^n \eta_{nt} (\widehat{A}_{nt} - A_{nt}) \right\} \\ &\quad \times \{ 1 + O_p(h) \} \\ &\equiv \{ T_{n41} + T_{n42} \} \{ 1 + O_p(h) \}. \end{aligned}$$

First,

$$\begin{aligned} T_{n41} &= \frac{1}{2} n^{-1} f(x_2)^{-1} \sum_{t=1}^n \dot{K}_h(X_{2,t} - x_2)^T D_{\theta} X_{2,t}(\theta^0) (\widehat{\theta} - \theta^0) A_{nt} \\ &\quad + O_p(n^{-1} h^{-d_2+2}) \\ &= O_p(n^{-1/2} h) + O_p(n^{-1} h^{-d_2+2}) = o_p(n^{-1/2} h^{-d_2/2}). \end{aligned}$$

Similarly, we can show that $T_{n42} = O_p(n^{-1/2} h) + O_p(n^{-1} h^{-d_2+2}) = o_p(n^{-1/2} h^{-d_2/2})$. This completes the proof.

A.3. Proof of (2.13) and (2.14)

Since the proof parallels that of Theorem 2.2, we only sketch the difference. Recall $S_n(x)$ is defined in A.11. When $x_{n2} = ch$, we can show that $S_n(x_{n2}) \xrightarrow{p} f(0)S_c$, where

$$S_c \equiv \begin{pmatrix} \mu_{c0} & \mu_{c1} \\ \mu_{c1} & \mu_{c2} \end{pmatrix}.$$

Following the proof of Lemma A.2, the dominant term in the asymptotic bias is determined by

$$\begin{aligned} \tilde{T}_{n21} &= \frac{1}{2}n^{-1} \sum_{t=1}^n e_1^T S_n^{-1}(x_{n2}) \left(\frac{1}{h} \frac{X_{2,t} - x_{n2}}{h} \right) \\ &\quad \times K_h(X_{2,t} - x_{n2}) A_{nt} \{1 + o_p(1)\} \\ &= \frac{1}{2}n^{-1} \frac{1}{f(0)(\mu_{c0}\mu_{c2} - \mu_{c1}^2)} \sum_{t=1}^n \left(\mu_{c2} - \mu_{c1} \frac{X_{2,t} - x_{n2}}{h} \right) \\ &\quad \times K_h(X_{2,t} - x_{n2}) A_{nt} \{1 + o_p(1)\} \\ &\xrightarrow{p} \frac{h^2}{2} \frac{\mu_{c2}^2 - \mu_{c1}\mu_{c3}}{\mu_{c0}\mu_{c2} - \mu_{c1}^2} [\sigma_p^2(x_1) \ddot{\sigma}_{np}^2(0) - \ddot{L}(0, \beta^0)]. \end{aligned}$$

Following the proof of Lemma A.1, the dominant term in the asymptotic variance is determined by

$$\begin{aligned} \sqrt{nh^{d_2}} \tilde{T}_{n11a} &= n^{-1/2} h^{d_2/2} \sum_{t=1}^n e_1^T S_n^{-1}(x_{n2}) \left(\frac{1}{h} \frac{X_{2,t} - x_{n2}}{h} \right) \\ &\quad \times K_h(X_{2,t} - x_{n2}) \xi_t + o_p(1) \\ &= n^{-1/2} h^{d_2/2} \frac{1}{f(0)(\mu_{c0}\mu_{c2} - \mu_{c1}^2)} \\ &\quad \times \sum_{t=1}^n \left(\mu_{c2} - \mu_{c1} \frac{X_{2,t} - x_{n2}}{h} \right) \\ &\quad \times K_h(X_{2,t} - x_{n2}) \xi_t + o_p(1) \\ &\equiv V_n + o_p(1). \end{aligned}$$

Noting that $\{[\mu_{c2} - \mu_{c1}(X_{2,t} - x_{n2})/h]K_h(X_{2,t} - x_{n2})\xi_t, \mathcal{F}_t\}$ is an mds, simple calculations show that $E[V_n] = 0$, and

$$\begin{aligned} \text{Var}(V_n) &= \frac{\int_{-c}^{\infty} (\mu_{c2} - \mu_{c1}z)^2 k^2(z) dz}{(\mu_{c0}\mu_{c2} - \mu_{c1}^2)^2} \\ &\quad \times [E(v_t^4 | X_{2,t} = x_2) - 1] f^{-1}(0) \sigma^4(x_1, 0) + o(1). \end{aligned}$$

This completes the proof.

A.4. Proof of Theorem 2.3

The proof parallels that of Theorem 2.1. The major difference lies in three aspects: (a) now we hold h as fixed; (b) we use the fact that $\sigma_{np}^2(\cdot) \equiv 1$ under the correct specification of the first-stage parametric conditional variance model; (c) the probability limit $G_n(\beta)$ now becomes

$$G_n(\beta) \equiv E\{[\sigma_p^2(x_1) - L((X_{2,t} - x_2), \beta)]^2 K_{ht}\} + E\{\xi_t^2 K_{ht}\}.$$

We proceed as in the proof of Theorem 2.1 until (A.7).

By Assumptions A1, A2, and A5, the Taylor expansion and the weak uniform law of large numbers, it is easy to show

that $\widehat{G}_{n1}(\beta) = \frac{1}{n} \sum_{t=1}^n [r_t^2 - L(X_{2,t} - x_2, \beta)]^2 K_{ht} + O_p(n^{-1/2}) = \overline{G}_{n1}(\beta) + o_p(1)$ uniformly in β , where $\overline{G}_{n1}(\beta) = E\{[r_t^2 - L(X_{2,t} - x_2, \beta)]^2 K_{ht}\}$. Now $\xi_t \equiv r_t^2 - \sigma_p^2(x_1)$, and

$$\begin{aligned} \overline{G}_{n1}(\beta) &= E\{[r_t^2 - L((X_{2,t} - x_2), \beta)]^2 K_{ht}\} \\ &= E\{[\xi_t + (\sigma_p^2(x_1) - L((X_{2,t} - x_2), \beta))]^2 K_{ht}\} \\ &= G_n(\beta), \end{aligned}$$

where the last equality follows from the fact that $\{\xi_t[\sigma_p^2(x_1) - L((X_{2,t} - x_2), \beta)]K_{ht}, \mathcal{F}_t\}$ is an mds. As in the proof of Theorem 2.1, $\widehat{G}_{n2}(\beta) = o_p(1)$ and $\widehat{G}_{n3}(\beta) = o_p(1)$ uniformly in β . Hence

$$\widehat{G}_n(\beta) = G_n(\beta) + o_p(1) \quad \text{uniformly in } \beta.$$

As in the proof of Theorem 2.1, choosing β to minimize $G_n(\beta)$ is equivalent to choosing $L(0, \beta)$ and $\dot{L}(\cdot, \beta)$ to minimize

$$\begin{aligned} G_n^*(\beta) &\equiv E\{[\sigma_p^2(x_1) - L(0, \beta) \\ &\quad - \dot{L}(\overline{C}_t(X_{2,t} - x_2), \beta)^T (X_{2,t} - x_2)]^2 K_{ht}\}, \end{aligned}$$

where \overline{C}_t is a $d_2 \times d_2$ matrix with elements that lie on the interval $[0, 1]$. Clearly, when $L(0, \beta) = \sigma_p^2(x_1)$ and $\dot{L}(\cdot, \beta) \equiv 0$, $G_n^*(\beta) = 0$ is minimized. Consequently, β^0 is uniquely determined by $L(0, \beta^0) = \sigma_p^2(x_1)$ and $\dot{L}(\cdot, \beta^0) \equiv 0$. Note that $L(0, \beta^0) = \Psi(\beta_0^0)$, $\dot{L}(x_2, \beta^0) = \dot{\Psi}(\beta_0^0 + \sum_{i=1}^{d_2} \beta_i^0 x_{2,i}) \beta_1^0$ and $\dot{\Psi}(\cdot) > 0$, thus we conclude that $\beta_0^0 = \Psi^{-1}(\sigma_p^2(x_1))$ and $\beta_1^0 = 0$.

A.5. Proof of Theorem 2.4

The proof parallels that of Theorem 2.2. The major difference lies in three aspects: (a) now we hold h as fixed; (b) we use the fact that $\sigma_{np}^2(\cdot) \equiv 1$ under the correct specification of the first-stage parametric conditional variance model; (c) we use the fact $\dot{L}(\cdot, \beta^0) \equiv 0$.

Let $R_n(x, \beta)$ be defined as in (A.8). By a Taylor expansion of the first-order conditions from minimizing $R_n(x, \beta)$, we have

$$\sqrt{n}(\widehat{\beta} - \beta^0) = - \left(\frac{1}{n} \frac{\partial^2 R_n(x, \widehat{\beta})}{\partial \beta \partial \beta^T} \right)^{-1} \frac{1}{\sqrt{n}} \frac{\partial R_n(x, \beta^0)}{\partial \beta}, \quad (\text{A.23})$$

where $\widehat{\beta}$ lies between $\widehat{\beta}$ and β^0 and thus converges to β^0 in probability by Theorem 2.3,

$$\begin{aligned} \frac{\partial R_n(x, \beta)}{\partial \beta} &= -2 \sum_{t=1}^n \{r_t^2 - L(\widehat{X}_{2,t} - x_2, \beta)\} \\ &\quad \times \dot{\Psi}_{\beta,t} \left(\frac{1}{\widehat{X}_t - x} \right) K_h(\widehat{X}_{2,t} - x_2), \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 R_n(x, \beta)}{\partial \beta \partial \beta^T} &= -2 \sum_{t=1}^n s_t(\beta) \\ &\quad \times \begin{pmatrix} 1 & \widehat{X}_{2,t} - x_2 \\ \widehat{X}_{2,t} - x_2 & (\widehat{X}_{2,t} - x_2)(\widehat{X}_{2,t} - x_2)^T \end{pmatrix} \\ &\quad \times K_h(\widehat{X}_{2,t} - x_2), \end{aligned}$$

where $s_t(\beta) \equiv [r_t^2 - L(\widehat{X}_{2,t} - x_2, \beta)] \dot{\Psi}_{\beta,t} - [\dot{\Psi}_{\beta,t}]^2$, $\dot{\Psi}_{\beta,t} \equiv \dot{\Psi}((1, (\widehat{X}_{2,t} - x_2)^T) \beta)$, $\dot{\Psi}_{\beta,t} \equiv \dot{\Psi}((1, (\widehat{X}_{2,t} - x_2)^T) \beta)$, and

$\dot{\Psi}(u)$ and $\ddot{\Psi}(u)$ denote the first and second derivatives of $\Psi(u)$ with respect to u , respectively. Note that Theorem 2.3 implies

$$L(\widehat{X}_{2,t} - x_2, \boldsymbol{\beta}^0) = L(0, \boldsymbol{\beta}^0) = \sigma_p^2(x_1), \quad (\text{A.24})$$

$$\dot{\Psi}_{\boldsymbol{\beta}^0,t} \equiv \dot{\Psi}(\boldsymbol{\beta}_0^0), \quad \text{and} \quad \ddot{\Psi}_{\boldsymbol{\beta}^0,t} \equiv \ddot{\Psi}(\boldsymbol{\beta}_0^0).$$

Let $N_\epsilon(\boldsymbol{\beta}^0)$ denote the ϵ -neighborhood of $\boldsymbol{\beta}^0$ where $\epsilon \equiv \epsilon(n) \rightarrow 0$ as $n \rightarrow \infty$. The proof is complete if we can show

$$\frac{1}{n} \frac{\partial^2 R_n(x, \tilde{\boldsymbol{\beta}})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T} - \frac{1}{n} \frac{\partial^2 R_n(x, \boldsymbol{\beta}^0)}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T} = o_p(1) \quad \text{uniformly in } \tilde{\boldsymbol{\beta}} \in N_\epsilon(\boldsymbol{\beta}^0), \quad (\text{A.25})$$

$$\frac{1}{n} \frac{\partial^2 R_n(x, \boldsymbol{\beta}^0)}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T} - 2[\dot{\Psi}(\boldsymbol{\beta}_0^0)]^2 \Sigma_{\boldsymbol{\beta}^0} = o_p(1), \quad (\text{A.26})$$

and

$$\frac{1}{\sqrt{n}} \frac{\partial R_n(x, \boldsymbol{\beta}^0)}{\partial \boldsymbol{\beta}} \xrightarrow{d} N(0, 4[\dot{\Psi}(\boldsymbol{\beta}_0^0)]^2 \Omega_{\boldsymbol{\beta}^0}). \quad (\text{A.27})$$

We first show (A.25). By Theorem 2.3, (A.24), and Assumptions A1–A5, we can show that $\sup_{\tilde{\boldsymbol{\beta}} \in N_\epsilon(\boldsymbol{\beta}^0)} |s_t(\tilde{\boldsymbol{\beta}}) - s_t(\boldsymbol{\beta}^0)| = o_p(1)$ uniformly in t . (A.25) then follows from this fact by applications of Taylor expansion and weak law of large numbers. Next, by Assumptions A2–A4, Taylor expansions and weak law of large numbers,

$$\begin{aligned} \frac{\partial^2 R_n(x, \boldsymbol{\beta}^0)}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T} &= -\frac{2}{n} \sum_{t=1}^n \{ [r_t^2 - \sigma_p^2(x_1)] \ddot{\Psi}(\boldsymbol{\beta}_0^0) - [\dot{\Psi}(\boldsymbol{\beta}_0^0)]^2 \} \\ &\quad \times \begin{pmatrix} 1 & & & \\ & X_{2,t} - x_2 & & \\ & & X_t - x & \\ & & & (X_{2,t} - x_2)(X_{2,t} - x_2)^T \end{pmatrix} \\ &\quad \times K_{ht} + o_p(1) \\ &= 2[\dot{\Psi}(\boldsymbol{\beta}_0^0)]^2 \Sigma_{\boldsymbol{\beta}^0} + o_p(1). \end{aligned} \quad (\text{A.28})$$

Now by (A.15), we make the following decomposition:

$$\begin{aligned} \frac{1}{\sqrt{n}} \frac{\partial R_n(x, \boldsymbol{\beta}^0)}{\partial \boldsymbol{\beta}} &= \frac{-2\dot{\Psi}(\boldsymbol{\beta}_0^0)}{\sqrt{n}} \sum_{t=1}^n \{ [r_t^2 - \sigma_p^2(x_1)] \tilde{X}_{2,t} K_h(\tilde{X}_{2,t} - x_2) \\ &\quad - 2\dot{\Psi}(\boldsymbol{\beta}_0^0) \sum_{j=1}^5 V_{n,j} \}, \end{aligned}$$

where $\tilde{X}_{2,t} \equiv (1, (\widehat{X}_{2,t} - x_2)^T)^T$, $V_{n,1} \equiv n^{-1/2} \sum_{t=1}^n \{ [r_t^2 - \sigma_p^2(x_1)] \tilde{X}_{2,t} K_h(\tilde{X}_{2,t} - x_2) \}$, $V_{n,2} \equiv n^{-1/2} \sum_{t=1}^n \{ \xi_{n1t} - r_t^2 \} \times \tilde{X}_{2,t} K_h(\tilde{X}_{2,t} - x_2)$, and $V_{n,j+1} \equiv n^{-1/2} \sum_{t=1}^n \xi_{n2j} \tilde{X}_{2,t} K_h(\tilde{X}_{2,t} - x_2)$, $j = 2, 3, 4$. We can show that

$$V_{n,1} = \frac{1}{\sqrt{n}} \sum_{t=1}^n \{ [r_t^2 - \sigma_p^2(x_1)] \} \begin{pmatrix} 1 & & & \\ & X_{2,t} - x_2 & & \end{pmatrix} K_{ht} + o_p(1),$$

$$V_{n,2} = \frac{1}{\sqrt{n}} \sum_{t=1}^n \sigma_p^2(x_1) (\mu_0(x_1) - \mu_0(X_{1,t}))^T$$

$$\begin{aligned} &\times (\widehat{\boldsymbol{y}} - \boldsymbol{y}^0) \begin{pmatrix} 1 & & & \\ & X_{2,t} - x_2 & & \end{pmatrix} K_{ht} + o_p(1), \\ V_{n,3} &= \frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\sigma_p^2(x_1) \dot{\sigma}_p^2(X_{1,t}, \boldsymbol{y}^0)^T}{\sigma_p^2(X_{1,t})} D_{\boldsymbol{\theta}} X_{1,t}(\boldsymbol{\theta}^0) \\ &\quad \times (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0) \begin{pmatrix} 1 & & & \\ & X_{2,t} - x_2 & & \end{pmatrix} K_{ht} + o_p(1), \\ V_{n,4} &= -\frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{2\varepsilon_t \sigma_p^2(x) [D_{\boldsymbol{\alpha}} g(U_t, \boldsymbol{\alpha}^0)]^T (\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^0)}{\sigma_p^2(X_{1,t})} \\ &\quad \times \begin{pmatrix} 1 & & & \\ & X_{2,t} - x_2 & & \end{pmatrix} K_{ht} + o_p(1) \\ &= o_p(1), \end{aligned}$$

and

$$\begin{aligned} V_{n,5} &= -\frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\sigma_p^2(x) \{ [D_{\boldsymbol{\alpha}} g(U_t, \boldsymbol{\alpha}^0)]^T (\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^0) \}^2}{\sigma_p^2(X_{1,t})} \\ &\quad \times \begin{pmatrix} 1 & & & \\ & X_{2,t} - x_2 & & \end{pmatrix} K_{ht} + o_p(1) \\ &= o_p(1). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{\sqrt{n}} \frac{\partial R_n(x, \boldsymbol{\beta}^0)}{\partial \boldsymbol{\beta}} &= \frac{-2\dot{\Psi}(\boldsymbol{\beta}_0^0)}{\sqrt{n}} \sum_{t=1}^n \{ [r_t^2 - \sigma_p^2(x_1)] \} \begin{pmatrix} 1 & & & \\ & X_{2,t} - x_2 & & \end{pmatrix} K_{ht} \\ &\quad + \frac{-2\dot{\Psi}(\boldsymbol{\beta}_0^0)}{n} \sum_{t=1}^n \sigma_p^2(x_1) \begin{pmatrix} 1 & & & \\ & X_{2,t} - x_2 & & \end{pmatrix} K_{ht} \\ &\quad \times \left\{ [0_{1 \times k_1}, (\mu_0(x_1) - \mu_0(X_{1,t}))^T] \right. \\ &\quad \left. - \frac{\dot{\sigma}_p^2(X_{1,t}, \boldsymbol{y}^0)^T}{\sigma_p^2(X_{1,t})} D_{\boldsymbol{\theta}} X_{1,t}(\boldsymbol{\theta}^0) \right\} n^{-1/2} \sum_{s=1}^n \varphi_s(\boldsymbol{\theta}^0) + o_p(1) \\ &= \frac{-2\dot{\Psi}(\boldsymbol{\beta}_0^0)}{\sqrt{n}} \sum_{t=1}^n \left\{ [r_t^2 - \sigma_p^2(x_1)] \begin{pmatrix} 1 & & & \\ & X_{2,t} - x_2 & & \end{pmatrix} K_{ht} \right. \\ &\quad \left. + \Upsilon \varphi_t(\boldsymbol{\theta}^0) \right\} + o_p(1) \\ &\xrightarrow{d} N(0, 4[\dot{\Psi}(\boldsymbol{\beta}_0^0)]^2 \Omega_{\boldsymbol{\beta}^0}). \end{aligned}$$

Consequently, $\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^0) \xrightarrow{d} N(0, [\dot{\Psi}(\boldsymbol{\beta}_0^0)]^{-2} \Sigma_{\boldsymbol{\beta}^0}^{-1} \Omega_{\boldsymbol{\beta}^0} \Sigma_{\boldsymbol{\beta}^0}^{-1})$.

[Received April 2008. Revised August 2008.]

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