Estimating Maximum Smoothness and Maximum Flatness Forward Rate Curve

Lim Kian Guan & Qin Xiao[1]

January 21, 2002

[1]Both authors are from the National University of Singapore, Centre for Financial Engineering. Research support from the Institute of High Performance Computing (IHPC) is gratefully acknowledged. Comments are welcome. All correspondences should be addressed to Prof. Lim Kian Guan, Director, Center for Financial Engineering, Kent Ridge Post Office P. O. Box 1130, Singapore 911105. E-mail: fbalimkg@nus.edu.sg.

Abstract

The yield and the forward rates are important to fixed income analysts. However, only a finite number of rates can be observed from the market at any point in time. This paper provides a complete system of estimating the term structure via the estimated forward rates. We derive the interesting result that fourth-order and quadratic polynomial spline functions obtain given the maximum smoothness and the maximum flatness estimation of the forward rate curve among all polynomial functional forms. We also compare our empirical results with discrete models using difference method and explicit solutions with matrix form. From the empirical results, our smoothness or flatness methods perform better than the standard cubic spline methods that are popularly used.

1 Introduction

The term structures of yield-to-maturity and forward rates are important to fixed income analysts. They are widely used to price interest rate derivatives. Hull and White (1993) show that by allowing the drift of instantaneous spot rate to be a function of time, single-factor term structure models such as Vasicek (1977) and Cox, Ingersoll, and Ross (CIR, 1985) can be made to fit current yield curve exactly. Heath, Jarrow and Morton (HJM, 1992) assume that the market risk preference is incorporated in the existing term structure of interest rates. Therefore, under the no-arbitrage condition, contingent claims can be priced when the yield curve is fitted to the market data.

Our concern here is that only a few points of the yield curve can be observed in the market at any one point in time. Some mathematical techniques are necessary to estimate the whole yield curve from the observed finite data. McCulloch (1975) uses the polynomial spline functions to fit the observed data. As is well known, the use of polynomial function to fit the entire yield curve may lead to unacceptable yield patterns. Vasicek and Fong (1982) use the exponential spline for the discount function, and choose the cubic form as the lowest odd order form from continuous derivatives. Delbaen and Lorimier (1992) introduce the discrete approach to estimate the yield curve by minimizing the difference between two adjacent forward rates. Frishing and Yamamula (1996) use a similar approach on the coupon bond prices, and Adams and Deventer (1994) employ a concept of maximum smoothness in the estimation.

In this paper, we introduce a complete system to estimate the term structure of yield using the corresponding maximum smoothness and also the maximum flatness forward curve. Both continuous and discrete models are developed. The maximum smoothness criterion in the continuous approach is defined as minimizing the total curvature of the curve. In the discrete model, sum of the squared second order central difference is used as a measure of the total curvature. For the maximum flatness criterion, the total slope of the curve is minimized, and the sum of the squared forward difference is used as the measure in the discrete model.

The potential problem is that all the models involve optimization of a large number of variables. This may be computationally cumbersome. However, all the optimization problems can transformed to quadratic forms. Then we can obtain the explicit solutions by the Lagrange-multiplier method. This approach allows for fast and tractable results.

In section 2, we derive the estimation under the maximum smoothness method. In section 3, we derive the estimation under the maximum flatness method. In section 4, we illustrate the estimation methods using data from the US Treasury. Section 5 contains the conclusions.

2 Continuous Model

2.1 Yield Curve and Forward Curve

Let $p(T)$ be the price of a zero-coupon bond with T periods to go before maturity. Thus $p(0) = 1$, and $p(t) > 0$ for all $t > 0$. Assume $\partial \ln p(t)/\partial t$ exists for all $t > 0$. The corresponding yield and instantaneous forward rates are $y(t)$ and $f(t)$ respectively. Then,

$$y(T) = -\frac{1}{T} \ln P(T) \tag{1}$$

and

$$p(T) = \exp\left(-\int_0^T f(u)du\right) \tag{2}$$

Taking the natural logarithm on both sides of equation (**??**) and eliminating $p(T)$ using equation (**??**), we obtain the relationship between the forward rates and the yield

$$\int_0^T f(u)du = y(T)T \tag{3}$$

Suppose that for the maturity date T_i $(i = 1, ..., n)$, the corresponding yield-to-maturity is known from the market as $y(T_i)$. Now we want to estimate the yields for other maturity dates and the corresponding instantaneous forward rates.

2.2 Maximum Smoothness Forward Curve

A maximum smoothness forward curve is one that minimizes the total curvature of the curve and which fits the observed data exactly. Assuming $f(\cdot) \in C^2[0, T_n]$, The general criterion for the total curvature is the integral of the squared second-order differential function

$$Z = \int_0^{T_n} f''^2(u)du. \tag{4}$$

Maximum smoothness obtains when Z is minimized on all C^2 functions defined on $[0, T_n]$. This expression is a common mathematical definition of smoothness. From equation (**??**), the constraints can be written as

$$\int_0^{T_i} f(t)dt = y(T_i)T_i \equiv y_i, \qquad i = 1, ..., n \tag{5}$$

Then the estimation becomes an optimization problem

$$\min_{f \in C^2(0,T_n)} Z$$

$$s.t. \quad \int_0^{T_i} f(t)dt = y_i, \qquad i = 1, ..., n$$

$$f(0) = r_0 \tag{6}$$

$$f'(0) = 0 \tag{7}$$

Integrating, we obtain

$$\int_0^T f(u)du = f(T)T - \int_0^T uf'(u)du \tag{8}$$

and

$$\int_0^T uf'(u)du = T^2 f'(T) - \int_0^T uf'(u)du - \int_0^T u^2 f''(u)du$$

so

$$\int_0^T uf'(u)du = \frac{1}{2}\left(T^2 f'(T) - \int_0^T u^2 f'(u)du\right). \tag{9}$$

Substituting (**??**) into (**??**), we get

$$\int_0^T f(u)du = f(T)T - \frac{1}{2}T^2 f'(T) + \frac{1}{2}\int_0^T u^2 f''(u)du. \tag{10}$$

Denoting

$$g(t) = f''(t) \qquad 0 \le t \le T_n,$$

then $g(\cdot) \in C[0, T_n]$. We have

$$f'(T) = \int_0^T g(u)du + f'(0) = \int_0^T g(u)du \tag{11}$$

and

$$f(T) = \int_0^T f'(u)du + r_0 = \int_0^T \int_0^t g(v)dvdu + r_0 \tag{12}$$

[1] Substituting (**??**) and (**??**) into (**??**), (**??**) becomes

$$y_i = \left(\int_0^{T_i}\int_0^t g(v)dvdu + r_0\right)T_i - \frac{1}{2}T_i^2\int_0^{T_i} g(u)du + \frac{1}{2}\int_0^{T_i} u^2 g(u)du \tag{13}$$

The maximum smoothness problem can be written as

$$\min_{g(\cdot)\in C[0,T_n]} \int_0^{T_n} g^2(u)du$$
$$s.t. \quad (13), \qquad i = 1, ..., n$$

let λ_i for $i = 1, ..., n$ be the Lagrange multipliers corresponding to the constraint (**??**). The objective function becomes

$$\min_{g\in C[0,T_n],\lambda_i} Z[g,\lambda] = \int_0^{T_n} g^2(u)du + \sum_{i=1}^n \lambda_i \left(T_i \int_0^{T_i}\int_0^t g(v)dvdu - \frac{1}{2}T_i^2\int_0^{T_i} g(u)du + \frac{1}{2}\int_0^{T_i} u^2 g(u)du + r_0 T_i - y_i\right)$$

[1] Here we involve an additional assumption that $f'(0) = 0$. We can also incorporate non-zero $f'(0)$, and the solution just become a bit more complicated.

We use the step function

$$I_t = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

to simplify the integral area

$$Z[g,\lambda] = \int_0^{T_n} \left(g^2(u) + \sum_{i=1}^{n} \lambda_i I_{T_i - u} \left(T_i \int_0^u g(v)dv - \frac{1}{2}T_i^2 g(u) + \frac{1}{2}u^2 g(u) \right) \right) du$$
$$+ \sum_{i=1}^{n} \lambda_i (r_0 T_i - y_i) \tag{14}$$

If g^* is the solution of the optimization problem, then

$$\frac{d}{d\sigma} Z[g^* + h\sigma] \bigg|_{\sigma=0} = 0 \tag{15}$$

for any continuous function $h(\cdot)$ defined on $[o, T_n]$, such that $h(u) = g(u) - g^*(u)$, where $g^*(u)$ is optimal. From (**??**) we have

$$\frac{d}{d\sigma} Z[g^* + h\sigma] \bigg|_{\sigma=0}$$
$$= \int_0^{T_n} \left(2g^*(u)h(u) + \sum_{i=1}^{n} \lambda_i I_{T_i - u} \left(T_i \int_0^u h(v)dv + \frac{1}{2}(u^2 - T_i^2)h(u) \right) \right) du$$
$$= \int_0^{T_n} \left(2g^*(u) + \sum_{i=1}^{n} \lambda_i I_{T_i - u} \frac{1}{2}(u^2 - T_i^2) \right) h(u)du$$
$$+ \int_0^{T_n} \left(\sum_{i=1}^{n} \lambda_i I_{T_i - u} T_i \right) \left(\int_0^u h(v)dv \right) du = 0 \tag{16}$$

Applying lemma 1 in appendix 1, (**??**) equals to zero if and only if

$$2g^*(u) + \sum_{i=1}^{n} \lambda_i I_{T_i - u} \frac{1}{2}(u^2 - T_i^2) = -\int_u^{T_n} \sum_{i=1}^{n} \lambda_i I_{T_i - v} T_i dv \tag{17}$$

For all $0 \leq u \leq T_n$. Thus

$$g^*(u) = -\frac{1}{4} \sum_{i=k+1}^{n} \lambda_i (T_i - u)^2 \qquad \text{when} \quad T_k < u \leq T_{k+1}, \quad k = 0, ..., n-1$$

Thus $g(u)$ is a continuous function in second-order polynomial form in each interval $[T_k, T_{k+1}]$. The maximum smoothness forward curve is a fourth-order polynomial spline function and is second-order continuous differentiable. This is summarized in the following proposition.

Proposition 1 *When $f(0) = r_0$ is known and $f'(0) = 0$, the function of the maximum smoothness forward curve is a fourth-order polynomial spline and is second-order continuously differentiable in the range $(0, T_n)$.*

Unlike Adams and Deventer (1994), proposition 1 shows that the cubic term could not be omitted in the polynomial spline function.

2.3 Algorithm

In this section we introduce how the optimization problem could be transformed to the quadratic form, and then obtain an explicit solution by Lagrange-multiplier method. We set the forward rate function as

$$f(t) = a_k T^4 + b_k t^3 + c_k t^2 + d_k t + e_k \qquad T_{k-1} < t \leq T_k, \quad k = 1, ..., n \tag{18}$$

Then

$$\begin{aligned}\int_{T_{k-1}}^{T_k} f''^2(t)dt &= \int_{T_{k-1}}^{T_k} \left(12a_k t^2 + 6b_k t + 2c_k\right)^2 dt \\ &= \frac{144}{5}\Delta_k^5 a_k^5 a_k^2 + 36\Delta_k^4 a_k b_k + 12\Delta_k^3 b_k^2 + 16\Delta_k^3 a_k c_k + 12\Delta_k^2 b_k c_k + 4\Delta_k^1 c_k^2 \\ &= \mathbf{x}_k^T \mathbf{h}_k \mathbf{x}_k\end{aligned}$$

where

$$\mathbf{x} = \begin{bmatrix} a_k \\ b_k \\ c_k \\ d_k \\ e_k \end{bmatrix}, \qquad \mathbf{h}_k = \begin{bmatrix} \frac{144}{5}\Delta_k^5 & 18\Delta_k^4 & 8\Delta_k^3 & 0 & 0 \\ 18\Delta_k^4 & 12\Delta_k^3 & 6\Delta_k^2 & 0 & 0 \\ 8\Delta_k^3 & 6\Delta_k^2 & 4\Delta_k^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\Delta_k^l = T_k^l - T_{k-1}^l, \quad l = 1, ..., 5.$$

Then the object function becomes

$$\min_{\mathbf{x}} \mathbf{x}^T \mathbf{h} \mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \qquad \mathbf{h} = \begin{bmatrix} \mathbf{h}_1 & & 0 \\ & \ddots & \\ 0 & & \mathbf{h}_n \end{bmatrix}$$

The constraints are as follows

1. Fitting the observed points:

$$\sum_{i=1}^{k} \left(\frac{1}{5}\Delta_i^5 a_i + \frac{1}{4}\Delta_i^4 b_i + \frac{1}{3}\Delta_i^3 c_i + \frac{1}{2}\Delta_i^2 d_i + \Delta_i^1 e_i \right) = y(T_i)T_i, \quad k = 1, ..., n \tag{19}$$

2. Continuity of the spline function at the joints:

$$(a_{k+1} - a_k)T_k^4 + (b_{k+1} - b_k)T_k^3 + (c_{k+1} - c_k)T_k^2 + (d_{k+1} - d_k)T_k$$
$$+(e_{k+1} - e_k) = 0, \quad k = 1, ..., n-1 \tag{20}$$

3. Continuity of the first-order differential of the spline function:

$$4(a_{k+1} - a_k)T_k^3 + 3(b_{k+1} - b_k)T_k^2 + 2(c_{k+1} - c_k)T_k + (d_{k+1} - d_k) = 0,$$
$$k = 1, ..., n-1 \tag{21}$$

4. Continuity of the second-order differential of the spline function:

$$12(a_{k+1}-a_k)T_k^2+6(b_{k+1}-b_k)T_k+2(c_{k+1}-c_k) = 0, \quad k = 1, ..., n-1 \tag{22}$$

5. Boundary conditions:

$$f(0) = r_0 : \quad e_1 = r_0 \tag{23}$$

$$f'(0) = 0 : \quad d_1 = 0 \tag{24}$$

All constraints (**??**) - (**??**) are linear with respect to $\mathbf{x}$. We can write the constraints in matrix form $\mathbf{A}\mathbf{x} = \mathbf{B}$, where $\mathbf{A}$ is a 4n-1 by 5n matrix, and $\mathbf{B}$ is a 4n-1 by 1 vector. Let $\lambda = [\lambda_1, \lambda_2, ..., \lambda_{n-1}]^T$ be the corresponding Lagrange multiplier vector to the constraints. The object function becomes

$$\min_{x,\lambda} Z(\mathbf{x}, \lambda) = \mathbf{x^T h x} + \lambda^{\mathbf{T}}(\mathbf{Ax} - \mathbf{B})$$

If $[\mathbf{x_0}, \lambda_\mathbf{0}]$ is the solution, we should have

$$\frac{\partial}{\partial \mathbf{x}} Z(\mathbf{x}, \lambda)\Big|_{\mathbf{x}_0, \lambda_0} = 2\mathbf{h}\mathbf{x}_0 + \mathbf{A}^T \lambda_0 = 0$$
$$\frac{\partial}{\partial \lambda} Z(\mathbf{x}, \lambda)\Big|_{\mathbf{x}_0, \lambda_0} = \mathbf{A}\mathbf{x}_0 - \mathbf{B} = 0$$

or

$$\begin{bmatrix} 2\mathbf{h} & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{B} \end{bmatrix}.$$

Then

$$\mathbf{x}_0 = \begin{bmatrix} I_{5n \times 5n} & 0_{5n \times (4n-1)} \end{bmatrix} \begin{bmatrix} 2\mathbf{h} & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \mathbf{B} \end{bmatrix}$$

The explicit solution of the parameter vector is obtained.

2.4 Discrete Model

The maximum smoothness forward curve can also be estimated under the discrete framework. The time-to-maturity is divided into intervals with equal length h, and the forward rate of the i^{th} interval is denoted f_i. We use the second-order central difference as the measure of the curvature

$$f''(t) \approx d_2[f_{\bar{t}}] \equiv \frac{f_{\bar{t}+1} - 2f_{\bar{t}} + f_{\bar{t}-1}}{h^2}$$

where $\bar{t} = t/h$. The numerical integral is used instead of ordinary integral

$$\int_0^T f(t)dt = \sum_{i=1}^{\bar{T}} f_i h$$

The maximum smoothness forward curve estimation becomes the following optimization problem

$$\min_{f_i} Z = \sum_{i=1} \bar{T}_n (d_2[f_i])^2 \tag{25}$$

$$s.t. \quad \sum_{i=1}^{\bar{T}_k} f_i h = y_k T_k \quad k = 1, ..., n \tag{26}$$

This problem has the properties that the object function is a convex function with respect to those forward rates, and the constraint set is a convex set. So the optimization has a unique global minimum.

In this discrete model, we note that the maximum smoothness forward curve is estimated without specifying of the underlying function form, so using this model we should get the maximum smoothness forward curve among all function forms.

The optimization problem (**??**) and (**??**) can also be transformed to quadratic form

$$\begin{aligned} &\min_{\mathbf{f}} \mathbf{f^T H f} \\ &s.t. \quad \mathbf{A f} = \mathbf{B} \end{aligned} \tag{27}$$

where

$$\mathbf{H} = \begin{bmatrix} 1 & -2 & 1 & & & & & \\ -2 & 5 & -4 & 1 & & & 0 & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & 1 & -4 & 6 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & 0 & & & 1 & -4 & 5 & -2 \\ & & & & & 1 & 2 & 1 \end{bmatrix}$$

The explicit solution can be obtained by the Lagrange Multiplier method as in the last section.

3 Maximum Flatness Forward Curve

A maximum flatness forward curve is one such that the total slope of the curve is minimized and the corresponding yield curve fits the observed points exactly. Assuming $f(\cdot) \in C^1[0, T_n]$ the general criterion for the total slope is the integral of the squared first-order differential equation

$$Z = \int_0^{T_n} f'^2(u)du$$

Maximum flatness obtains when Z is minimized on all $C^1[0, T_n]$ functions.

Proposition 2 *When $f(0) = r_0$ is known, the function of the maximum flatness forward curve should be a second-order polynomial spline, which is continuously differentiable in the range $(0, T_n)$.*

The proof of Proposition 2 is given in the Appendix.

Delbaen and Lorimier (1992) and Frishing and Yamamura (1996) first discuss the use of the discrete case for estimating maximum flatness forward curve. We use the first-order forward difference

$$d_1[f_i] = \frac{f_{i+1} - f_i}{h}$$

as the approximation of slope. The object function is

$$\min_{f_i} Z = \sum_{i=1}^{\bar{T}_n} (d_1[f_i])^2$$

The constraints are that the observed points of yields should be fitted, which is the same as (**??**).

Both the continuous and discrete approach can be transformed to the quadratic form. Thus the explicit solution for the parameters in the continuous case and forward rates in the discrete case can be obtained, using the Lagrange multiplier method.

4 Illustration

As an illustration of the application of our methods, we apply monthly Treasury rates from January 1996 to December 1998 with maturity of three, six month, one, two, three, five, seven, and ten years that are obtained from the Federal Reserve Bank of Chicago. The forward rates and yield surface estimated by the discrete maximum smoothness method are shown in Figures 1 and 2.

(Figures 1 and 2 about here)

4.1 Comparison between Continuous and Discrete Models

To compare the different approaches, we use the interest rates on Jan $2^n d$ 1997, which is a representative type of term structure. The yields with different maturities are listed in Table 1.

(Table 1 about here)

The term structure of forward rates and yields are estimated using both the maximum smoothness and the maximum flatness method. Since we specify the functional form of the forward rate to be fourth-order or second-order polynomial spline functions in the continuous maximum smoothness or the flatness methods, while there is no need of such specification in the discrete model, it will be interesting to compare the estimations of the continuous and the discrete methods. Figure 3 and 4 show the comparison.

(Figures 3, 4 about here)

It is obvious that the forward rate curves by continuous and discrete approaches are almost the same in both maximum smoothness and maximum flatness methods. The percentage differences between the forward rates of continuous and discrete maximum smoothness method are within 2%, and those of the maximum flatness method are within 1%.

4.2 Comparison between Cubic Spline and Max. Smoothness Method

We compare our maximum smoothness continuous method with the cubic spline method.

(Figures 5, 6 about here)

Figure 5 shows the forward rate surface implied in the term structure of the yields, which is estimated by the cubic spline method. The surface appears rocky because of the dramatic changes in individual forward curves over time. Figure 6 is the comparison of the yields and forward rates between the cubic-spline and maximum smoothness methods. The forward rate curve of the maximum smoothness method is much smoother and more reasonable than that of the cubic spline.

5 Conclusions

This paper introduces methods to estimate the forward rate curve and the corresponding yield curve with maximum smoothness and maximum flatness respectively. We define the integral of the squared second-order derivative of the curve function as a measure of smoothness, and the integral of the squared first-order derivative as a measure of flatness. We also use the sum of the squared second or first order difference of the forward rates as the measure of smoothness or flatness in the discrete approach.

We prove that the maximum smoothness forward curve is a polynomial spline function. We also prove that the maximum flatness forward curve is a second-order polynomial spline function. Furthermore, explicit solutions for the smoothness and flatness approaches, and the continuous and discrete approaches are given. The transformation to quadratic form allows for tractable solutions.

Empirical illustration shows that the forward rate curves by continuous and discrete methods are almost similar, either using maximum smoothness or maximum flatness criterion. We also see that our model performs better than the cubic spline method.

Table 1: Yields of different maturities on Jan. 2^{nd}, 1997

Maturity T	Yield $y(T)$ (%)
3 months	5.17
6 months	5.31
1 year	5.61
2 years	6.01
3 years	6.16
5 years	6.33
7 years	6.47
10 years	6.58

Figure 1: Forward Rate Surface by Discrete Maximum Smoothness Method

We use the monthly US Treasury rates from January of 1996 to December of 1998 with maturity of three, six month, one, two, three, five, seven, and ten years.

Figure 2: The Corresponding Yield Surface by discrete Maximum Smoothness Method

We use the monthly US Treasury rates from January of 1996 to December of 1998 with maturity of three, six month, one, two, three, five, seven, and ten years.

Figure 3: Comparison of Forward Rate Curves between Continuous and Discrete Maximum Smoothness Method

The yields with different maturities on Jan. 2^{nd} 1997 are listed in table 1.

Figure 4: Comparison of Forward curves between Continuous and Discrete Maximum Flatness Method

The yields with different maturities on Jan. 2^{nd} 1997 are listed in table 1.

Figure 5: Forward Rate Surface by Cubic Spline Method

We use the monthly US Treasury rates from January of 1996 to December of 1998 with maturity of three, six month, one, two, three, five, seven, and ten years.

Figure 6: Comparison of Forward Curves between Cubic Spline and Maximum Smoothness Method

The yields with different maturities on Jan. 2^{nd} 1997 are listed in table 1.

Appendix 1 Proof of Lemma 1

Lemma 1 *Given* $A(\cdot) \in C[a,b]$ *and* $B(\cdot)$ *is integrable, then*

$$\int_a^b A(u)h(u)du + \int_a^b B(u)\left(\int_a^u h(v)dv\right)du = 0 \qquad \text{(A-1)}$$

for any $h(\cdot) \in C[a,b]$ *if and only if*

$$A(u) = -\int_a^b B(v)dv \quad \forall u \in [a,b] \qquad \text{(A-2)}$$

proof. (**??**) can be written as

$$\begin{aligned}\int_a^b A(u)h(u)du + \int_a^b \int_a^u B(u)h(v)dvdu \\ &= \int_a^b A(v)h(v)dv + \int_a^b h(v)\left(\int_v^b B(u)du\right)dv \\ &= \int_a^b h(v)\left(A(v) + \int_v^b B(u)du\right)dv = 0\end{aligned}$$

for any continuous function $h(\cdot)$ defined on $[a,b]$. Noticing that

$$A(v) + \int_v^b B(u)du$$

is a continuous function on $[a,b]$, therefore (**??**) obtains if and only if (**??**) holds.

Q.E.D.

Appendix 2 Proof of Proposition 2

The maximum flatness forward curve is a first-order differentiable function f that satisfies the optimization problem

$$\min_{f\in C^1(0,T_n)} \int_0^{T_n} f'^2(u)du \tag{A-3}$$

$$s.t. \quad \int_0^{T_i} f(s)ds = y_i \qquad i = 1, ..., n \tag{A-4}$$

Integrating, we obtain

$$\int_0^T f(u)du = f(T)T - \int_0^T uf'(u)du. \tag{A-5}$$

Denoting $g(t) = f'(t)$ for $0 \le t \le T_n$, we have

$$f(T) = \int_0^T g(u)du + r_0 \tag{A-6}$$

Substituting (**??**) into (**??**) concerning the n known point, the constraints become

$$y_i = T_i\left(\int_0^{T_i} g(u)du + r_0\right) - \int_0^{T_i} ug(u)du \quad i = 1, ..., n \tag{A-7}$$

The optimization problem can be written as

$$\min_{g(\cdot)\in C[0,T_n]} \int_0^{T_k} g^2(u)du$$

$$s.t. \qquad (\textbf{??}).$$

Let λ_i $(i = 1, ..., n)$ be the Lagrange multipliers corresponding to the constraints (**??**), the object function is

$$\begin{aligned}
&\min_{g\in C[0,T_n],\lambda} Z \\
&= \int_0^{T_n} g^2(u)du + \sum_{i=1}^{n} \lambda_i \left(T_i\left(\int_0^{T_i} g(u)du + r_0\right) - \int_0^{T_i} ug(u)du - y_i\right) \\
&= \int_0^{T_n} \left(g^2(u) + \sum_{i=1}^{n} \lambda_i(T_i - u)I_{T_i-u}g(u)\right) du + \sum_{i=1}^{n} \lambda_i(r_0T_i - y_i)
\end{aligned} \tag{A-8}$$

If $[g^*, \lambda]$ is the solution of the optimization problem (**??**), then

$$\frac{\partial}{\partial \sigma} Z[g^* + \sigma h]\bigg|_{\sigma=0} = 0$$

holds for any continuous function $h(\cdot)$ defined on $[0, T_n]$ such that $h(u) = g(u) - g^*(u)$ where $g^*(u)$ is optimal. We have

$$\frac{\partial}{\partial \sigma} Z[g^* + \sigma h]\bigg|_{\sigma=0} = \int_0^{T_n} \left(2g^*(u) + \sum_{i=1}^{n} \lambda_i (T_i - u) I_{T_i - u} \right) h(u) du$$

From Lemma 1 in appendix 1, putting $B(u) \equiv 0$, $[g^*, \lambda_0]$ must satisfy

$$2g^*(u) + \sum_{i=1}^{n} \lambda_i (T_i - u) I_{T_i - u} = 0.$$

Thus

$$g(u) = -\frac{1}{2} \sum_{i=k+1}^{n} \lambda_i (T_i - u) \quad \text{for} \quad T_k < u \leq T_{k+1}$$

Q.E.D.