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An Approximation Pricing Algorithm in an Incomplete Market: A Differential Geometric Approach

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Abstract The minimal distance equivalent martingale measure (EMM) defined in Goll and Rüschendorf (2001) is the arbitrage-free equilibrium pricing measure. This paper provides an algorithm to approximate its density and the fair price of any contingent claim in an incomplete market. We first approximate the infinite dimensional space of all EMMs by a finite dimensional manifold of EMMs. A Riemannian geometric structure is then shown on the manifold. An optimization algorithm on the Rieman-

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nian manifold becomes the approximation pricing algorithm. The financial interpretation of the geometry is also given in terms of pricing model risk.

Key words Incomplete markets, asset pricing, Riemannian manifold, entropy, cross entropy

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1 Introduction

Pricing in incomplete markets is a key area of current research in quantitative finance. The fair derivative price was suggested by Davis (1997), using an economic approach of marginal rate of substitution. The problem of utility maximization in incomplete markets was studied by Karatzas, Lehoczky, Shreve, and Xu (1991), Kramkov and Schachermayer (1999), Cvitanić, Schachermayer, and Wang (2001), using the convex duality and martingale approach. It is shown in Goll and Rüschendorf (2001) that both approaches give rise to the same pricing measure, the minimal distance martingale measure Q^* over the space of all equivalent martingale measures (EMMs) $\mathcal{M}$. In order to compute the fair price, the optimal hedging strategy and the optimal portfolio, we need to compute the density of Q^*. Although the existence and uniqueness of Q^* are well studied by Bellini and Frittelli (2002), and Kramkov and Schachermayer (2001), little is known on the computation or approximation of the Radon-Nikodym density $\frac{dQ^*}{dP}$ with respect to the empirical measure P. The primary contribution of this paper is to provide an algorithm to approximate $\frac{dQ^*}{dP}$ using the method from differential geometry.

In Section 2 we present our generic incomplete market model and the main problem. The new subject of differential geometry of EMMs is studied in the subsequent sections.

In Section 3, using Wiener-Itô Chaos expansions, we approximate the infinite dimensional space $\mathcal{M}$ by a n-dimensional manifold of EMMs $\mathcal{S}$ where

n can be arbitrary large. The motivation for such finite dimensional approximation comes from the industrial implementation of incomplete market pricing models using computers. In our terminology, the negative of the distance to P is called the entropy. An approximate problem is formulated as maximization of the entropy over the finite dimensional space $\mathcal{S}$.

In Section 4, we introduce the concept of cross entropy as the pseudo-distance between EMMs which is new to the finance literature. In the main theorem, a result in Rao (1987) is extended to show that there is a natural Riemannian geometric structure on $\mathcal{S}$. By this theorem, the approximate problem becomes an optimization problem on a Riemannian manifold. In Section 5, an important application of the theorem, the approximation pricing algorithm, is provided by the gradient descent method on a Riemannian manifold.

The financial interpretation of the geometry of EMMs is given in terms of a quantitative measure of pricing model risk in Section 6. We show that the Riemannian distance between two EMMs is related to the pricing model risk between two pricing systems. The solution to the approximate problem is also shown to be the minimal pricing model risk approximation to Q^*. Section 7 provides some examples.

The focus of this paper is to provide a theoretical foundation for the pricing algorithm and a methodology for pricing in incomplete markets.

To keep this paper short, we refer all the detailed examples, formulations and proofs to the first author's doctorial dissertation Gao (2002).

2 The Pricing Problem in An Incomplete Market

2.1 The Incomplete Financial Market Model

The market consists of two tradable securities. One is the numéraire bank account. The price of the other security after dividing by the numéraire is denoted by Z_t and is assumed to be a bounded semimartingale with continuous sample path on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ with the finite time-horizon T. On this probability space, there is a 2-dimensional Brownian motion $(B_t, W_t)^\top$. While the Brownian B_t drives the tradable security Z_t, the other independent Brownian motion W_t drives some non-tradable process. Examples of the non-tradable processes arise in the general stochastic volatility models, weather derivatives and energy market models.

If Z_t is locally bounded, then the results of this paper still hold true if we replace an equivalent martingale measure (EMM) with an equivalent local martingale measure. Our theory could be applied readily to multi-dimensional Brownian motion and multiple tradable assets.

Let $\mathcal{M}$ be the space of all EMMs. Let the predictable process ϕ_t be the market price of risk (MPR) process and satisfy $E_P \int_0^T \phi_s^2 ds < \infty$. No-arbitrage condition fixes ϕ_t and we call ϕ_t the *complete market price of risk* corresponding to the "hedgeable" random factor B_t. Let another predictable process η_t satisfy $\int_0^T \eta_s^2 ds < \infty$ P a.s.. We use the following notation

$$\mathcal{E}\Big(\int_0^T \eta_s dW_s\Big) \triangleq \exp\Big(\int_0^T \phi_s dB_s - \frac{1}{2}\int_0^T \phi_s^2 ds + \int_0^T \eta_s dW_s - \frac{1}{2}\int_0^T \eta_s^2 ds\Big).$$

Let $Q_\eta \in \mathcal{M}$ with $\frac{dQ_\eta}{dP} = \mathcal{E}\left(\int_0^T \eta_s dW_s\right)$. By the Girsanov theorem, $\widetilde{B}_t = B_t - \int_0^t \phi_s ds$ and $\widetilde{W}_t = W_t - \int_0^t \eta_s ds$ are components of a two-dimensional Q_η-Brownian motion. We call η_t the *incomplete market price of risk* corresponding to the "unhedgeable" random factor W_t. Clearly, every suitable η_t gives a parameterization of the Radon-Nikodym density of an EMM with respect to P. So $\mathcal{M}$ is infinite dimensional. We will identify an EMM Q with its Radon-Nikodym density $\frac{dQ}{dP}$ in the space $\mathcal{M}$.

Example 1 (Stochastic Volatility Heath-Jarrow-Morton Model) We do not assume the complete market conditions C.5 and C.6 of Heath, Jarrow, and Morton (1992). Under their conditions C.1 to C.4, the forward rate dynamics are modelled as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dB_t,$$

$$d\sigma(t,T) = \beta(t,T)dt + \nu(t,T)dW_t,$$

where $\sigma(t,T) > 0$ P a.s.. The HJM no-arbitrage condition fixes ϕ_t as

$$\phi_t = \int_t^T \sigma(t,s)ds - \frac{\alpha(t,T)}{\sigma(t,T)}.$$

The price of the zero coupon bond after dividing by the numéraire is $Z(t,T)$. In this case, $Z_t = Z(t,T)$.

2.2 Arbitrage-free Equilibrium Pricing

The preference structure of the investors is described by the terminal utility function $U : \mathbb{R} \to [-\infty, \infty)$ for discounted terminal wealth at time T.

Our definition of the utility function follows Definition 3.4.1 of Karatzas and Shreve (1998). We assume further that the utility function $U(.)$ is twice continuously differentiable, strictly concave, and strictly increasing. The inverse of U' denoted by I, exists and is continuously differentiable with $U'(\infty) = \lim_{x\to\infty} U'(x) = 0$. The Fenchel-Legendre transformation yields the dual function $\widetilde{V}(z) \triangleq \sup_{x\in\mathbb{R}}\{U(x) - xz\}, z > 0$. The conjugate dual function satisfies $\widetilde{V}(z) = U(I(z)) - zI(z)$ and it is twice continuously differentiable, strictly decreasing, strictly convex, and satisfies $\widetilde{V}' = -I$.

Following the duality and martingale approach, in order to solve the terminal utility maximization problem in an incomplete market, we are interested in its dual optimization problem

$$\min_{Q\in\mathcal{M}} E_P\left[\widetilde{V}(y\frac{dQ}{dP})\right] \tag{1}$$

where $y > 0$ is related to the initial wealth x. See Goll and Rüschendorf (2001) and reference therein for more details. If the optimal measure Q_y^* of the above dual problem exists, then under some technical conditions (see Corollary 5.3 of Goll and Rüschendorf (2001)), the *arbitrage-free equilibrium price* in the sense of Davis (1997) of a contingent claim with discounted terminal payoff C is $E_{Q_y^*}C$.

If $\widetilde{V}$ is of the form

$$\widetilde{V}(yz) = a(y)V(z) + b(y)z + c(y),\ \forall\ z, y > 0 \tag{2}$$

for some functions of $a(y), b(y), c(y)$ with $a(y) > 0$, and twice continuously differentiable function $V(z)$, then Q_y^* is independent of y or the initial wealth. In this case of (2), we say that the dual function $\widetilde{V}$ is *homogeneous in initial wealth*. Examples of such utility functions are the ones with linear risk tolerance such as the exponential utility, the log utility, the power hyperbolic absolute risk aversion (HARA) utility, constant relative risk aversion (CRRA) utility.

When (2) does not hold, we could define $V(z) \triangleq \widetilde{V}(yz)$ for $y > 0$ being fixed in Problem 1. The subsequent theory holds for both definitions of the function V.

Definition 1 *The* entropy H_V *for the strictly convex function* V *is a function on* $\mathcal{M}$ *defined as*

$$H_V(Q) \triangleq \begin{cases} -E_P V(\frac{dQ}{dP}), & \text{if } E_P \left| V(\frac{dQ}{dP}) \right| < \infty, \\ -\infty, & \text{otherwise.} \end{cases}$$

Our definition of the entropy is related to the negative of the distance with respect to P in Goll and Rüschendorf (2001). The negative sign comes from the convention of information theory. The principle of minimal distance to P is reformulated as the *principle of maximum entropy.*

With the above notations, our main problem is

$$\max_{Q \in \mathcal{M}} H_V(Q). \tag{3}$$

3 An Approximate Problem

3.1 Wiener-Itô Chaos Expansions

We will approximate $\mathcal{M}$, the infinite dimensional domain of optimization for Problem (3), with its finite dimensional subset using Wiener-Itô chaos expansions. We then consider the more tractable approximate problem of which the optimization domain is the finite dimensional subset.

Let us recall the formulations of Wiener-Itô chaos expansions for the multi-dimensional Brownian motion $(W_t^1, \ldots, W_t^m)^\top$. Let $h_n(x), n = 0, 1, \cdots$ be the Hermite polynomials and $\{e_n\}_{n=0}^{\infty}$ be the basis of $L^2[0,T]$ using the Legendre polynomials. Define $e_j^i(s,l) \triangleq e_j(s)\delta_l^i, 1 \le i,l \le m$. Using the notations and results of Nualart (1995) example 1.1.2 and section 4 of Aase, Øksendal, Privault, and Ubøe (2000), the set of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k)$ of nonnegative integers $\mathbb{N} \cup \{0\}$ is denoted by $\mathcal{T}$. Let $\mathcal{T}^m$ be the set of all m-tuples $\Gamma = (\gamma^{(1)}, \ldots, \gamma^{(m)})$ of multi-indices $\gamma^{(i)} \in \mathcal{T}$. A typical element $\Gamma \in \mathcal{T}^m$ has the following representation

$$\Gamma = \Big((\alpha_{K_1}^{(1)}, \cdots, \alpha_{M_1}^{(1)}), \cdots, (\alpha_{K_m}^{(m)}, \cdots, \alpha_{M_m}^{(m)})\Big).$$

Letting $|\Gamma| = \sum_{i=1}^{m} \sum_{j=K_i}^{M_i} \alpha_j^{(i)}$, $\otimes$ be the tensor product and $\widehat{\otimes}$ be the symmetric tensor product, the Wiener-Itô chaos expansion of $G \in L^2(P)$ is:

$$G(\omega) = \sum_{\Gamma \in \mathcal{T}^m} c_\Gamma \, |\Gamma|! \sum_{i_1, \ldots, i_{|\Gamma|}=1}^{m} \int_0^T \int_0^{t_{|\Gamma|}} \cdots \int_0^{t_2} (e_{K_1}^1)^{\otimes \alpha_{K_1}^{(1)}} \widehat{\otimes} \cdots \widehat{\otimes} (e_{M_m}^m)^{\otimes \alpha_{M_m}^{(m)}}$$
$$((t_1, i_1), \ldots, (t_{|\Gamma|}, i_{|\Gamma|})) dW_{t_1}^{i_1} \cdots dW_{t_{|\Gamma|}}^{i_{|\Gamma|}}.$$

3.2 Finite Dimensional Approximation

In our case, $m = 2, W_t^1 = B_t, W_t^2 = W_t$. To simplify notations, we write the Wiener-Itô chaos expansion of $\int_0^T \eta_s dW_s \in L^2(P)$ as

$$\int_0^T \eta_s dW_s = \sum_{i=1}^{\infty} \theta_i \int_0^T \widetilde{\xi}_s^{(i)} dW_s$$

with limit in $L^2(P)$, $\theta_i \in \mathbb{R}$. The processes $\widetilde{\xi}_s^{(i)}$ involve multiple iterated Itô integrals. Examples of the first few terms of $\widetilde{\xi}_s^{(i)}$ are

$$\begin{aligned}
&1, s, s^2, \ldots \\
&W_s, s\int_0^s t_1 dW_{t_1}, s^2 \int_0^s t_1^2 dW_{t_1}, \ldots \\
&sW_s, s^2 W_s, \int_0^s t_1 dW_{t_1}, \int_0^s t_1^2 dW_{t_1} \cdots \\
&B_s, sB_s, s^2 B_s, \cdots
\end{aligned}$$

Let $K, i \in \mathbb{N}$ and $s \in [0,T]$. If $E_P \exp\left(\int_0^T \widetilde{\xi}_s^{(i)} dW_s\right) < \infty$, we simply define $\xi_s^{(i),K} \triangleq \widetilde{\xi}_s^{(i)}$; otherwise, define $\xi_s^{(i),K} \triangleq \max\left\{\min\left\{\widetilde{\xi}_s^{(i)}, K\right\}, -K\right\}$. Hence, $\xi_s^{(i),K}$ satisfy the Novikov condition or the Kazamaki condition.

Let $n \in \mathbb{N}$, we define

$$\mathcal{M}^{(n,K)} \triangleq \left\{ Q_\theta \sim P \Big| \frac{dQ_\theta}{dP} = \mathcal{E}\Big(\int_0^T \sum_{i=1}^n \theta_i \xi_s^{(i),K} dW_s\Big), \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \right\}.$$

By construction and the fact that each $\int_0^T \widetilde{\xi}_s^{(i)} dW_s$ is a product of Hermite polynomials of some Gaussian random variables, $\mathcal{M}^{(n,K)}$ is an n-dimensional subspaces of $\mathcal{M}$.

Proposition 1 (Approximation using Wiener-Itô Chaos Expansions) *For any* $Q \in \mathcal{M}$*, the following statements hold.*

(a) For K being fixed, $\mathcal{M}^{(n,K)} \subseteq \mathcal{M}^{(n+1,K)}, \forall\, n \in \mathbb{N}$.

(b) There exist a sequence of EMMs $\left\{Q^{(k)}\right\}_{k=1}^{\infty}$ in $\bigcup_{n=1}^{\infty}\bigcup_{K=1}^{\infty}\mathcal{M}^{(n,K)}$ such that $\frac{dQ^{(k)}}{dP} \to \frac{dQ}{dP}$ for P almost surely.

(c) If we identify an EMM Q with its Radon-Nikodym density $\frac{dQ}{dP}$, then the union of the finite dimensional subspaces $\bigcup_{n=1}^{\infty}\bigcup_{K=1}^{\infty}\mathcal{M}^{(n,K)}$ is dense in $\mathcal{M}$ in the sense of P almost sure convergence.

(d) For any bounded contingent claim with discounted terminal payoff C, the sequence of prices evaluated using the sequence of EMMs in (b) also converges to the price evaluated using Q, that is $\lim_{k\to\infty} E_{Q^{(k)}}[C] = E_Q[C]$.

Proof Part (a) is obvious from the definition. The key to the rest of the proof is the fact that we can extract an almost sure converging subsequence from an $L^2(P)$ or $L^1(P)$ converging sequence of random variables. From this fact, we can show that $\forall\, \epsilon > 0$, there exist $n, K \in \mathbb{N}$ and $Q^{(\epsilon)} \in \mathcal{M}^{(n,K)}$ such that

$$\left|\frac{dQ^{(\epsilon)}}{dP} - \frac{dQ}{dP}\right| < \epsilon \tag{4}$$

for P almost surely. Part (b) and (c) then follow easily from (4). To show (d), we use Theorem 18.5 of Jacod and Protter (2000) by replacing their density functions with the Radon-Nikodym densities. □

From Proposition 1, for any $\epsilon > 0$, there exists some large enough n, K and an EMM $Q^{(\epsilon)}$ in the finite dimensional subset $\mathcal{M}^{(n,K)}$ such that

$$\left|E_{Q^*}[C] - E_{Q^{(\epsilon)}}[C]\right| < \epsilon.$$

So we can approximate the fair price $E_{Q^*}[C]$ using $E_{Q^{(\epsilon)}}[C]$.

From now on, we will fix n, K. For simplicity of notation, write $\mathcal{S}$ for $\mathcal{M}^{(n,K)}$ and ξ_s^i for $\xi_s^{(i),K}$.

From the definition of $\mathcal{S}$, the incomplete market price of risk processes of any $Q \in \mathcal{S}$ will be of the form $\eta_t = \sum_{i=1}^n \theta_i \xi_t^i$. It means that incomplete MPR is linearly dependent on n processes ξ_t^i which are Wiener chaoses up to certain order. In practice, the choice of n, K and ξ_t^i can be made through empirical calibration so that the Wiener chaos terms affecting the incomplete MPR significantly will be selected. Similar calibration has been used by Brace, Gatarek, and Musiela (1997) to approximate processes related to interest rate derivatives with order 0 and 1 Wiener chaoses in a complete market setting.

We are interested in solving the following approximate problem

$$\min_{Q \in \mathcal{S}} H_V(Q) \tag{5}$$

and use the solution to approximate Q^*. Since continuous-time finance model is an approximation to the real financial market, the original Problem (3) is also a kind of "approximate" problem. Hence, it makes sense both mathematically and practically to consider the approximate Problem (5).

4 Differential Geometry of Equivalent Martingale Measures

4.1 Finite Dimensional Manifold of Equivalent Martingale Measures

We can view $\mathcal{S}$ as a finite dimensional manifold. The key observation is that a Radon-Nikodym density plays the same role as a distribution function in differential geometry of statistical distributions (also called information geometry). See, for example, Amari and Nagaoka (2000) for an introduction to information geometry. Each point on this abstract manifold is an EMM Q_θ or the Radon-Nikodym density $\frac{dQ_\theta}{dP}$. The coordinate of the point is $\theta \in \mathbb{R}^n$. For the study of the topology and differential geometry of $\mathcal{S}$, we impose some technical regularity assumptions on V. In most applications, these assumptions are satisfied due to the nature that ξ_t^i are related to polynomials of Gaussian random variables.

Let the parameter space Θ in the following assumptions be $\mathbb{R}^n$. Let $\theta = (\theta_1, \ldots, \theta_n)^\top \in \Theta$, $\xi_t = (\xi_t^1, \ldots, \xi_t^n)^\top$. Let $\|\xi_t\|$ be the $\mathbb{R}^n$ norm of ξ_t. If $\theta^1, \theta^2 \in \mathbb{R}^n$ are two vectors such that $(\theta^1)_i \leq (\theta^2)_i$ for all i, then we write $\theta^1 \leq \theta^2$.

Assumption 1

$$E_P \exp\Big(\frac{1}{2}\int_0^T \phi_t dB_t + \frac{1}{2}\sum_{i=1}^n \theta_i \int_0^T \xi_t^i dW_t\Big) < \infty, \ \forall\ \theta \in \Theta.$$

Assumption 2

$$E_P \left| V(\frac{dQ_\theta}{dP}) \right| < \infty, \ \ and \ \ E_P V'(\frac{dQ_\theta}{dP})^2 < \infty, \forall\ \theta \in \Theta.$$

Assumption 3 *V''' exists and is continuous. For* $\forall\ \theta^1, \theta^2 \in \Theta$, $-\infty < \theta^1 < \theta^2 < \infty$,

$$\sup_{\theta^1 \le \theta \le \theta^2} E_{Q_\theta} \int_0^T \|\xi_t\|^{16}\, dt < \infty,$$

$$\sup_{\theta^1 \le \theta \le \theta^2} E_{Q_\theta} V'(\frac{dQ_\theta}{dP})^2 < \infty,$$

$$\sup_{\theta^1 \le \theta \le \theta^2} E_{Q_\theta} V''(\frac{dQ_\theta}{dP})^8 < \infty,$$

$$\sup_{\theta^1 \le \theta \le \theta^2} E_{Q_\theta} V'''(\frac{dQ_\theta}{dP})^2 < \infty, \ \textit{and}$$

$$\sup_{\theta^1 \le \theta \le \theta^2} E_{Q_\theta} \Big(\frac{dQ_\theta}{dP}\Big)^8 < \infty.$$

In general, let $\Theta \subseteq \mathbb{R}^n$. If $\xi_t^i, i = 1, \ldots, n$ are n nontrivial known processes that satisfy Assumptions 1, 2 and 3, then the set

$$\left\{ Q_\theta \sim P \Big| \frac{dQ_\theta}{dP} = \mathcal{E}\Big(\int_0^T \sum_{i=1}^n \theta_i \xi_s^i dW_s \Big), \theta = (\theta_1, \ldots, \theta_n) \in \Theta \right\}$$

is called *a n-dimensional manifold of equivalent martingale measures.*

The n-dimensional manifold of EMMs $\mathcal{S}$ is an exponential family and is not a convex set of measures.

4.2 A New Kind of Topology

To solve Problem (5), $H_V(Q_\theta)$ can be viewed as a real function on $\mathbb{R}^n$ and standard gradient descent method in Euclidean space could be applied. The

Euclidean gradient vector of H_V can be computed as $(\nabla H_V)_i = \frac{\partial}{\partial \theta_i} H_V(Q_\theta)$. Why do we need to study the differential geometry of EMMs?

The reason is that the Euclidean geometric structure of EMMs does not capture much financial information related to the problem of entropy maximization. In this case, the Euclidean squared distance between Q_{θ^1} and Q_{θ^2} is $\sum_{i=1}^n (\theta_i^1 - \theta_i^2)^2$. This distance does not capture any distributional difference between the two Radon-Nikodym densities and the entropies which are crucial to Problem (5). That is why we need to develop a non-Euclidean geometric model. We proceed by introducing a new kind of pseudo-distance between EMMs.

Let the set of EMMs with finite entropies be

$$\mathcal{M}_V \triangleq \left\{ Q \in \mathcal{M} \middle| \quad E_P \left| V(\frac{dQ}{dP}) \right| < \infty \right\}.$$

The *Bregman Difference* for a convex function V is defined as

$$\Delta_V(y, x) \triangleq V(y) - V(x) - V'(x)(y - x).$$

Definition 2 *The* cross entropy *or the* Bregman distance *between two points in $\mathcal{M}_V$ is defined as*

$$D_V(Q_1, Q_2) \triangleq E_P \Delta_V(\frac{dQ_1}{dP}, \frac{dQ_2}{dP}).$$

The definition of the cross entropy is equivalent to the definitions in Rao (1987), and Rao and Nayak (1985). From the strict convexity of V, $D_V(Q, \widetilde{Q}) = 0$ if $Q = \widetilde{Q}$; $D_V(Q, \widetilde{Q}) > 0$ if $Q \neq \widetilde{Q}$.

Example 2 In the mean-variance hedging approach of Schweizer (1996), the problem is to minimize $E_P\Big[\Big(\frac{dQ}{dP}\Big)^2\Big]$. This corresponds to $V(z) = z^2$ in our framework. In this case, the cross entropy is the mean square difference of the densities

$$D_V(Q_1, Q_2) = E_P\Big(\frac{dQ_1}{dP} - \frac{dQ_2}{dP}\Big)^2.$$

Since the Bregman difference measures the difference between x and y together with the difference between $V(x)$ and $V(y)$, the cross entropy measures both the differences in distributions of $V(\frac{dQ}{dP})$ and $\frac{dQ}{dP}$ simultaneously. Therefore, the cross entropy as a reasonable measure of the pseudo-distance or dissimilarity between two Radon-Nikodym densities is mathematically clear.

4.3 Riemannian Geometric Structure

As shown in the following theorem, the cross entropy defined in the infinite dimensional space $\mathcal{M}_V$ induces a Riemannian geometric structure on the finite dimensional manifold of EMMs $\mathcal{S}$.

Theorem 1 (Riemannian Metric Theorem) *Suppose Assumption 1, 2, 3 hold true. For any point $Q_{\theta_0} \in \mathcal{S}$, let $f(\theta) = D_V(Q_{\theta_0}, Q_\theta)$. Then*

$$\frac{\partial}{\partial\theta_i} f(\theta)\Big|_{\theta=\theta_0} = 0, \quad 1 \le i \le n,$$
$$\Big(\frac{\partial^2 f}{\partial\theta_i\partial\theta_j}\Big)\Big|_{\theta=\theta_0} = g_{ij}(\theta_0), \; 1 \le i, j \le n,$$

where

$$g_{ij}(\theta) = E_P\Big[V''(\frac{dQ_\theta}{dP})\frac{\partial}{\partial\theta_i}\Big(\frac{dQ_\theta}{dP}\Big)\frac{\partial}{\partial\theta_j}\Big(\frac{dQ_\theta}{dP}\Big)\Big] \tag{6}$$

and $g_{ij}(\theta)$ is continuous in θ. Hence, from Taylor formula, we have

$$D_V(Q_\theta, Q_{\theta+d\theta}) = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij} d\theta_i d\theta_j + o(\|d\theta\|^2).$$

Proof (Riemannian Metric Theorem) From the property of the cross entropy we know that $f(\theta) \geq 0$ and $f(\theta_0) = 0$. Since the differentiable cross entropy function obtains the minimal value zero at $\theta = \theta_0$, we have $\frac{\partial}{\partial\theta_i} f(\theta)\Big|_{\theta=\theta_0} = 0,\ 1 \leq i \leq n$. It is proved in Gao (2002) that under Assumption 2 and 3, we can exchange the operators E_P and $\frac{\partial}{\partial\theta_i}$. Hence, the following computations are justified.

$$\begin{aligned}
&\Big(\frac{\partial^2 f}{\partial\theta_i\partial\theta_j}\Big)\Big|_{\theta=\theta_0} = \frac{\partial^2}{\partial\theta_i\partial\theta_j} E_P\Big[V(\frac{dQ_{\theta_0}}{dP}) - V(\frac{dQ_\theta}{dP}) + V'(\frac{dQ_\theta}{dP})\Big(\frac{dQ_\theta}{dP} - \frac{dQ_{\theta_0}}{dP}\Big)\Big] \\
&= \frac{\partial}{\partial\theta_i} E_P\Big[-V'(\frac{dQ_\theta}{dP})\frac{\partial}{\partial\theta_j}(\frac{dQ_\theta}{dP}) + V''(\frac{dQ_\theta}{dP})\Big(\frac{\partial}{\partial\theta_j}(\frac{dQ_\theta}{dP})\Big)\Big(\frac{dQ_\theta}{dP} - \frac{dQ_{\theta_0}}{dP}\Big) \\
&\qquad\qquad + V'(\frac{dQ_\theta}{dP})\frac{\partial}{\partial\theta_j}(\frac{dQ_\theta}{dP})\Big] \\
&= \frac{\partial}{\partial\theta_i} E_P\Big[V''(\frac{dQ_\theta}{dP})\Big(\frac{\partial}{\partial\theta_j}(\frac{dQ_\theta}{dP})\Big)\Big(\frac{dQ_\theta}{dP} - \frac{dQ_{\theta_0}}{dP}\Big)\Big]\Big|_{\theta=\theta_0} \\
&= E_P\Big[V''(\frac{dQ_\theta}{dP})\Big(\frac{\partial}{\partial\theta_j}(\frac{dQ_\theta}{dP})\Big)\Big(\frac{\partial}{\partial\theta_i}(\frac{dQ_\theta}{dP})\Big)\Big]\Big|_{\theta=\theta_0} \\
&= g_{ij}(\theta_0).
\end{aligned}$$

Also $g_{ij}(\theta)$ is continuous in θ from Gao (2002). □

Definition 3 *The* Riemannian metric of the manifold of EMMs $\mathcal{S}$ *is defined by the elements g_{ij} in (6) of Theorem 1.*

As shown in Gao (2002), the metric is indeed positive definite.

Example 3 (Fisher Information Metric of Exponential Utility) We consider the exponential utility $U(x) = -e^{-x}, x > 0$. We can check that (2) holds and $V''(z) = \frac{1}{z}$. The Riemannian metric is $g_{ij} = E_P\left[\frac{dQ_\theta}{dP}\frac{\partial}{\partial\theta_i}(\log\frac{dQ_\theta}{dP})\frac{\partial}{\partial\theta_j}(\log\frac{dQ_\theta}{dP})\right]$. This is also called the Fisher Information metric which plays a central role in the differential geometry of statistics and econometrics.

4.4 Existence and Uniqueness

We have translated Problem (5) into an optimization problem on a Riemannian manifold. Let Q_{θ^*} be the maximum entropy point over $\mathcal{S}$. In general, the existence and uniqueness of Q_{θ^*} can be proved using results from Riemannian geometry. We will not discuss the geometric conditions in this paper. However, for some special utility functions, the usual coercive functional argument is sufficient to ensure the existence and uniqueness without any geometric considerations.

Proposition 2 (Existence and Uniqueness) *Let* Θ *be* $\mathbb{R}^n$ *and* $U'(\infty) \triangleq \lim_{x\to\infty} U'(x) = 0$. *Suppose Assumption 1, 2, 3 hold true. If*

(1) the relative risk aversion is not more than one, that is

$$R(x) \triangleq -\frac{xU''(x)}{U'(x)} \leq 1, \ \forall \ x > 0;$$

(2) ξ_t^i *are linearly independent in the sense that*

$$\sum_{k=1}^{n} b_k \xi_t^k = 0, \ \forall \ t \quad \Rightarrow b_k = 0, \quad k = 1, \ldots, n;$$

(3)

$$E_P \int_0^T \xi_s^i \xi_s^j ds < \infty, \forall \ 1 \leq i, j \leq n;$$

then there exists a unique maximum entropy point Q_{θ^*} over $\mathcal{S}$.

Proof Without loss of generality, we can assume $y = 1$. It suffixes to show the existence and uniqueness of the point attaining the minimum of $h(\theta) \triangleq E_P \widetilde{V}(\frac{dQ_\theta}{dP})$.

From Lemma 4.2 and Lemma 12.6 of Karatzas, Lehoczky, Shreve, and Xu (1991), if $R(x) \leq 1,\ \forall\, x > 0$, and $U'(\infty) = 0$, then $w \mapsto \widetilde{V}(e^w)$ is convex in w and $\widetilde{V}(0) = U(\infty) = \infty$ where $\widetilde{V}(0) = \lim_{z \to 0} \widetilde{V}(z)$. Using this fact, we can show that $h(\theta)$ is coercive in θ. By Jensen's inequality, $h(\theta)$ is bounded from below. Hence, $h(\theta)$ is a coercive continuous functional bounded from below on a Hilbert space. By Theorem 1.6 of Grossinho and Tersian (2001), there exist a point attaining the infimum of $h(\theta)$. The uniqueness follows from the positive definiteness of the Euclidian Hessian function $\frac{\partial^2 h(\theta)}{\partial \theta_i \theta_j}$ and Taylor formula. □

5 An Approximation Pricing Algorithm

We assume the existence and uniqueness of Q_{θ^*} in this section. The objective is to compute or approximate $\frac{dQ_{\theta^*}}{dP}$ or θ^* for the approximation pricing. With our geometrical view, Q_{θ^*} is a *critical point* of the entropy functional H_V on $\mathcal{S}$. The critical point does not depend on the Riemannian metric g_{ij}. However, the gradient of the entropy ∇H_V does depend on g_{ij}. The gradient on the Riemannian manifold is given by

$$(\nabla H_V)_i = \sum_{j=1}^{n} g^{ij} \frac{\partial H_V}{\partial \theta_j}(Q_\theta),\ 1 \leq i \leq n$$

where g^{ij} is the matrix inverse of g_{ij}. See Jost (1998) for details. By Cauchy-Schwarz inequality, the gradient of the entropy functional is in the direction of maximum local increase of H_V.

The gradient descent algorithm is the simplest method to approximate θ^* from an initial coordinate θ_0 iteratively. Once θ^* is known, we can compute $\frac{dQ_{\theta^*}}{dP}$ as well as the price by taking expectation. Let $Q_\theta \in \mathcal{S}$ and C be the discounted terminal payoff of a contingent claim, suppose we can compute $E_{Q_\theta}[C]$ once θ is known. Assume further that $E_{Q_\theta}[C]$ is continuous in θ.

Algorithm 1 (Approximation Pricing Algorithm) *(1) Pick an initial coordinate* $\theta_0 = (\theta_0)_i, 1 \leq i \leq n$.

(2) Compute $\frac{\partial H_V}{\partial \theta_i}$ *at the current state* θ*. If* $\left|\frac{\partial H_V}{\partial \theta_i}\right| \leq \varepsilon, \forall\ i$*, where* ε *is some fixed precision, then stop and compute* $E_{Q_\theta}[C]$ *as the price.*

(3) Otherwise, compute g_{ij} *and invert it to* g^{ij} *to get* $\upsilon_i = \sum_{j=1}^n g^{ij} \frac{\partial H_V}{\partial \theta_j}$*. Move to the next coordinate* $\theta + \tau\upsilon$*, where* τ *maximizes* H_V *along the line connecting* θ *and* $\theta + \upsilon$*.*

(4) Repeat step 2 at this new point.

Each iteration of the algorithm involves the computations of $\frac{\partial H_V}{\partial \theta_i}, g_{ij}, g^{ij}$. They can be computed via numerical integration due to the fact that ξ_t^i are related to products of Hermite polynomials of Gaussian random variables. The detailed numerical implementation and the empirical testings of the algorithm are in Gao (2002) and Gao, Lim, and Ng (2002).

Remark 1 If $Q^* \in \mathcal{S}$, then $Q_{\theta^*} = Q^*$ and the approximation pricing algorithm gives the exact fair price.

6 Financial Interpretation of the Geometry

6.1 Cross Entropy and Pricing Model Risk

Let y be the fixed constant in Problem (1) and $Q_1, Q_2 \in \mathcal{M}_V$. Write $X_i \triangleq I(y\frac{dQ_i}{dP}), i = 1, 2$. The financial interpretation of the cross entropy is based on the following duality result.

Proposition 3 *(a) If $V(z) = \widetilde{V}(yz)$, then*

$$E_P \Delta_V(\frac{dQ_1}{dP}, \frac{dQ_2}{dP}) = E_P \Delta_{-U}(X_2, X_1). \tag{7}$$

(b) If (2) holds, then

$$E_P \Delta_V(\frac{dQ_1}{dP}, \frac{dQ_2}{dP}) = \frac{1}{a(y)} E_P \Delta_{-U}(X_2, X_1), \tag{8}$$

where the function $a(.)$ is from (2).

Proof The proof follows directly from the definitions. □

We want to interpret the right hand side of (7) or (8) as a reasonable measure of pricing model risk between the two pricing models $E_{Q_1}[.]$ and $E_{Q_2}[.]$.

Suppose an agent (a bank, for example) uses $E_{Q_i}[.], i = 1$ or 2 as the pricing model and tries to maximized the utility of discounted terminal wealth from some budget set of portfolios which are not necessarily self-financing. The agent faces the following constraint utility maximization problem:

$$\max_{X \in \mathcal{C}} E_P U(X)$$

where $\mathcal{C} = \left\{ X \in L^1(Q_i) \middle| E_{Q_i}[X] \leq x_i \right\}$ and the initial wealth x_i is set such that $E_{Q_i}\left[I(y\frac{dQ_i}{dP})\right] = x_i$. By Theorem 3.59 of Föllmer and Schied (2002), $X_i = I(y\frac{dQ_i}{dP})$ is the discounted terminal wealth of the optimal portfolio that solves the above problem. We call $I(y\frac{dQ_i}{dP})$ the *implied optimal wealth* of the pricing model $E_{Q_i}[.]$. The implied optimal wealth is unique to each pricing model.

A reasonable measure of pricing model risk should answer the question: What is lost if an agent uses a wrong model? Since the negative of the utility can be used as a convex loss function, the loss can be measured in terms of the expected differences in both the implied optimal wealths and the loss functions of the implied optimal wealths. The Bregman difference function Δ_{-U} can capture both differences simultaneously. This leads to the following definition.

Definition 4 *The* pricing model risk *of the pricing model $E_{Q_1}[.]$ relative to the one of $E_{Q_2}[.]$ is defined as*

$$d_y(Q_1, Q_2) \triangleq E_P \Delta_{-U}(X_2, X_1)$$

where $X_i = I(y\frac{dQ_i}{dP}), i = 1, 2$ are the implied optimal wealths.

With this definition and Proposition 3, the cross entropy between Q_1 and Q_2 is proportional to the pricing model risk between $E_{Q_1}[.]$ and $E_{Q_2}[.]$. From Theorem 1, we also have a financial interpretation of the Riemannian geometry of $\mathcal{S}$. The infinitesimal Riemannian square distance $ds^2 =$

$\sum_{i,j=1}^{n} g_{ij} d\theta_i d\theta_j$ is proportional to the infinitesimal pricing model risk of choosing Q_θ relative to $Q_{\theta+d\theta}$ as the pricing measure.

6.2 Minimal Pricing Model Risk Approximation

We are now in a position to give the financial justification of the approximation of Q^* using Q_{θ^*}. The solution of Problem (5), Q_{θ^*}, is the minimal pricing model risk approximation to the solution of Problem (3), Q^*.

Proposition 4 (Minimization of Pricing Model Risk) *Suppose Assumptions 1, 2, 3 hold true. Assume the maximum entropy point Q^* over $\mathcal{M}$ exists, $Q^* \in \mathcal{M}_V$, $V'(\frac{dQ^*}{dP}) \in L^1(Q^*)$ and $V'(\frac{dQ^*}{dP}) \in L^4(P)$, then $Q_{\theta^*} \in \mathcal{S}$ attains the maximum entropy over $\mathcal{S}$ if and only if Q_{θ^*} has the minimum pricing model risk relative to Q^* over $\mathcal{S}$. That is, $H_V(Q_{\theta^*}) = \max_{Q\in\mathcal{S}} H_V(Q)$ if and only if $d_y(Q_{\theta^*}, Q^*) = \min_{Q\in\mathcal{S}} d_y(Q, Q^*)$.*

Proof The proof is based on Theorem 5.1 of Goll and Rüschendorf (2001). From their theorem, $V'(\frac{dQ^*}{dP}) = c + \int_0^T \varphi_s dZ_s$ and $\int_0^\cdot \varphi_s dZ_s$ is a Q^*-martingale for some Z-integrable predictable process φ. For any $Q_\theta \in \mathcal{S}$, let $\widetilde{B}_t = B_t - \int_0^t \phi_s ds$ be the Q_θ-Brownian motion. Then there exists a predictable process $\widetilde{\varphi}_s$ such that $V'(\frac{dQ^*}{dP}) = c + \int_0^T \widetilde{\varphi}_s d\widetilde{B}_s$. Using the regularity assumptions of the proposition, we can show that $E_{Q_\theta} V'(\frac{dQ^*}{dP}) = E_{Q^*} V'(\frac{dQ^*}{dP})$. Thus, $D_V(Q_\theta, Q^*) = E_P V(\frac{dQ_\theta}{dP}) - E_P V(\frac{dQ^*}{dP})$. Hence Q_{θ^*} attains the minimum of $E_P V(\frac{dQ_\theta}{dP})$ over $\mathcal{S}$ if and only if it attains the minimum of $D_V(Q_\theta, Q^*)$ over $\mathcal{S}$. Since $D_V(.,.)$ and $d_y(.,.)$ are proportional and the proposition is proved.

□

Remark 2 If $Q^* \in \mathcal{S}$, then the statement of Proposition 4 is trivially true.

7 Examples and Relationship to the Minimal Martingale Measure

When $\theta = 0$ where 0 could be the zero vector, the EMM Q_θ corresponds to the minimal martingale measure defined by Föllmer and Schweizer (1991).

Example 4 (one-dimensional manifold of EMMs) Suppose the complete MPR ϕ_t is independent of the Brownian motion W. We specify $n = 1$ and a zero order chaos approximation of the incomplete MPR. That is, $\xi_t^1 \equiv 1$ or $\eta_t = \theta$. Then the maximum entropy measures over $\mathcal{S}$ of exponential, log, HARA utilities are all given by the minimal martingale measure, Q_θ where $\theta = 0$. For the proofs, see Gao (2002).

Example 5 (two-dimensional manifold of EMMs) Suppose the complete MPR $\phi_s = \phi W_s$ for some constant ϕ. We specify a deterministic incomplete MPR $\eta_s = \theta_1 + \theta_2 s$ which is linear in time. The utility function is of HARA type $U(x) = \frac{1}{b-1}\Big(a + bx\Big)^{1-\frac{1}{b}}, 0 < b < 1$ so that the risk tolerance $-\frac{U'(x)}{U''(x)} = a + bx$. The dual function is homogeneous in initial wealth and of the form (2) with $V(z) = -z^{1-b}$. The entropy is $H_V(Q) = E_P\left(\frac{dQ}{dP}\right)^{1-b}$. Write $F(W) = \phi \int_0^T W_s dB_s - \frac{1}{2}\phi^2 \int_0^T W_s^2 ds$. The minimal martingale measure $Q_{(0,0)}$ is not the optimal point since

$$\frac{\partial H_V}{\partial \theta_1}(0,0) = (1-b)E_P\Big[e^{(1-b)F(W)} W_T\Big] \neq 0,$$
$$\frac{\partial H_V}{\partial \theta_2}(0,0) = (1-b)E_P\Big[e^{(1-b)F(W)}\Big(\int_0^T s dW_s\Big)\Big] \neq 0.$$

We can choose $Q_{(0,0)}$ as the initial point and apply Algorithm 1.

8 Summary

Our methodology of contingent claim pricing in an incomplete market can be summarized as follows:

1. specify some parametric form of the complete MPR process ϕ_t, calibrate the parameters using the drift and volatility of the tradable securities;
2. specify a utility function and the form of the incomplete MPR processes, i.e. n, K and $\xi_t^i, i = 1, \ldots, n$;
3. approximate the parameters θ^* from the principle of maximum entropy using Algorithm 1 on Riemannian manifold $\mathcal{S}$;
4. use these parameters to price all contingent claims.

The approximation pricing algorithm provides a method for tractable industrial implementation of the arbitrage-free equilibrium pricing model in incomplete markets. The discovery of the Riemannian geometrical structure of EMMs shows a new direction in building the pricing theory of incomplete markets.

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