Minimum Investment Requirements, Financial Market Globalization, and Symmetry Breaking*

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Abstract

We incorporate wealth heterogeneity and the minimum investment requirements in the model of Matsuyama (2004, Econometrica) and provide a complete characterization of symmetry breaking. In particular, we identify the *extensive margin of investment* as a key channel through which the interest rate may respond positively to capital accumulation, or equivalently, the interest rate can be higher in the rich than in the poor countries. Then, financial market globalization may lead to "uphill" capital flows from the poor to the rich countries, which widens the initial cross-country income gap and leads to income divergence among inherently identical countries, a phenomenon that Matsuyama calls symmetry breaking.

Keywords: financial frictions, financial market globalization, minimum investment requirements, symmetry breaking
JEL Classification: E44, F41

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1 Introduction

Matsuyama (2004) shows that, in the presence of fixed investment requirements (FIR, hereafter) and financial frictions, countries which are inherently identical except for the initial income converge to the same, unique, and stable steady state under international financial autarky (IFA, hereafter), while financial market globalization (FMG, hereafter) may lead to "symmetry breaking", i.e., the initially rich (poor) countries may converge to a new steady state with the income higher (lower) than that under IFA. He mentions in subsection 7.1 and 7.2 that symmetry breaking may also arise in the presence of wealth inequality and the minimum investment requirements (MIR, hereafter), while a complete characterization of multiple steady states is *"hopelessly complicated"*.

In this paper, we formally prove Matsuyama's conjecture by providing a complete, analytical characterization of symmetry breaking in a generalized model with wealth heterogeneity and MIR. Meanwhile, we show that, given financial frictions and MIR (or FIR), the *extensive margin of investment*[1] becomes a key channel through which the interest rate may respond *positively* to capital accumulation under IFA, or equivalently, the interest rate can be higher in the rich than in the poor countries under IFA. Thus, FMG may lead to "uphill" capital flows from the poor to the rich countries, which widens the cross-country income gap and leads to symmetry breaking.

The intuition is as follows. Suppose that the world economy consists of a continuum of countries which only differ in the initial income. In each country, some agents have both the technology and the funds for investment, and they are called entrepreneurs; without either the technology or the funds, others have to lend out their net wealth and are called households. If the interest rate is below the marginal rate of return to investment, entrepreneurs prefer to finance the investment projects with external funds. However, they are subject to borrowing constraints and have to put the own funds in the project. The higher the entrepreneurial net wealth, the more they can borrow and invest.

Under IFA, all countries converge to the same, unique steady state, if the production function has the decreasing marginal product of capital (MPK, hereafter). Along the convergence path, capital accumulation raises the individuals' income and net wealth, which affects the credit market and the interest rate.

In the absence of FIR, the higher the individual's net wealth, the more each entrepreneur (household) borrows (lends). This way, capital accumulation affects the credit market only on the *intensive* margin. Meanwhile, the higher aggregate investment reduces the marginal rate of return to investment, which is called the *neoclassical effect*. It reduces the entrepreneurial pledgeable value per unit of investment and dampens the expansion of their credit demand. Thus, the rise in the credit demand is dominated by that in the credit supply so that the interest rate responds negatively to capital accumulation, or equivalently, the interest rate is lower in the rich than in the poor country under IFA. Under FMG, financial capital flows are "downhill" from the rich to the poor, narrowing

[1] Aggregate investment depends on the investment size of individual investors (the intensive margin) as well as the mass of investors (the extensive margin).

the cross-country income gap and inducing countries with different initial incomes to converge to the same steady state. Thus, FMG does not lead to symmetry breaking.

In the presence of FIR, the individual's investment project has the positive output if it reaches a fixed size. The higher individual's net wealth not only reduces (raises) the individual entrepreneur's (household's) credit demand (supply) but also allows more agents to become entrepreneurs with leveraged investment. This way, capital accumulation affects the credit market on the intensive and *extensive* margins. In particular, the aggregate credit demand declines (rises) while the aggregate credit supply rises (declines) on the intensive (extensive) margin. On the credit demand side, the overall expansion in the aggregate credit demand is identical as in the absence of the FIR. On the credit supply side, the intensive-margin effect is the same as in the absence of the FIR, while the extensive-margin effect is new. If the supply-side extensive-margin effect dominates the demand-side neoclassical effect, the rise in the credit supply is dominated by that in the credit demand so that the interest rate responds positively to capital accumulation, or equivalently, the interest rate is higher in the rich than in the poor country under IFA. Under FMG, financial capital flows are "uphill" from the poor to the rich, widening the cross-country income gap and inducing countries with the different initial incomes to converge to the different steady states. Thus, FMG leads to symmetry breaking.

To sum up, due to financial frictions and FIR, the interest rate may respond positively to capital accumulation through the extensive-margin channel, which then causes the "uphill" financial capital flows and symmetry breaking. This mechanism is also at work in the presence of the MIR. Matsuyama (2005, 2007, 2008, 2012, 2013), Kikuchi (2008), Kikuchi and Stachurski (2009), apply the mechanism of symmetry breaking to the issues on credit traps, credit cycles, endogenous fluctuations, inequality, and other implications of credit market imperfections. However, it is not quite clear how to empirically test the theoretical conditions that support this mechanism in these papers. Here, we propose an empirically testable hypothesis, i.e., symmetry breaking is more likely if the real interest rate responses to income changes is positive and sufficiently large. A comprehensive empirical investigation is beyond the scope of this paper and is left for future research.

The rest of the paper is structured as follows. Section 2 sets up the model with financial frictions and MIR. Sections 3 and 4 analyze the allocation under IFA and under FMG, respectively. Section 5 concludes with some final remarks. The appendix compares the results in the models with and without FIR as well as provides the technical proofs.

2 The Model

The world economy consists of a continuum of countries, indexed by $i \in [0, 1]$. Countries are inherently identical except for the initial income level. In each country, a continuum of agents indexed by $j \in [0, 1]$ are born every period and live for two periods, young and old; the population size of each generation is constant at one; agents have the labor endowment when young and consume when old; agent j is endowed with $l_j = \frac{\theta+1}{\theta}\frac{1}{\epsilon_j}$ units of labor, where $\epsilon_j \in (1, \infty)$ follows the Pareto distribution with the cumulative distribution function $G(\epsilon_j) = 1 - \epsilon_j^{-\theta}$ and $\theta > 1$. Agents supply the labor endowment inelastically to the market and the aggregate labor supply is constant at $L = \int_1^\infty l_j dG(\epsilon_j) = 1$.

A final good is internationally tradable and chosen as the numeraire. The final good can be consumed or used to produce capital goods, which becomes available in the next period. Capital goods are non-tradable and can be combined with labor to produce final goods contemporaneously. Capital fully depreciates after the production. The markets for final goods, capital goods, and labor are perfectly competitive. Thus, the productive factors are rewarded with their respective marginal products. There is no uncertainty in the model economy. Y_t^i denotes aggregate output of final goods, $L = 1$ and K_t^i denote the aggregate inputs of labor and capital goods, w_t^i and q_t^i denote the wage rate and the price of capital goods in country i and period t. To sum up,

$$Y_t^i = \left(\frac{K_t^i}{\alpha}\right)^\alpha \left(\frac{L}{1-\alpha}\right)^{1-\alpha}, \qquad \text{where} \qquad \alpha \in (0,1), \tag{1}$$

$$q_t^i K_t^i = \alpha Y_t^i \qquad \text{and} \qquad w_t^i L = (1-\alpha) Y_t^i. \tag{2}$$

Each agent is endowed with one project to produce capital goods subject to the MIR. Consider agent j born in country i and period t. As shown in the left panel of figure 1, the agent can invest $m_{j,t}^i$ units of final goods in period t and produce $k_{j,t+1}^i = Rm_{j,t}^i$ units of capital goods in period $t+1$, if its investment size is no less than a specific value, $m_{j,t}^i \geq \mathbb{m}_t^i$; otherwise, the output is zero.[2] The MIR has the function form $\mathbb{m}_t^i = \mathfrak{m}(Y_t^i)^{1-\sigma}$ with $\mathfrak{m} > 0$. As shown in the right panel of figure 1, the MIR is constant at $\mathfrak{m}$ for $\sigma = 1$, and it is linear in aggregate income for $\sigma = 0$. This function form allows for the possibility that the MIR may differ in the rich and in the poor country.[3]

Agents can save the labor income $n_{j,t}^i = w_t^i l_j$ either by producing capital goods at the marginal rate of return $q_{t+1}^i R$ or lending to the market at the gross interest rate r_t^i. The interest rate cannot exceed the marginal rate of return $r_t^i \leq q_{t+1}^i R$; otherwise, nobody would produce capital goods. Matsuyama (2004) calls it the profitability constraints.

Let us start with the case of $r_t^i < q_{t+1}^i R$. If agent j can meet the MIR, it prefers to finance its investment, $m_{j,t}^i$, with loans. However, due to limited commitment, it can only borrow up to a fraction λ of its investment return in the present value and has to use its own funds as equity capital to cover the gap,

$$b_{j,t}^i \leq \lambda \frac{q_{t+1}^i R m_{j,t}^i}{r_t^i}, \quad \text{and} \quad m_{j,t}^i - b_{j,t}^i \leq n_{j,t}^i, \tag{3}$$

where $\lambda \in (0,1)$ reflects the level of financial development.[4] Let $\psi_{j,t}^i \equiv \frac{m_{j,t}^i - b_{j,t}^i}{m_{j,t}^i}$ denote the agent's equity-investment ratio. In period $t+1$, it gets the investment return, $q_{t+1}^i R m_{j,t}^i$,

[2]Despite the nonconvex individual production set, Matsuyama (2007, 2008) argues that assuming a continuum of agents convexifies the aggregate production set.

[3]We assume the dependence of the MIR on aggregate income purely for the analytical purpose. As shown in section 3, for $\sigma = 0$, capital accumulation does not affect the mass of investors and hence, the aggregate investment responds only on the intensive margin; for $\sigma = 1$, capital accumulation affects the mass of investors and hence, aggregate investment responds on the intensive and extensive margins. By comparing the results in the two settings, we can highlight the role of the extensive-margin channel.

[4]Matsuyama (2008) shows that the strategic default a là Hart and Moore (1994) can give rise to this

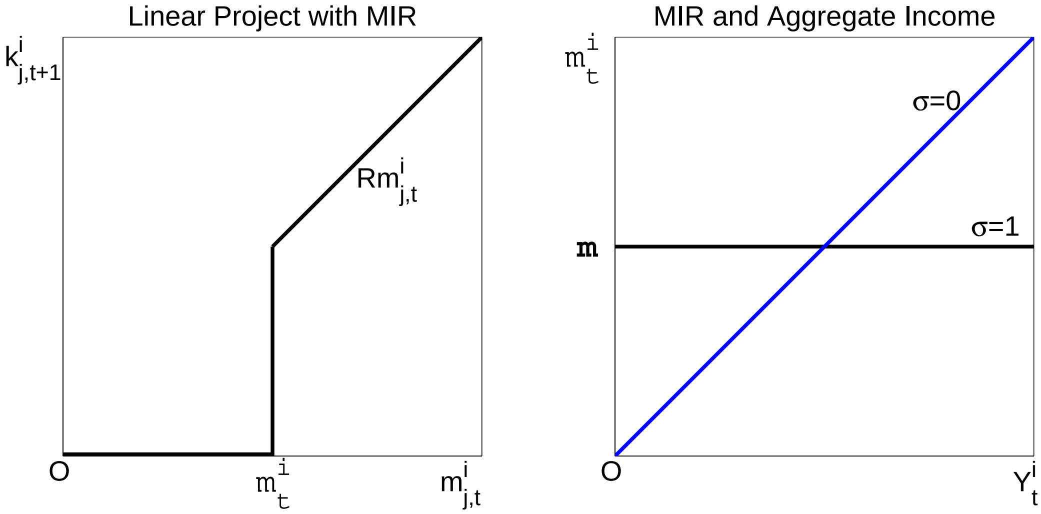

Figure 1: Individual Investment Project, MIR, and Aggregate Income

repays the debt, $r^i_t b^i_{j,t}$, and consumes the rest. The equity rate is defined as the rate of return to its equity capital, $\Omega^i_{j,t} \equiv \frac{q^i_{t+1} R m^i_{j,t} - r^i_t b^i_{j,t}}{m^i_{j,t} - b^i_{j,t}}$. Use the borrowing constraint (3) to get,

$$\psi^i_{j,t} \geq 1 - \lambda \frac{q^i_{t+1} R}{r^i_t}, \tag{4}$$

$$\Omega^i_{j,t} = q^i_{t+1} R + (q^i_{t+1} R - r^i_t)(\frac{1}{\psi^i_{j,t}} - 1). \tag{5}$$

The leverage effect $(q^i_{t+1}R - r^i_t)(\frac{1}{\psi^i_{j,t}} - 1)$ depends positively on the spread $(q^i_{t+1}R - r^i_t)$ and the debt-equity ratio $(\frac{1}{\psi^i_{j,t}} - 1)$. Given the positive spread $q^i_{t+1}R > r^i_t$, the agent maximizes the leverage effect by borrowing to the limit. Thus, the equality sign holds for (4) and $\psi^i_{j,t}$ is independent of agent-j's net wealth. The positive leverage effect, $\Omega^i_t > q^i_{t+1}R > r^i_t$, induces the agent to invest its entire labor income as equity capital, $m^i_{j,t} - b^i_{j,t} = n^i_{j,t}$.

If $r^i_t = q^i_{t+1}R$, the leverage effect vanishes. Then, the agent does not borrow to the limit or invest its entire labor income, i.e., $m^i_{j,t}$ and $\psi^i_{j,t}$ are indeterminate. To sum up,

$$\psi^i_{j,t} \begin{cases} = \psi^i_t \equiv 1 - \lambda \frac{q^i_{t+1}R}{r^i_t}, & \text{wealth-independent} \quad \text{if } r^i_t < q^i_{t+1}R; \\ \geq 1 - \lambda \frac{q^i_{t+1}R}{r^i_t}, & \text{indeterminate,} \quad \text{if } r^i_t = q^i_{t+1}R; \end{cases} \tag{6}$$

$$\Omega^i_{j,t} = \Omega^i_t = \begin{cases} q^i_{t+1}R + (q^i_{t+1}R - r^i_t)(\frac{1}{\psi^i_t} - 1) > q^i_{t+1}R, & \text{if } r^i_t < q^i_{t+1}R; \\ q^i_{t+1}R, & \text{if } r^i_t = q^i_{t+1}R; \end{cases} \tag{7}$$

$$m^i_{j,t} \begin{cases} = \frac{n^i_{j,t}}{\psi^i_t} = \frac{w^i_t}{\psi^i_t}\frac{\theta+1}{\theta\epsilon_j}, & \text{and } \frac{\partial m^i_{j,t}}{\partial \epsilon_j} < 0, \quad \text{if } r^i_t < q^i_{t+1}R; \\ \leq \frac{n^i_{j,t}}{\psi^i_t}, & \text{indeterminate,} \quad \text{if } r^i_t = q^i_{t+1}R. \end{cases} \tag{8}$$

If $r^i_t < q^i_{t+1}R$, there exists a cutoff value $\underline{\epsilon}^i_t$. The agents with $\epsilon_j \in (1, \underline{\epsilon}^i_t]$ can meet the MIR, $m^i_{j,t} = \frac{w^i_t}{\psi^i_t}\frac{\theta+1}{\theta\epsilon_j} \geq \mathtt{m}^i_t$, and are called *entrepreneurs*. Their total mass is $\tau^i_t = 1 - (\underline{\epsilon}^i_t)^{-\theta}$.

form of the borrowing constraints.

The cutoff value is determined by the marginal entrepreneur with $\epsilon_j = \underline{\epsilon}_t^i$ and $m_{j,t}^i = \mathbb{m}_t^i$,

$$\frac{w_t^i}{\psi_t^i}\frac{1+\theta}{\theta\underline{\epsilon}_t^i} = \mathfrak{m}(Y_t^i)^{1-\sigma}, \Rightarrow \underline{\epsilon}_t^i = \frac{(w_t^i)^\sigma}{\psi_t^i \mathbb{F}}, \text{ where } \mathbb{F} \equiv \frac{\theta\mathfrak{m}}{(1-\alpha)^{1-\sigma}(\theta+1)}. \tag{9}$$

When young, entrepreneurs use the entire labor income, $n_{j,t}^i$, and the loan $b_{j,t}^i = n_{j,t}^i(\frac{1}{\psi_t^i} - 1)$ to finance their investment; when old, they consume, $c_{j,t+1}^{i,e}$, and exit from the economy,

$$n_{j,t}^i = w_t^i l_j \quad \text{and} \quad c_{j,t+1}^{i,e} = n_{j,t}^i \Omega_t^i. \tag{10}$$

The agents with $\epsilon_j > \underline{\epsilon}_t^i$ cannot meet the MIR and are called *households*. Their total mass is $1 - \tau_t^i = (\underline{\epsilon}_t^i)^{-\theta}$. When young, households lend out the entire labor income $n_{j,t}^i$; when old, they consume, $c_{j,t+1}^{i,h}$, and exit from the economy,

$$n_{j,t}^i = w_t^i l_j \quad \text{and} \quad c_{j,t+1}^{i,h} = n_{j,t}^i r_t^i. \tag{11}$$

We analyze the economic allocation under two scenarios: (1) IFA where agents are allowed to borrow or lend domestically, (2) FMG where agents are allowed to borrow or lend domestically as well as internationally.[5]

Let M_t^i, D_t^i, and S_t^i denote the aggregate investment, the aggregate credit demand and supply. Under IFA, the markets for capital goods and credit clear domestically,[6]

$$K_{t+1}^i = \int_1^{\underline{\epsilon}_t^i} R m_{j,t}^i dG(\epsilon_j) = RM_t^i, \text{ where } M_t^i \equiv \int_1^{\underline{\epsilon}_t^i} m_{j,t}^i dG(\epsilon_j), \tag{12}$$

$$D_t^i \equiv \int_1^{\underline{\epsilon}_t^i} (m_{j,t}^i - n_{j,t}^i) dG(\epsilon_j), \ S_t^i \equiv \int_{\underline{\epsilon}_t^i}^{\infty} n_{j,t}^i dG(\epsilon_j), \ D_t^i = S_t^i, \Rightarrow M_t^i = w_t^i. \tag{13}$$

If $r_t^i = q_{t+1}^i R$, the agents who can meet the MIR may not invest their entire labor income or borrow to the limit. Despite the indeterminate individual investment size, aggregate saving is fully invested into capital goods in equilibrium, $K_{t+1}^i = RM_t^i = Rw_t^i$.

Definition 1. *Under IFA, a market equilibrium in country i is a set of allocations of agents, $\{n_{j,t}^i, m_{j,t}^i, c_{j,t}^{i,e}, c_{j,t}^{i,h}, \psi_{j,t}^i\}$, and aggregate variables, $\{Y_t^i, K_t^i, M_t^i, q_t^i, w_t^i, r_t^i, \Omega_t^i, \underline{\epsilon}_t^i\}$, satisfying equations* (1)-(2), (6)-(13).

Under FMG, let ϕ_t^i denote the ratio of capital outflows over domestic saving in country i, with negative values indicating the case of capital inflows. The equilibrium conditions are identical as under IFA except for the domestic and world credit market conditions.

$$M_t^i = w_t^i(1 - \phi_t^i), \tag{14}$$

$$\int_0^1 w_t^i \phi_t^i di = 0. \tag{15}$$

Definition 2. *Under FMG a market equilibrium in country i is a set of allocations of agents, $\{n_{j,t}^i, m_{j,t}^i, c_{j,t}^{i,e}, c_{j,t}^{i,h}, \psi_{j,t}^i\}$, and aggregate variables, $\{Y_t^i, K_t^i, M_t^i, q_t^i, w_t^i, \Omega_t^i, \underline{\epsilon}_t^i, \phi_t^i\}$, satisfying equations* (1)-(2), (6)-(12), (14), *the interest rate is equalized across countries $r_t^i = r_t^*$, and the world interest rate r_t^* is determined by equation* (15).

[5]Following Matsuyama (2004), we exclude FDI flows by assumption. von Hagen and Zhang (2014a,b) analyze the joint determination of financial capital flows and FDI flows.

[6]According to the Walras' law, the market for final goods clears in equilibrium.

3 Equilibrium Allocation Under IFA

Without loss of generality, we suppress the country index i for the scenario of IFA. Let X_A denote the steady-state value of variable X_t under IFA. Given the fixed aggregate labor input $L = 1$, equation (2) implies that $w_t = (1-\alpha)Y_t$. Thus, we can use the wage as a proxy for aggregate income in the following analysis.

Combine equation (12)-(13) to get $K_{t+1} = Rw_t$ and the law of motion for wage[7] is

$$w_{t+1} = \frac{(1-\alpha)}{L}Y_{t+1} = \left(\frac{Rw_t}{\rho}\right)^{\alpha}, \tag{16}$$

which is concave and crosses the 45° line once and only one from left with $w_A = \left(\frac{R}{\rho}\right)^{\rho}$.

Proposition 1. *Under IFA, there exists a unique, stable steady state in each country.*

As a collection of autarkic countries, the world economy has a unique, stable steady state under IFA which is symmetric in the sense that, independent of the initial income, all countries end up in the long run with the same income at $Y_A = \frac{w_A}{1-\alpha}$.

Although financial frictions and the MIR do not affect the dynamics and the steady state under IFA, they may fundamentally change the dynamic stability of the world economy under FMG. In the following, we analyze the interest rate response to capital accumulation under IFA, which is critical for us to understand the consequences of FMG.

Interest Rate Response to Capital Accumulation

Iff $r_t < q_{t+1}R$, the borrowing constraints are strictly binding and the interest rate is a function of w_t defined by equations (17)-(18).[8]

$$r_t = \frac{\lambda}{1-\psi_t}q_{t+1}R = \frac{\lambda}{1-\psi_t}w_t^{-(1-\alpha)}R^{\alpha}\rho^{1-\alpha} \tag{17}$$

$$\frac{\psi_t}{(1-\psi_t)^{\frac{1}{1+\theta}}} = \frac{w_t^{\sigma}}{\mathbb{F}}. \tag{18}$$

Iff $r_t = q_{t+1}R$, the borrowing constraints are slack and the interest rate is,

$$r_t = q_{t+1}R = w_t^{-(1-\alpha)}R^{\alpha}\rho^{1-\alpha}. \tag{19}$$

As mentioned in section 2, the zero spread $r_t = q_{t+1}R$ leads to the indeterminate $m_{j,t}$ and $\psi_{j,t}$. For analytical simplicity, we focus on an equilibrium where all entrepreneurs still invest their entire labor income and choose the same ψ_t determined by equation (18).

Define $\Lambda \equiv \frac{\lambda^{\frac{1}{1+\theta}}}{1-\lambda}(1-\alpha)(1+\frac{1}{\theta})$ as a function of $\lambda \in (0,1)$ and $\frac{\partial\Lambda}{\partial\lambda} > 0$.

Lemma 1. *Iff $\psi_t \in (1-\lambda, 1)$ or equivalently $\mathfrak{m} \le Y_t^{\sigma}\Lambda$, the borrowing constraints are slack; iff $\psi_t \in (0, 1-\lambda]$ or equivalently $\mathfrak{m} \ge Y_t^{\sigma}\Lambda$, the borrowing constraints are binding.*

[7]Using the law of motion for wage simplifies our dynamic and steady-state analysis. Alternatively, one can also use the law of motion for capital, but the analysis of FMG becomes more complicated.

[8]See the proof of lemma 1 in appendix B for derivation.

If the borrowing constraints are slack, the interest rate declines in aggregate income, due to the neoclassical effect (the decreasing MPK).

$$\ln r_t = \underbrace{\ln q_{t+1}R}_{\text{neoclassical effect}} = -(1-\alpha)\ln w_t + \ln R^{\alpha}\rho^{1-\alpha}, \text{ and } \frac{\partial \ln r_t}{\partial \ln w_t} = -(1-\alpha) < 0. \quad (20)$$

If the borrowing constraints are binding, the interest rate may rise in aggregate income. We identify the relevant condition by analyzing the credit market equilibrium.

Use equations (3) and (13) to rewrite the aggregate credit demand and supply as $D_t = \frac{\lambda q_{t+1}R}{r_t}M_t = \frac{\lambda q_{t+1}R}{r_t}w_t$ and $S_t = w_t \underline{\epsilon}_t^{-(1+\theta)}$, which are affected by various factors,

$$\ln D_t = \underbrace{\ln w_t}_{\text{net-wealth effect}} + \underbrace{\ln q_{t+1}R}_{\text{neoclassical effect}} + \underbrace{\ln \lambda}_{\text{financial-development effect}} - \underbrace{\ln r_t}_{\text{interest-rate effect}} \quad (21)$$

$$\ln S_t = \ln w_t - (1+\theta)\ln \underline{\epsilon}_t = \underbrace{\ln w_t}_{\text{net-wealth effect}} + \underbrace{\left(1+\frac{1}{\theta}\right)\ln(1-\tau_t)}_{\text{supply-side extensive-margin effect}}. \quad (22)$$

According to equation (21), a rise in the interest rate reduces the present value of the entrepreneurs' pledgeable investment return so that the credit demand curve is downward sloping, $\frac{\partial D_t}{\partial r_t} < 0$; according to equation (22), since households supply their labor income inelastically to the credit market, the credit supply curve is vertical $\frac{\partial S_t}{\partial r_t} = 0$. Besides, the aggregate credit demand and supply are affected by the following factors.

- The net-wealth effect: the higher the aggregate income, the higher the agents' labor income and net wealth, the higher the aggregate credit demand and supply.

- The neoclassical effect: the higher the aggregate investment in period t, the lower the MPK in period $t+1$, the lower the period-t pledgeable value of the individual entrepreneur's investment return, the lower the credit demand.

- The financial-development effect: the higher the level of financial development, the more the individual entrepreneur can borrow, the higher the credit demand.

- The supply-side extensive-margin effect: the larger the mass of households $1-\tau_t$, the higher the aggregate credit supply.

Figure 2 shows the credit market equilibrium under IFA. The downward-sloping credit demand curve D_t and the vertical credit supply curve S_t cross at point E with the equilibrium interest rate at r_t. If aggregate income rises marginally from Y_t to $\tilde{Y}_t$, the aggregate saving rises proportionally from $w_t = (1-\alpha)Y_t$ to $\tilde{w}_t = (1-\alpha)\tilde{Y}_t$.

Combine equations (9) and (13) to get

$$\underline{\epsilon}_t[1-(\underline{\epsilon}_t)^{-(1+\theta)}] = \frac{w_t^{\sigma}}{\mathbb{F}}, \Rightarrow \operatorname{sgn}\left(\frac{\partial \underline{\epsilon}_t}{\partial w_t}\right) = \operatorname{sgn}(\sigma). \quad (23)$$

For $\sigma = 0$, higher Y_t raises the MIR $\mathbb{m}_t = \mathbb{m}Y_t$ and the individual's net wealth $n_{j,t} = l_j w_t = l_j(1-\alpha)Y_t$ in the equal proportions. Thus, according to equation (23), the

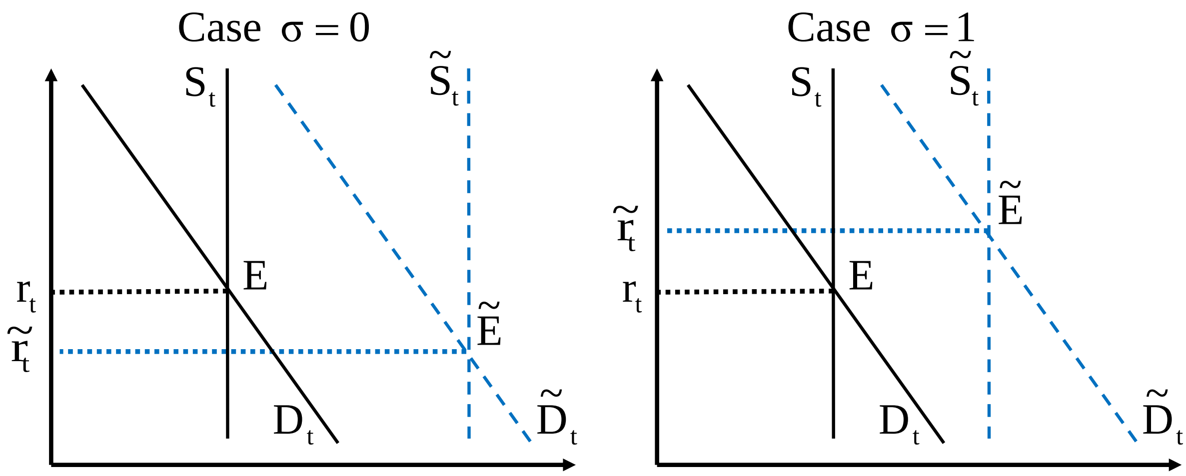

Figure 2: The Credit Market Response to An Increase in Aggregate Income

cutoff value is constant at $\underline{\epsilon}_t = \underline{\epsilon}_A$ and so is the mass of entrepreneurs $\tau_t = \tau_A = 1 - \underline{\epsilon}_A^{-\theta}$. Then, the higher individual's net wealth affects the aggregate credit demand and supply only on the **intensive margin**. According to equations (21) and (22), the net-wealth effect raises the credit supply and demand in the equal proportions, while the neoclassical effect reduces the credit demand. Let $\Delta \ln X_t \equiv \ln \tilde{X}_t - \ln X_t$ denote the percentage change in variable X_t. Combining equations (21) and (22), the net-wealth effect cancels out and the interest rate is driven by the neoclassical effect on the credit-demand side,

$$\begin{aligned} &\Delta \ln D_t = \Delta \ln w_t + \Delta \ln q_{t+1} R - \Delta \ln r_t, \quad \Delta \ln S_t = \Delta \ln w_t, \\ &\Delta \ln D_t = \Delta \ln S_t, \quad \Rightarrow, \quad \Delta \ln r_t = \underbrace{\Delta \ln q_{t+1} R}_{\text{the neoclassical effect (-)}} . \end{aligned} \tag{24}$$

As shown in the left panel of figure 2, the rightward shift of the credit demand curve is dominated by that of the credit supply curve. Then, the credit market equilibrium moves from point E to Ẽ with a lower interest rate $\tilde{r}_t < r_t$.

For $\sigma = 1$, the MIR is constant at $\mathfrak{m}_t = \mathfrak{m}$ and the higher individual's net wealth allows more agents to meet the MIR and invest as entrepreneurs, i.e., $\frac{\partial \underline{\epsilon}_t}{\partial Y_t} > 0$ and $\frac{\partial \tau_t}{\partial Y_t} > 0$, according to equation (23). Then, the aggregate credit demand and supply respond on the **intensive and the extensive margins**. In particular, the decline in the mass of households reduces the credit supply on the extensive margin. Combining equations (21) and (22), the interest rate is affected by two factors,

$$\Delta \ln r_t = \underbrace{\Delta \ln q_{t+1} R}_{\text{neoclassical effect (-)}} - \underbrace{\Delta \left(1 + \frac{1}{\theta}\right) \ln(1 - \tau_t)}_{\text{supply-side extensive-margin effect (-)}} . \tag{25}$$

If the supply-side extensive-margin effect dominates the demand-side neoclassical effect, the rightward shift of the credit supply curve is dominated by that of the credit demand curve. In this case, the credit market equilibrium moves from point E to Ẽ with a higher interest rate, $\tilde{r}_t > r_t$, as shown in the right panel of figure 2.

Lemma 2. *For $\sigma = 1$, ψ_t rises in Y_t under IFA. Given $\lambda \in (0, \tilde{\lambda}_A)$, the interest rate rises in aggregate income, if $\psi_t \in (\tilde{\psi}_A, 1-\lambda)$, where $\tilde{\psi}_A \equiv \frac{1}{\frac{2-\alpha}{1-\alpha} - \frac{1}{1+\theta}}$ and $\tilde{\lambda}_A \equiv 1 - \tilde{\psi}_A$.*
For $\sigma = 0$, the interest rate strictly declines in aggregate income under IFA.

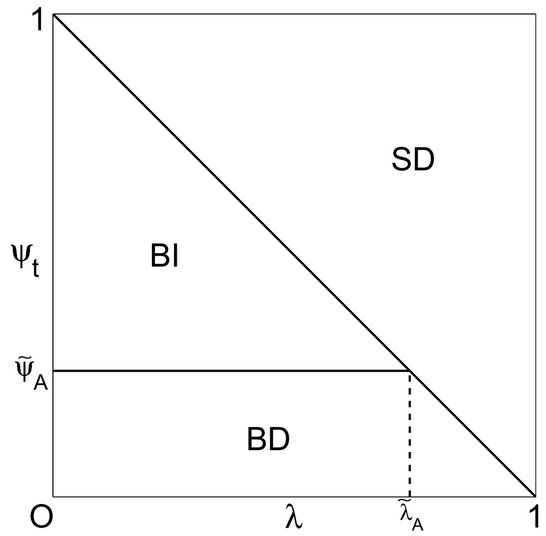

Figure 3: Interest Rate Patterns in the $\{\lambda, \psi_t\}$ Space under IFA

Figure 3 shows the interest rate pattern in the $\{\lambda, \psi_t\}$ space under IFA. According to lemma 1, for (λ, ψ_t) in region SD, the borrowing constraints are ***s***lack and the interest rate, which coincides with the social rate of return, ***d***eclines in aggregate income, due to the neoclassical effect; for (λ, ψ_t) in region BI (BD), the borrowing constraints are ***b***inding and the interest rate ***i***ncreases (***d***eclines) in aggregate income, as the supply-side extensive-margin effect dominates (is dominated by) the neoclassical effect.

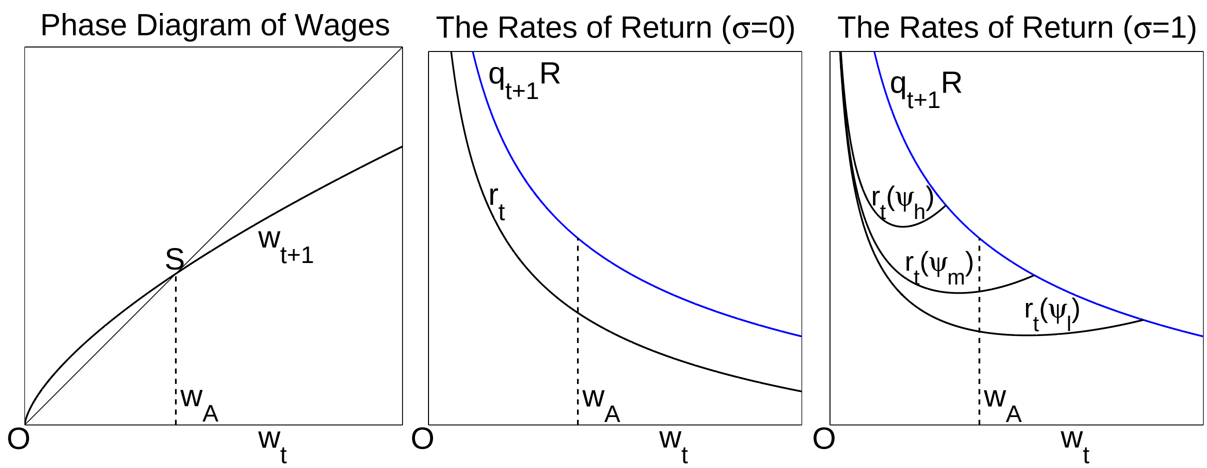

Figure 4: Dynamics of Wage and Interest Rate under IFA

Figure 4 shows proposition 1 and lemma 2 graphically. In the left panel, the law of motion for wage is concave and crosses the 45° line once and only once from the left at point S with the steady-state wage w_A. Thus, financial frictions and the MIR do not affect the uniqueness and stability of the steady state under IFA. In the middle panel, for $\sigma = 0$, the interest rate, which is proportional to the marginal rate of return to investment, declines in w_t, due to the neoclassical effect. In the right panel, for $\sigma = 1$ and $\lambda \in (0, \tilde{\lambda}_A)$, the interest rate is a non-monotonic function of w_t, due to the interactions of the supply-side extensive-margin effect and the neoclassical effect.

Consider the interest rate response to w_t^i around the steady state. Let $Z \equiv \frac{R}{\rho}\left(\frac{1+\theta}{\theta\mathfrak{m}}\right)^{\frac{1}{\rho}}$. Combine equations (16) and (18) to get ψ_A as a function of the parameters $\{R, \mathfrak{m}, \rho, \theta\}$,

$$\frac{\psi_A}{(1-\psi_A)^{\frac{1}{1+\theta}}} = \frac{w_A}{\mathbb{F}} = \left(\frac{R}{\rho}\right)^{\rho} \frac{1+\theta}{\theta\mathfrak{m}} = Z^{\rho}. \tag{26}$$

If the parameter configuration makes $\psi_A = \psi^h \in (1-\lambda, 1)$ or $\psi_A = \psi^l \in (0, \tilde{\psi}_A)$, i.e., in region SD or BD of figure 3, the interest rate declines in w_t around the steady state; if the parameter configuration makes $\psi_A = \psi^m \in (\tilde{\psi}_F, 1-\lambda)$, i.e., in region BI of figure 3, the interest rate rises in w_t around the steady state. See the right panel of figure 4.

As shown in section 4, the positive interest rate response to capital accumulation under IFA is key to Matsuyama's symmetry breaking result.[9]

4 Equilibrium Allocation Under FMG

From period $t = 0$ on, agents in country i are allowed to borrow or lend abroad. As a small open economy, country i takes the world interest rate as given $r_t^i = r^*$. Without loss of generality, we assume that $r^* = r_A$, where $r_A = \frac{\lambda}{1-\psi_A}\rho$ ($r_A = \rho$) if the borrowing constraints are binding (slack) in the autarkic steady state. In this case, the autarkic steady state is still a steady state under FMG, but it may not be stable or unique.

Under FMG, there exists a threshold value $\bar{w}_F$ such that, given $r_t^i = r^*$, for $w_t^i \in (0, \bar{w}_F)$, the borrowing constraints are binding, $\psi_t^i \in (0, 1-\lambda)$, and the aggregate dynamics of country i are characterized by $\{w_t^i, \psi_t^i, \underline{\epsilon}_t^i\}$ satisfying equations (9), (27)-(28),[10]

$$w_{t+1}^i = \left[\frac{R}{\rho} w_t^i \frac{1-(\underline{\epsilon}_t^i)^{-(1+\theta)}}{\psi_t^i}\right]^{\alpha}, \tag{27}$$

$$r_t^i = \frac{\lambda}{1-\psi_t^i} q_{t+1}^i R = \frac{\lambda}{1-\psi_t^i} (w_{t+1}^i)^{-\frac{1}{\rho}} R = r^*. \tag{28}$$

For $w_t^i > \bar{w}_F$, the borrowing constraints are slack and the law of motion for wage is flat at $w_{t+1}^i = \left(\frac{R}{r^*}\right)^{\rho}$. One can solve for $\bar{w}_F$ by putting $\psi_t^i = 1-\lambda$ in equations (9), (27)-(28).

Consider first the case of $\sigma = 0$. Suppose that country i is below the autarkic steady state in period $t = 0$, i.e., $Y_0^i < Y_A$. If it were still under IFA in period $t = 0$, the interest rate would be higher than the world interest rate, $r_t^i > r_A = r^*$, according to lemma 2. FMG leads to capital inflows to country i in period $t = 0$, which raises its domestic

[9] As long as the aggregate production function has the decreasing MPK, i.e., $f'(k) > 0$ and $f''(k) < 0$, where $k \equiv \frac{K}{L}$, the neoclassical effect and the supply-side extensive margin effect exist in the presence of the MIR and financial frictions. Thus, our assumption of the Cobb-Douglas production function is not essential for the positive interest rate response to capital accumulation.

[10] See the proof of Proposition 2 in appendix B for the derivation.

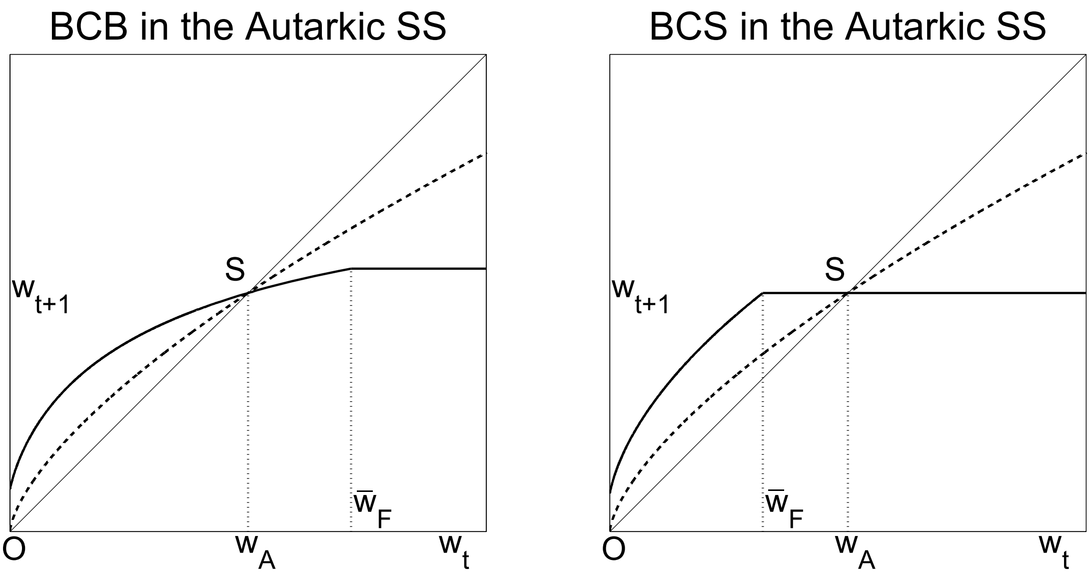

Figure 5: Phase Diagrams of Wages under FMG versus under IFA: $\sigma = 0$

investment. By the same logic, if $Y_0^i > Y_A$, FMG leads to capital outflows from country i, which reduces its domestic investment. In both circumstances, capital flows make the law of motion for wage flatter around the autarkic steady state than under IFA, which speeds up the convergence. The solid (dashed) curve in figure 5 shows the law of motion for wage under FMG and under IFA, respectively. The left (right) panel shows the case where the **b**orrowing **c**onstraints are **b**inding (**s**lack) in the autarkic steady state.

Proposition 2. *For $\sigma = 0$, the autarkic steady state is still the unique, stable steady state under FMG. For $\sigma = 1$, FMG may lead to multiple steady states if $\lambda \in (0, \hat{\lambda}_F)$, where $\hat{\lambda}_F \equiv \frac{\alpha + \frac{1-\alpha}{1+\theta} + \sqrt{(2-\alpha-\frac{1-\alpha}{1+\theta})^2 - 4(1-\alpha)}}{2}$.*

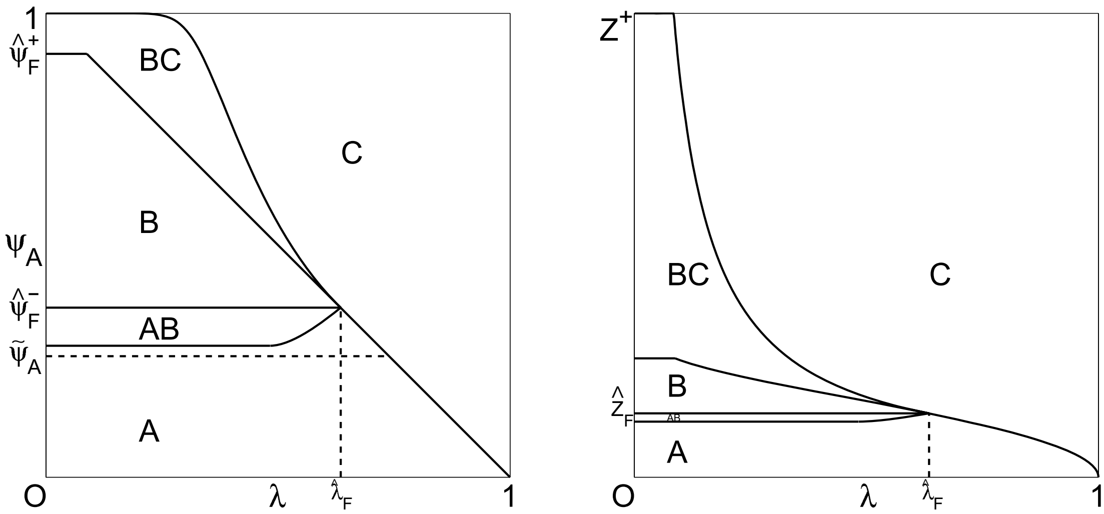

Figure 6: Parameter Configuration for Symmetry Breaking under FMG: $\sigma = 1$

Consider the case of $\sigma = 1$. Figure 6 shows the parameter configuration for multiple steady states under FMG in the (λ, ψ_A) space and in the $\{\lambda, Z\}$ space, respectively.[11] The dashed line in the left panel shows the threshold value $\tilde{\psi}_A$ defined in lemma 2. The solid (dash) curves in figure 7 show the laws of motion for wage under FMG (IFA), with the parameter configuration in the five regions of figure 6, respectively.

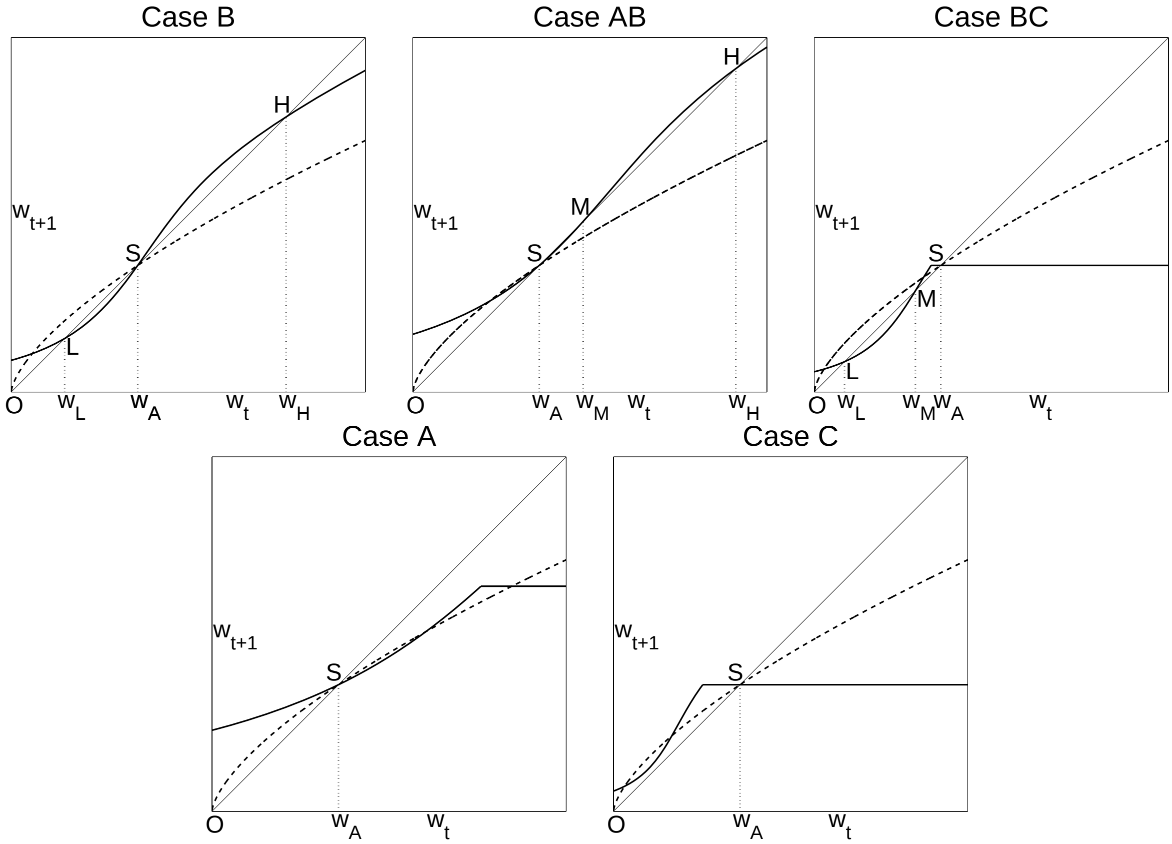

Figure 7: Phase Diagrams of Wage under FMG versus under IFA: $\sigma = 1$

Consider the parameter configuration in region B of figure 6. As shown in the upper-left panel of figure 7, if the country's initial income is higher (lower) than in the autarkic steady state $w_0^i > w_A$ $(w_0^i < w_A)$, it converge to a new stable steady state H (L) with $w_H^i > w_A$ $(w_L^i < w_A)$ under FMG. The intuition is as follows.

[11] At first sight, it seems wrong to claim that the left panel shows the parameter configuration, because ψ_A on the vertical axis is not a parameter. In fact, as the steady-state value of an endogenous variable, ψ_A is defined by equation (26) as an implicit function of the parameters $\{R, \mathfrak{m}, \rho, \theta\}$ and reflects the changes in these parameters. One can use equation (26) to map the diagram from the $\{\lambda, \psi_A\}$ space (the left panel) to the $\{\lambda, Z\}$ space (the right panel). As both λ and ψ_A can be measured empirically, the diagram in the $\{\lambda, \psi_A\}$ space can be interpreted more meaningfully than in the $\{\lambda, Z\}$ space.
Matsuyama (2004) normalizes the FIR at unity and shows in figure 5 the parameter configuration in the $\{\lambda, R\}$ space. Then, he analyzes the impact of the productivity, R, on symmetry breaking. In our model, we introduce $\mathfrak{m}$ as a free parameter for the MIR. Technically, it is $\frac{R^\rho}{\mathfrak{m}}$ that matters for symmetry breaking. If the combination of R and $\mathfrak{m}$ gives the same value of Z, ψ_A does not change and neither does the parameter configuration in the $\{\lambda, \psi_A\}$ and $\{\lambda, Z\}$ spaces. One can map the diagram from the $\{\lambda, Z\}$ space to the $\{\lambda, R\}$ space or to the $\{\lambda, \mathfrak{m}\}$ space to analyze the impacts of R or $\mathfrak{m}$.
According to appendix A and the proofs of propositions 2 and 4, the model with the FIR in Matsuyama (2004) is a limiting case of our model with $\sigma = 1$ and $\theta \to \infty$. In particular, figure 10 is a limiting case of figure 6 with $\theta \to \infty$.

For the parameter configuration $\{\lambda, \psi_A\}$ in region BI of figure 3, the interest rate responds positively to capital accumulation in the steady state under IFA, which dampens the expansion of the entrepreneurial borrowing capacity. By decoupling the interest rate from the domestic credit market condition $r_t^i = r^*$, FMG eliminates this dampening effect and allows more agents to meet the MIR and invest as entrepreneurs, which amplifies the response of domestic investment on the extensive margin to capital accumulation. For the parameter configuration in region B of figure 6, the FMG-driven amplification effect dominates the neoclassical effect so that the slope of the law of motion for wage exceeds unity and the autarkic steady state is unstable. Here, *the positive interest rate response to capital accumulation under IFA is key to the multiple steady states under FMG.*

Starting from region B of figure 6, let us reduce $\mathfrak{m}$ so that ψ_A rises[12] and the parameter configuration moves upwards into region BC where the borrowing constraints are slack in the autarkic steady state. The autarkic interest rate coincides with the marginal rate of return to investment and declines in aggregate income, $r_t^i = q_{t+1}^i R$, due to the neoclassical effect. Given $r^* = r_A = \rho$, FMG makes the law of motion for wage flat at the autarkic steady state with $w_{t+1}^i = \left(\frac{R}{r^*}\right)^\rho = \left(\frac{R}{\rho}\right)^\rho = w_A$ and hence, the autarkic steady state is locally stable. However, for $w_t^i \ll w_A$, ψ_t^i enters into region BI of figure 3 where the interest rate responds positively to income change. Then, FMG affects domestic investment in the same way as in case B. The upper-right panel of figure 7 shows that, besides the stable autarkic steady state S, there are another stable steady state L and an unstable steady state M with $w_L < w_M < w_A$.

Starting from region B of figure 6, let us raise $\mathfrak{m}$ so that ψ_A declines and the parameter configuration moves downwards into region AB where the borrowing constraints are binding in the autarkic steady state. In region AB, the interest rate response to income change is slightly positive around the autarkic steady state. Thus, the FMG-driven amplification effect is dominated by the neoclassical effect so that the autarkic steady state is still stable. However, for $w_t^i \gg w_A$, ψ_t^i enters into region B of figure 6 where the interest rate response to income change is strongly positive. Then, FMG affects domestic investment in the same way as in case B. The upper-middle panel of figure 7 shows that, besides the stable autarkic steady state S, there are another stable steady state H and an unstable steady state M with $w_H > w_M > w_A$.

For the parameter configuration in region A and C of figure 6, FMG does not generate multiple steady states. See the lower panels of figure 7.

To sum up, although the initial income level does not matter for the economic dynamics and the steady state under IFA, it does matter under FMG. In case B, starting with the income level *slightly* higher (lower) than that in the autarkic steady state, a small open economy converges to a new, stable steady state with the income much higher (lower) than in the autarkic steady state; in case AB (BC), starting with an income *sufficiently* higher (lower) than that in the autarkic steady state, a small open economy converges to a new, stable steady state with the income much higher (lower) than in the

[12] According to equation (26), ψ_A declines in $\mathfrak{m}$.

autarkic steady state. This way, FMG amplifies the initial cross-country income gap.

Incorporate this mechanism into a world economy where countries are inherently identical except for the initial income level. For the parameter configuration in region B, the world economy has a unique, symmetric, stable steady state under IFA where all countries have the same income in the long run, independent of the initial income level; FMG inevitably destabilizes this symmetric, autarkic steady state and leads to the asymmetric, stable steady state where the initially rich (poor) countries have the income higher (lower) than in the autarkic steady state.

Comparison with Matsuyama (2004)

Matsuyama (2004) assumes that agents have the same labor endowment and the same indivisible investment project with the FIR. In the presence of financial frictions, aggregate investment adjusts only on the extensive margin and the agents who can borrow and invest are randomly determined by lottery. We extend Matsuyama's model in two aspects. First, we replace his assumption of the FIR with the MIR so that, in the presence of financial frictions, aggregate investment adjusts on the intensive and the extensive margins; second, we introduce the heterogeneous labor endowment so that the individual's net wealth becomes the criterion for allocating the loans.

Matsuyama (2004) shows that in the case of symmetry breaking, the borrowing constraints must be slack in the rich country under FMG, and hence, the equity premium is always zero there, $\Omega_t^i - r_t^i = 0$. As shown in the proof of proposition 2, for $\sigma = 1$, the law of motion for wage under FMG may consist of three subfunctions, i.e., a convex part, a concave part and a flat part.[13] Thus, in the case of symmetry breaking, the borrowing constraints can be binding in the rich country under FMG, and hence, the equity premium can be positive $\Omega_t^i - r_t^i > 0$, in the rich country, but smaller than that in the poor country. Compare the upper panels of figure 7 with those in figure 11. Thus, one may test the tightness of the borrowing constraints across countries by empirically estimating the spread between the equity rate and the interest rate.

For $\mathfrak{m} = 1$ and $\sigma = 1$, the MIR is constant at one; for $\theta \to \infty$, the labor endowment distribution degenerates into a unit mass at $l_j = 1$ so that agents have the same labor income. In this limiting case, our model is analytically identical as that of Matsuyama (2004) and the right panel of figure 6 is the same as figure 5 of Matsuyama (2004).[14]

5 Final Remark

This paper highlights the extensive margin of investment as a key channel through which the interest rate may respond positively to capital accumulation, given financial frictions

[13] One can prove that, if w_t^i is in the flat part, the borrowing constraints are slack; otherwise, the borrowing constraints are binding.

[14] For the comparison purpose, we replicate the results of Matsuyama (2004) in appendix A. Note that figure 5 of Matsuyama (2004) has a technical error. See the appendix for further discussion.

and the MIR. Then, FMG may lead to "uphill" capital flows from the poor to the rich countries, which widens the initial cross-country income gap and leads to income divergence. The model developed in this paper can be applied to other related issues where even very small exogenous heterogeneities may lead to large heterogeneities in endogenous variables. Zhang (2014) introduces the two-sector Heckscher-Ohlin feature into the current setting and shows that trade integration may also lead to symmetry breaking and affect the direction of financial capital flows. By decomposing the project investment into the tangible and intangible parts, Zhang (2013a) shows that FMG may amplify the cross-country differences in investment tangibility and productivity.

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Appendix

A FIR and Symmetry Breaking: Matsuyama (2004)

In this section, we replicate the results of Matsuyama (2004) and highlight the role of the extensive-margin channel by comparing the settings with and without the FIR. We also prove that the setting with the FIR is analytically a limiting case of the generalized model with $\sigma = 1$ and $\theta \to \infty$. See Zhang (2013b) for the more detailed analysis.

A.1 Model Settings with and without the FIR

The model setting differs from the generalized model in sector 2 only in two aspects:

- each agent has one unit of labor endowment, and
- the individual projects are subject to the FIR.

For the comparison purpose, we also introduce a model setting with no FIR.

In the first setting, a fraction $\tau \in (0, 1)$ of agents who are born in period t and country i have the technology to convert m_t^i units final goods in period t to $k_{t+1}^i = Rm_t^i$ units of capital goods in period $t+1$, and they are called entrepreneurs. Without the technology, other agents can only lend their labor income and are called households. The mass of entrepreneurs τ is exogenous, while the individual investment size m_t^i is endogenous. Then, aggregate investment takes place on the *intensive margin*, $K_{t+1}^i = \tau Rm_t^i$. With **n**o investment size requirements, it is called setting **N**.

In the second setting, each agent is endowed with an indivisible project to transform $\mathfrak{m}$ units of final goods in period t into $R\mathfrak{m}$ units of capital goods in period $t + 1$.[15] If $w_t^i < \mathfrak{m}$, an agent must borrow $\mathfrak{m} - w_t^i$ to start its project and the aggregate saving is not sufficient to allow all agents to start the projects. According to Matsuyama (2004), random credit rationing allows a fraction $\tau_t^i \in (0, 1)$ of agents to get the loan $\mathfrak{m} - w_t^i$ to start the projects and they are called entrepreneurs, while other agents can only lend the labor income and they are called households. Different from setting **N**, the individual investment size $\mathfrak{m}$ is exogenous, while the mass of entrepreneurs τ_t^i is endogenous.[16] Thus, aggregate investment takes place only on the *extensive margin*, $K_{t+1}^i = \tau_t^i R\mathfrak{m}$. With **f**ixed investment size requirements at the individual level, it is called setting **F**.

Figure 8 shows the individual investment function in the two settings. In setting **N**, the individual project is linear, $k_{t+1}^i = Rm_t^i$. In setting **F**, the project output is zero for the input $m_t^i \in [0, \mathfrak{m})$ and it is constant at $R\mathfrak{m}$ for the input $m_t^i \geq \mathfrak{m}$. For simplicity, we use τ_t^i and m_t^i to denote the mass of entrepreneurs and the individual investment size in the model description. Setting **N** is characterized by the fixed mass of entrepreneurs, $\tau_t^i = \tau$, while setting **F** is characterized by the fixed project size, $m_t^i = \mathfrak{m}$.

[15] Matsuyama (2004) implicitly normalizes the individual project size at $\mathfrak{m} = 1$, while we allow $\mathfrak{m}$ to be a free parameter and analyze its impacts on symmetry breaking.

[16] Although the FIR results in the non-convexity of the individual production set, Matsuyama (2007, 2008) argues that assuming a continuum of homogeneous agents convexifies the production set.

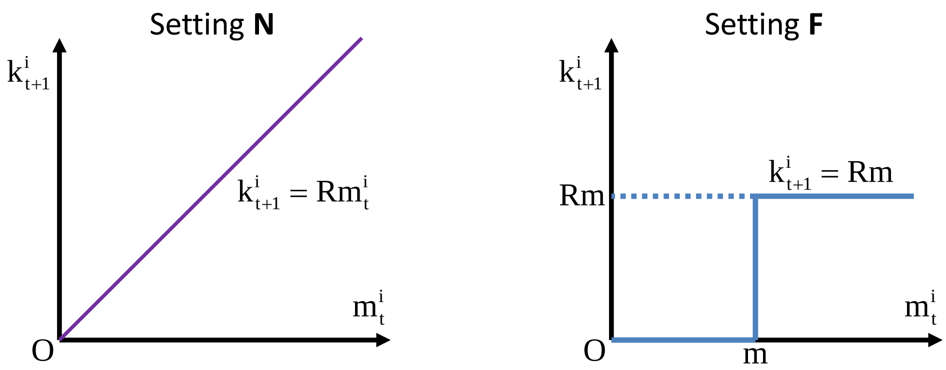

Figure 8: Individual Projects in the Two Settings

Entrepreneurs are subject to the borrowing constraints (3). Since households are homogeneous and so are entrepreneurs, we drop the subscript j.

Under IFA, the markets for capital goods and credit clear each period.

$$K^i_{t+1} = \tau^i_t R m^i_t, \tag{29}$$

$$\tau^i_t(m^i_t - w^i_t) = (1 - \tau^i_t)w^i_t. \tag{30}$$

Definition 3. *Under IFA, a market equilibrium in country i is a set of allocations of agents, $\{m^i_t, c^{i,e}_t, c^{i,h}_t, \psi^i_t\}$, and aggregate variables, $\{Y^i_t, K^i_t, q^i_t, w^i_t, r^i_t, \Omega^i_t, \tau^i_t\}$, satisfying equations* (1)*-*(2)*,* (6)*-*(8)*,* (10)*-*(11)*,* (29)*-*(30)*.*
*$\tau^i_t = \tau$ is exogenous in setting **N**, while $m^i_t = \mathfrak{m}$ is exogenous in setting **F**.*

Under FMG, the equilibrium conditions are identical as under IFA except for the domestic and world credit market conditions.

$$\tau^i_t(m^i_t - w^i_t) = (1 - \tau^i_t)w^i_t - \phi^i_t w^i_t, \tag{31}$$

$$\int_0^1 w^i_t \phi^i_t di = 0. \tag{32}$$

Definition 4. *Under FMG, a market equilibrium in country i is a set of allocations of agents, $\{m^i_t, c^{i,e}_t, c^{i,h}_t, \psi^i_t\}$, and aggregate variables, $\{Y^i_t, K^i_t, q^i_t, w^i_t, \Omega^i_t, \tau^i_t, \phi^i_t\}$, satisfying equations* (1)*-*(2)*,* (6)*-*(8)*,* (10)*-*(11)*,* (29)*, and* (31)*, and the interest rate is equalized across countries $r^i_t = r^*_t$ and the world interest rate r^*_t is determined by equation* (32)*.*
*$\tau^i_t = \tau$ is exogenous in setting **N**, while $m^i_t = \mathfrak{m}$ is exogenous in setting **F**.*

A.2 Equilibrium Allocation under IFA

For simplicity, we suppress the country index i for the scenario of IFA.

In setting **N**, according to equations (29) and (30), the equity-investment ratio is constant at $\psi_t = \frac{w_t}{m_t} = \tau$ and domestic investment is fully financed by domestic saving $K_{t+1} = R\tau m_t = R w_t$. The law of motion for wage is characterized by equation (16).

In setting **F**, for $w_t < \mathfrak{m}$, aggregate saving is too low to allow all agents to run their projects. According to equations (29) and (30), the mass of entrepreneurs and the equity-investment ratio are endogenous, $\tau_t = \psi_t = \frac{w_t}{\mathfrak{m}} < 1$, and domestic investment is fully financed by domestic saving $K_{t+1} = R\tau_t m = Rw_t$. Then, the law of motion for wage is the same as in setting **N**. For $w_t \geq \mathfrak{m}$, all agents self-finance their projects, $\tau_t = \psi_t = 1$. Given the fixed project size, the aggregate output of capital goods is constant at $K_{t+1} = R\mathfrak{m}$ and the law of motion for wage is flat at $w_{t+1} = \left(\frac{R\mathfrak{m}}{\rho}\right)^{\alpha}$.

Proposition 3. *In setting **N**, $\psi_t = \tau$; in setting **F**, $\psi_t = \tau_t = \frac{w_t}{\mathfrak{m}}$.*
In both settings, there exists a unique, stable steady state in each country under IFA;
iff $\psi_t \in (0, 1 - \lambda]$, the borrowing constraints are binding;
iff $\psi_t \in (1 - \lambda, 1]$, the borrowing constraints are slack.

Proposition 3 is essentially the same as proposition 1 and lemma 1.

Interest Rate Response to Capital Accumulation

If the borrowing constraints are slack, the interest rate is equal to the marginal rate of return to investment and declines in aggregate income. See equation (20).

Let us then consider the case of the binding borrowing constraints.

In setting **N**, combine equations (6), (16), (29)-(30) to get

$$r_t = \frac{\lambda}{1 - \psi_t} q_{t+1} R = \frac{\lambda}{1 - \tau} w_t^{-(1-\alpha)} R^{\alpha} \rho^{1-\alpha}. \tag{33}$$

Higher aggregate income raises the agents' labor income. With the extensive margin mute, entrepreneurs (households) to borrow (lend) more. Due to the neoclassical effect, the interest rate declines in aggregate income, as in the generalized model with $\sigma = 0$.

In setting **F**, combine equations (6), (16), (29)-(30) to get

$$r_t = \frac{\lambda}{1 - \psi_t} q_{t+1} R = \frac{\lambda}{1 - \tau_t} w_t^{-(1-\alpha)} R^{\alpha} \rho^{1-\alpha}. \tag{34}$$

Higher aggregate income raises the agents' labor income, which allows more agents to get the required loans and invest as entrepreneurs $\tau_t = \frac{w_t}{\mathfrak{m}}$. Then, the decline in the mass of households $(1 - \tau_t)$ reduces the aggregate credit supply on the extensive margin. If the supply-side extensive-margin effect dominates the neoclassical effect, the interest rate rises in aggregate income, as in the generalized model with $\sigma = 1$.

Lemma 3. *In setting **N**, the interest rate declines in aggregate income under IFA.*
*In setting **F**, ψ_t rises in Y_t under IFA. Given $\lambda \in (0, \tilde{\lambda}_A)$, the interest rate rises in aggregate income, if $\psi_t \in (\tilde{\psi}_A, 1 - \lambda)$, where $\tilde{\psi}_A \equiv \frac{1-\alpha}{2-\alpha}$ and $\tilde{\lambda}_A \equiv 1 - \tilde{\psi}_A$.*

Lemma 3 is essentially the limiting case of lemma 2 with $\theta \to \infty$.

Figure 9 shows the interest rate pattern in the $\{\lambda, \psi_t\}$ space in setting **F**, which is the limiting case of figure 3 with $\theta \to \infty$. The analysis follows that in section 3.

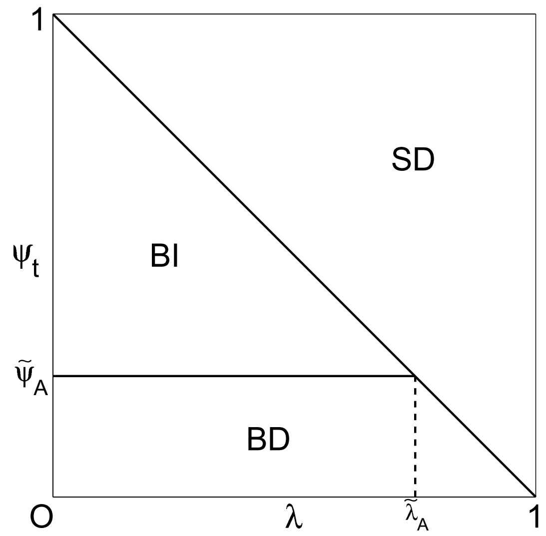

Figure 9: Interest Rate Patterns in the $\{\lambda, \psi_t\}$ Space under IFA

A.3 Equilibrium Allocation under FMG

From period $t = 0$ on, agents in country i are allowed to borrow or lend abroad. As a small open economy, country i takes the world interest rate as given $r_t^i = r^*$. Without loss of generality, we assume that $r^* = r_A$, where $r_A = \frac{\lambda}{1-\psi_A}\rho$ ($r_A = \rho$) if the borrowing constraints are binding (slack) in the autarkic steady state, with $\psi_A = \tau$ ($\psi_A = \frac{w_A}{\mathfrak{m}}$) in setting **N** (**F**).

In setting **N**, given the negative interest rate response to income change under IFA, FMG leads to financial capital inflows (outflows), if $Y_0^i < Y_A$ ($Y_0^i > Y_A$), which dampens the response of aggregate investment to income changes. The FMG-driven dampening effect reinforces the neoclassical effect, which ensures the uniqueness and the stability of the steady state under FMG, as in the generalized model with $\sigma = 0$.

In setting **F**, as the interest rate may respond positively to income change under IFA and FMG may amplify the response of aggregate investment to income changes. If the FMG-driven amplifying effect dominates the neoclassical effect, multiple steady states may arise, as in the generalized model with $\sigma = 1$.

Proposition 4. *Under FMG, the autarkic steady state is still the unique, stable steady state in setting* ***N****, while multiple steady states may arise in setting* ***F*** *if* $\lambda \in (0, \hat{\lambda}_F)$*, where* $\hat{\lambda}_F \equiv \alpha$*.*

Proposition 4 is essentially the limiting case of proposition 2 with $\theta \to \infty$.

Define an auxiliary parameter $Z \equiv \frac{R}{\rho}\left(\frac{1}{\mathfrak{m}}\right)^{\frac{1}{\rho}}$. Under IFA, the steady-state value of ψ_t^i is a function of the parameters $\{R, \mathfrak{m}, \rho\}$

$$\psi_A = \frac{w_A}{\mathfrak{m}} = \left(\frac{R}{\rho}\right)^{\rho} \frac{1}{\mathfrak{m}} = Z^{\rho}. \tag{35}$$

Figure 10 shows the parameter configuration for multiple steady states under FMG in the (λ, ψ_A) space and in the $\{\lambda, Z\}$ space, respectively. The dashed line in the left panel shows the threshold value $\tilde{\psi}_A$ defined in lemma 3. As shown in the proof of propositions 2 and 4, figure 10 is the limiting case of figure 6 with $\theta \to \infty$.

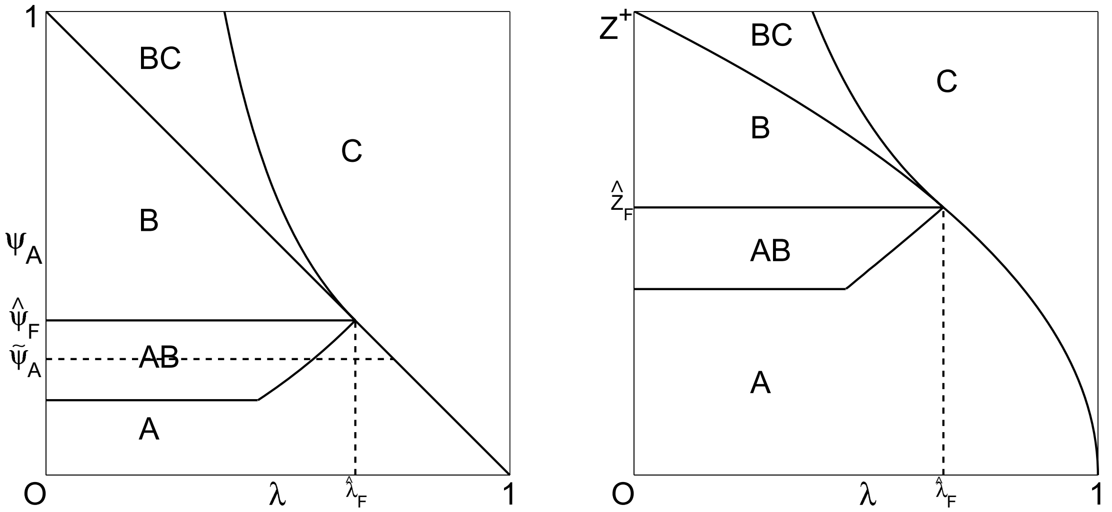

Figure 10: Parameter Configuration for Symmetry Breaking under FMG: Setting **F**

The right panel of figure 10 is almost identical as figure 5 of Matsuyama (2004) except for the boundary between region AB and A. By definition, the mass of entrepreneurs cannot exceed the total mass of population in each generation, $\tau_t^i \leq 1$. Taking that into account, the boundary between region AB and A is characterized by a piecewise function with two subfunctions.[17] This result is confirmed in the generalized model with $\sigma = 1$ in section 4. Thus, there is a technical error in figure 5 of Matsuyama (2004).

The solid (dash) curves in figure 11 show the laws of motion for wage under FMG (IFA) in setting **F**, with the parameter configuration in the five regions of figure 10, respectively. The analysis follows that in section 4.

As shown in the proof of proposition 4, the law of motion for wage under FMG consists of the convex part for $w_t^i \in (0, \bar{w}_F)$ and the flat part for $w_t^i > \bar{w}_F$.[18] Thus, in the case of symmetry breaking, if the country ends up in the stable steady state with the higher (lower) income level, the borrowing constraints must be slack (binding). For example, as shown in the upper-left panel of figure 11, the law of motion for wage in case B is flat at point H and upward-sloping at point L, where $w_H > w_L$.

To sum up, the model with the FIR in Matsuyama (2004) can be regarded as the limiting case of the generalized model with $\sigma = 1$ and $\theta \to \infty$.

[17] See the proof of proposition 4 for the analytical characterization of the two subfunctions.

[18] One can prove that, if w_t^i is in the flat part, the borrowing constraints are slack; otherwise, the borrowing constraints are binding.

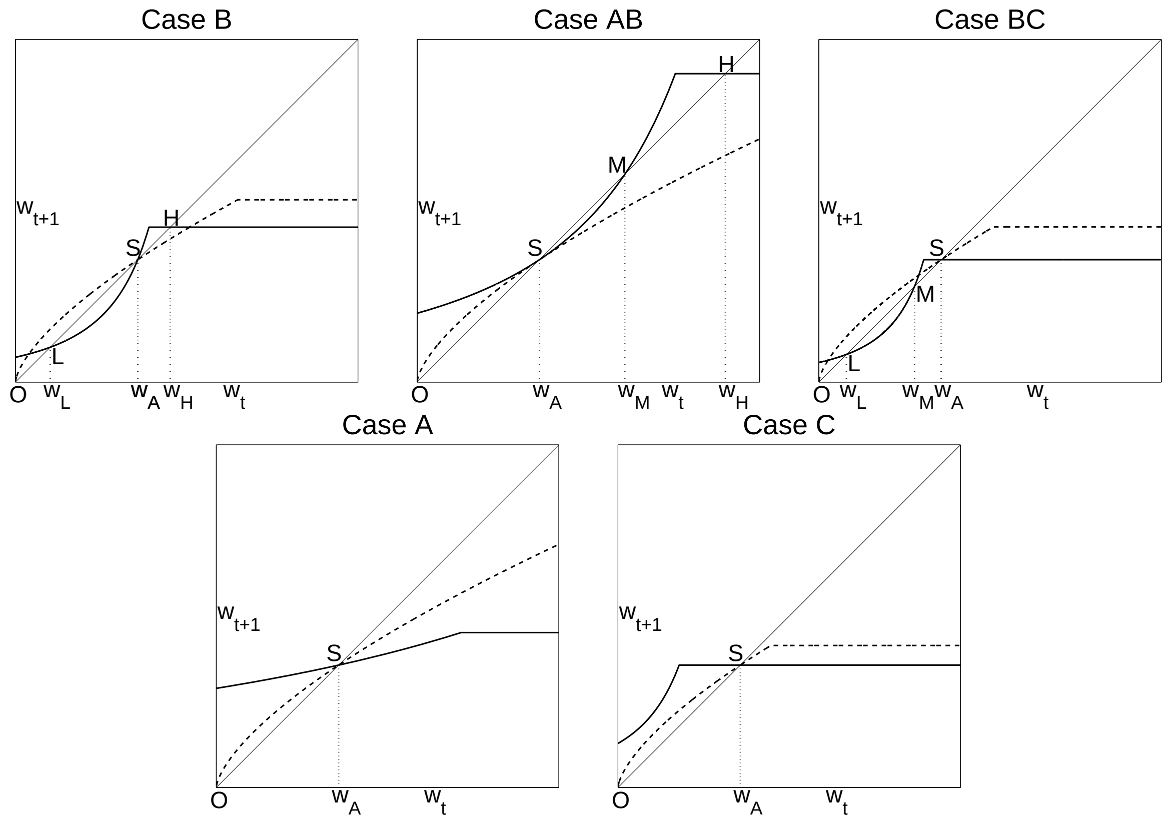

Figure 11: Phase Diagrams of Wage under FMG versus under IFA: Setting **F**

B Proofs

Proof of Proposition 1

Proof. According to equation (16), the law of motion for wage is log-linear, $\ln w_{t+1} = \alpha \ln w_t + \alpha \ln \frac{R}{\rho}$, with a slope less than unity, $\alpha < 1$. Thus, there exists a unique and stable steady state. $\square$

Proof of Lemma 1

Proof. The proof consists of two steps.

Step 1: solution to the interest rate with the binding borrowing constraints

Combine equations (1)-(2) to get the factor price equation,

$$q_t^{\alpha} w_t^{1-\alpha} = 1. \tag{36}$$

Rewrite equation (6) as $r_t = \frac{\lambda}{1-\psi_t} q_{t+1} R$. Combine it with equations (16) and (36) to get equation (17).

According to the credit market clearing equation,

$$D_t = w_t(\frac{1}{\psi_t} - 1)[1 - \epsilon_t^{-(1+\theta)}], \;\; S_t = w_t \epsilon_t^{-(1+\theta)}, \;\; D_t = S_t, \;\; \Rightarrow \;\; 1 - \epsilon_t^{-(1+\theta)} = \psi_t. \tag{37}$$

Combine equations (9) and (37) to get equation (18).

Thus, one can use equations (17)-(18) to solve ψ_t and r_t as the functions of w_t.

Step 2: condition for the binding borrowing constraints

Use equation (17) to rewrite the condition for the binding borrowing constraints from $r_t \leq q_{t+1}R$ into $\psi_t \leq 1-\lambda$.

Combine $w_t = (1-\alpha)Y_t$ with equation (18) to get ψ_t as a function of Y_t,

$$\sigma \ln Y_t = \ln \psi_t - \frac{1}{1+\theta}\ln(1-\psi_t) + \ln \mathfrak{m} + \ln \frac{\theta}{(\theta+1)(1-\alpha)}, \tag{38}$$

$$\frac{\partial \ln \psi_t}{\partial \ln \mathfrak{m}} = \frac{-1}{1+\frac{1}{1+\theta}\frac{\psi_t}{1-\psi_t}} < 0, \quad \frac{\partial \ln \psi_t}{\partial \ln Y_t} = \frac{\sigma}{1+\frac{1}{1+\theta}\frac{\psi_t}{1-\psi_t}}, \quad \Rightarrow \quad \mathrm{sgn}\left(\frac{\partial \psi_t}{\partial Y_t}\right) = \mathrm{sgn}(\sigma). \tag{39}$$

In the boundary case where the borrowing constraints are weakly binding with $\psi_t = 1-\lambda$, equation (38) can be rewritten as $\mathfrak{m} = (Y_t)^\sigma \Lambda$.

Taking into account equations (39), for $\sigma = 0$ and $\sigma = 1$, the condition for the binding borrowing constraints $\psi_t < 1-\lambda$ can be restated as $\mathfrak{m} > (Y_t)^\sigma \Lambda$ and the condition for the slack borrowing constraints $\psi_t > 1-\lambda$ can be restated as $\mathfrak{m} < (Y_t)^\sigma \Lambda$. □

Proof of Lemma 2

Proof. If $\psi_t \in (1-\lambda, 1)$, the borrowing constraints are slack and, according to equation (20), $\frac{\partial \ln r_t}{\partial \ln w_t} = -(1-\alpha) < 0$.

If $\psi_t \in (0, 1-\lambda)$, the borrowing constraints are binding. Rewrite equation (17) as

$$\ln r_t = -(1-\alpha)\ln w_t - \ln(1-\psi_t) + \ln \lambda R^\alpha \rho^{1-\alpha}, \tag{40}$$

$$\Rightarrow \quad \frac{\partial \ln r_t}{\partial \ln w_t} = -\underbrace{(1-\alpha)}_{\text{neoclassical effect}} + \underbrace{\frac{\sigma}{\frac{1}{\psi_t} - \frac{\theta}{1+\theta}}}_{\text{supply-side extensive-margin effect}}. \tag{41}$$

For $\sigma = 0$, $\frac{\partial \ln r_t}{\partial \ln w_t} = \alpha - 1 < 0$.

For $\sigma = 1$, $\frac{\partial \psi_t}{\partial Y_t} > 0$, according to equation (39). Let $\tilde{\psi}_A \equiv \frac{1}{\frac{2-\alpha}{1-\alpha} - \frac{1}{1+\theta}}$. According to equation (41), for $\psi_t \in (0, \tilde{\psi}_A)$, $\frac{\partial \ln r_t}{\partial \ln w_t} < 0$; for $\psi_t \in (\tilde{\psi}_A, 1-\lambda)$, $\frac{\partial \ln r_t}{\partial \ln w_t} > 0$.

□

Proof of Proposition 2

Proof. The proof consists of three steps. For simplicity, we suppress the country index i.

Step 1: derive the model solutions (9), (27)-(28) **under FMG**

For $w_t \in (0, \bar{w}_F)$, $\psi_t \in (0, 1-\lambda)$ and the borrowing constraints are binding. Combine equations (8) and (12) to get $K_{t+1} = RM_t = R\frac{1-\underline{\epsilon}_t^{-(1+\theta)}}{\psi_t} w_t$. Combine it with equations (1)-(2) to get the law of motion for wage (27). Combine equations (6), (36) and $r_t = r^*$ to get equation (28). Equation (9) defines the cutoff value $\underline{\epsilon}_t$. Thus, given w_t, one can use equations (9), (27)-(28) to solve $\{\psi_t, \underline{\epsilon}_t, w_{t+1}\}$ simultaneously.

Step 2: the shape of the law of motion for wage under FMG

Under FMG, the law of motion for wage is piecewise. Let $\mu_{t+1} \equiv \frac{r_t}{q_{t+1}R}$ denote the rate-of-return wedge. Given $r_t = r^*$ under FMG, for $w_t > \bar{w}_F$, the borrowing constraints are slack, $\mu_{t+1} = 1$, and the law of motion for wage is flat at $w_{t+1} = \bar{w}_{t+1} \equiv \left(\frac{R}{r^*}\right)^\rho$; for

$w_t \in (0, \bar{w}_F)$, the borrowing constraints are binding and the law of motion for wage is implicitly defined by $\{w_t, \psi_t, \underline{\epsilon}_t, \mu_{t+1}\}$ satisfying equations (42),

$$\mu_{t+1} = \frac{\lambda}{1-\psi_t},\ w_{t+1}^{-\frac{1}{\rho}} R\mu_{t+1} = r_t = r^*,\ w_t^{\sigma} = \psi_t \underline{\epsilon}_t \mathbb{F},\ \frac{w_{t+1}}{w_t} = \frac{1-\underline{\epsilon}_t^{-(1+\theta)}}{\psi_t \mu_{t+1}} \frac{r^*}{\rho}, \tag{42}$$

$$\frac{\partial \mu_{t+1}}{\partial \psi_t} = \frac{\lambda}{(1-\psi_t)^2} > 0,\ \ \frac{\partial \psi_t}{\partial w_t} = \frac{\$ + \sigma(1-\$)}{\mathbb{G}+1} \frac{\psi_t}{w_t} > 0, \tag{43}$$

where $\$ \equiv \frac{1-\underline{\epsilon}_t^{-(1+\theta)}}{1+\theta \underline{\epsilon}_t^{-(1+\theta)}}$ and $\mathbb{G} \equiv (1+\rho)\frac{\psi_t}{1-\psi_t}\$$. The positive mass of entrepreneurs $\tau_t = 1 - \underline{\epsilon}_t^{-\theta} > 0$ gives $\underline{\epsilon}_t > 1$ and hence, $\$ \in (0,1)$. Given $\frac{\partial \psi_t}{\partial w_t} > 0$, for $w_t \to 0$, $\psi_t \to 0$ so that $\mu_{t+1} \to \lambda$ and $w_{t+1} \to \underline{w}_{t+1} \equiv \left(\frac{R\lambda}{r^*}\right)^{\rho}$. Thus, the law of motion for wage has a positive intercept on the vertical axis at $\underline{w}_{t+1}$. Let $\mathbb{Z} \equiv 1 - \psi_t - \frac{\$}{1-\$} - (1+\rho)\theta\psi_t \$^2$.

$$\mathbb{J} \equiv \frac{\partial w_{t+1}}{\partial w_t} = \frac{\rho[\$ + \sigma(1-\$)]}{\mathbb{G}+1} \frac{\psi_t}{1-\psi_t} \frac{w_{t+1}}{w_t} > 0, \text{ if } \sigma \geq 0; \tag{44}$$

$$\text{for } \sigma = 0,\ \mathbb{H} \equiv \frac{\partial^2 w_{t+1}}{\partial w_t^2} = -\left[\frac{1-\$}{\mathbb{G}\$}(2+\theta\$) + \frac{\rho+\mathbb{G}}{\mathbb{G}} \frac{\psi_t}{1-\psi_t}\right] \frac{\$}{\mathbb{G}+1} \frac{\mathbb{J}}{w_t} < 0; \tag{45}$$

$$\text{for } \sigma = 1,\ \mathbb{H} \equiv \frac{\partial^2 w_{t+1}}{\partial w_t^2} = \mathbb{Z} \frac{1-\$}{\mathbb{G}+1} \frac{1+\rho}{\rho} \frac{1}{1-\psi_t} \frac{\mathbb{J}^2}{w_{t+1}} \quad \Rightarrow \quad sgn\,(\mathbb{H}) = sgn(\mathbb{Z}). \tag{46}$$

In the case of $\sigma = 0$, the law of motion for wage is piecewise with a positive intercept on the vertical axis at $\underline{w}_{t+1}$, concave for $w_t \in (0, \bar{w}_F]$, and flat at $\bar{w}_{t+1}$ for $w_t > \bar{w}_F$.

In the case of $\sigma = 1$,

$$\frac{\partial \mathbb{Z}}{\partial w_t} = -\left\{\frac{[1+(1+\rho)\theta\$_t^2]\psi_t}{(\mathbb{G}+1)w_t} + \frac{(1-\$)(1+\theta\$)\mathbb{G}}{(\mathbb{G}+1)w_t}\left[\frac{1}{(1-\$)^2} + 2\theta(1-\psi_t^i)\mathbb{G}\right]\right\} < 0.$$

Given $\frac{\partial \psi_t}{\partial w_t} > 0$, for $w_t \to 0$, $\psi_t \to 0$, so that $\mathbb{Z} > 0$ and the law of motion for wage is convex. Since $\frac{\partial \mathbb{Z}}{\partial w_t} < 0$, it is possible that, for $w_t \to \bar{w}_F$, $\psi_t \to 1-\lambda$ so that $\mathbb{Z} < 0$ and the law of motion for wage becomes concave. Let $\breve{w}_t$ define the threshold value such that $\mathbb{Z} = 0$, i.e., the inflection point of the law of motion for wage. There are two cases.

- Case 1: if $\breve{w}_t > \bar{w}_F$, the law of motion for wage is piecewise with a positive intercept on the vertical axis at $\underline{w}_{t+1}$, convex for $w_t \in (0, \bar{w})$, and flat at $\bar{w}_{t+1}$ for $w_t > \bar{w}_F$.

- Case 2: if $\breve{w}_t < \bar{w}_F$, the law of motion for wage is piecewise with a positive intercept on the vertical axis at $\underline{w}_{t+1}$, convex for $w_t \in (0, \breve{w})$, concave for $w_t \in (\breve{w}, \bar{w}_F)$, and flat at $\bar{w}_{t+1}$ for $w_t > \bar{w}_F$.

Step 3: the threshold values for multiple steady states under FMG

For $\sigma = 0$, the law of motion for wage under FMG has a concave-flat shape so that there exists a unique, stable steady state. See figure 5.

For $\sigma = 1$, the law of motion for wage under FMG has a convex-flat or convex-concave-flat shape so that multiple steady states may arise, as shown in figure 7. Given $\sigma = 1$ and $r^* = r_A$, we derive as follows the threshold values for the five regions of figure 6.

Case 1: consider the upper-right triangle of figure 6 where the borrowing constraints are slack at the autarkic steady state with $r_A = \rho$. Given $r^* = r_A = \rho$, the law of motion for wage at the autarkic steady state (S) is flat so that the autarkic steady state is still stable under FMG. Compare the upper-right and the lower-right panels of figure 7. The boundary between region BC and C of figure 6 is defined as the case where the law of motion is tangent with the 45° line at point M, i.e., $w^i_{t+1} = w^i_t = w_M < w_A$, $r_M = r^* = \rho$, and $\mathbb{J}_M \equiv \frac{\partial w_{t+1}}{\partial w_t}\|_{w_M} = 1$. Let $\mathbb{D}_M \equiv 1 - \underline{\epsilon}_M^{-(1+\theta)}$ and $\mathbb{N} \equiv \lambda$. Combine the three conditions with equations (27)-(28) to get

$$w_M < w_A, \ \Rightarrow \ \frac{\psi_M \underline{\epsilon}_M}{\psi_A \underline{\epsilon}_A} = \frac{w_M}{w_A} = \left(\frac{\lambda}{1-\psi_M}\right)^{\rho} \ \text{ and } \ \psi_M < \psi_A, \tag{47}$$

$$r_M = \frac{\lambda}{1-\psi_M}\frac{\psi_M \rho}{1-\underline{\epsilon}_M} = \rho, \ \Rightarrow \ \mathbb{D}_M = \frac{\mathbb{N}\psi_M}{1-\psi_M} < \psi_M, \tag{48}$$

$$\mathbb{J}_M = \frac{\rho}{(1+\rho)\mathbb{S}_M + \frac{1-\psi_M}{\psi_M}} = 1, \ \Rightarrow \ 1 - \frac{1}{\psi_M(\rho+1)} = \mathbb{S}_M = \frac{\mathbb{D}_M}{1+\theta(1-\mathbb{D}_M)}. \tag{49}$$

Combine equations (48) and (49) to get

$$\left[1 + \frac{1}{\rho(\theta+1)}\right]\mathbb{D}_M^2 - \left[\frac{\mathbb{N}}{\rho(1+\frac{1}{\theta})} + 1\right]\mathbb{D}_M + \frac{\mathbb{N}}{\rho} = 0. \tag{50}$$

$\mathbb{D}_M$ is a root of equation (50).[19] Combine the solution to $\mathbb{D}_M$ with equation (48) to solve for ψ_M and $\underline{\epsilon}_M = (1-\mathbb{D}_M)^{-\frac{1}{1+\theta}}$. Plug them and $\underline{\epsilon}_A = (1-\psi_A)^{-\frac{1}{1+\theta}}$ in equation (47) to solve ψ_A as a function of λ, which defines the boundary between region BC and C.

Let us consider the limiting case of $\theta \to \infty$. Equation (50) has two roots, i.e., $\mathbb{D}_M = 1$ and $\mathbb{D}_M = \frac{\lambda}{\rho}$. Combine $\mathbb{D}_M = 1$ with equation (48) to get $\psi_M = \frac{1}{1+\lambda} < 1$, which violates the condition of $\mathbb{D}_M < \psi_M < 1$. Thus, the true solution should be $\mathbb{D}_t = \frac{\lambda}{\rho}$. Combine it with equation (48) to get $\psi_M = 1 - \alpha$. Additionally, $\underline{\epsilon}_M = (1-\mathbb{D}_M)^{-\frac{1}{1+\theta}} = 1$ and $\underline{\epsilon}_A = (1-\psi_A)^{-\frac{1}{1+\theta}} = 1$. Inserting ψ_M, $\underline{\epsilon}_M$, and $\underline{\epsilon}_A$ in equation (47) to get the boundary condition for region BC and C in the limiting case of $\theta \to \infty$,

$$\psi_A = (1-\alpha)\left(\frac{\alpha}{\lambda}\right)^{\rho}, \ \text{ and } \ \lambda < \alpha. \tag{51}$$

Case 2: consider the lower-left triangle of figure 6 where the borrowing constraints are binding at the autarkic steady state with $\psi_A \in (0, 1-\lambda)$, $\mu_A \in (\lambda, 1)$, and $r_A = \rho\mu_A < \rho$. As shown in the upper-left panel of figure 7, given $r_t = r^* = r_A$ under FMG, case B arises if $\mathbb{J}_A \equiv \frac{\partial w_{t+1}}{\partial w_t}\|_{w_A} > 1$. The solution is

$$\begin{aligned}
&\psi_A \in (\hat{\psi}_F^-, \hat{\psi}_F^+), \ \text{ and } \ \psi_A \in (0, 1-\lambda),\\
\text{where } \ &\hat{\psi}_F^- = \frac{\left(2-\alpha-\frac{1-\alpha}{1+\theta}\right) - \sqrt{(2-\alpha-\frac{1-\alpha}{1+\theta})^2 - 4(1-\alpha)}}{2},\\
&\hat{\psi}_F^+ = \frac{\left(2-\alpha-\frac{1-\alpha}{1+\theta}\right) + \sqrt{(2-\alpha-\frac{1-\alpha}{1+\theta})^2 - 4(1-\alpha)}}{2},
\end{aligned}$$

[19] According to equation (50), there are two roots for $\mathbb{D}_t$. However, only one root satisfies the condition of $\mathbb{D}_M < \psi_M < \psi_A$.

which defines the border of region B in figure 6.

Let us consider the limiting case of $\theta \to \infty$. The solution is

$$\psi_A \in (1-\alpha, 1) \;\text{ and }\; \psi_A \in (0, 1-\lambda). \tag{52}$$

Case 3: consider the region with $\psi_A < \hat{\psi}_F^-$ in figure 6. Since $\mathbb{J}_A < 1$, the autarkic steady state is still stable under FMG. Compare the upper-middle and the lower-left panel of figure 7. As proved above, the law of motion for wage can be either convex or convex-concave for $w_t \in (0, \bar{w}_F)$. Taking that into account, FMG may lead to multiple steady states in two subcases.

- Case 3.1: multiple steady states arise if the kink point of the law of motion for wage is on or above the 45° line. Given $r^* = r_A = \rho\mu_A$, the **k**ink point is characterized by $w_t = \bar{w}_F$, $w_{t+1} = \bar{w}_{t+1} \equiv \left(\frac{R}{r^*}\right)^\rho$, $\psi_t = \psi_K \equiv 1-\lambda$, $\mu_{t+1} = \mu_K = 1$. As the boundary case, the kink point is on the 45° line, i.e., $\bar{w}_{t+1} = \bar{w}_F$. Combine them with equations (42) to get,

$$\bar{w}_{t+1}^{\frac{1}{\rho}} = \frac{R}{r^*} = \frac{R}{\rho\mu_A}, \qquad \bar{w}_F = \mathbb{F}\psi_K\underline{\epsilon}_K = \mathbb{F}(1-\lambda)\underline{\epsilon}_K \tag{53}$$

$$\frac{r^*}{\rho}\frac{1-\underline{\epsilon}_K^{-(1+\theta)}}{\mu_K\psi_K} = \frac{\bar{w}_{t+1}}{\bar{w}_F} = 1 \;\Rightarrow\; \underline{\epsilon}_K = \left(\frac{\mu_A}{\mu_A - 1 + \lambda}\right)^{\frac{1}{1+\theta}} \tag{54}$$

$$\left(\frac{R}{\rho\mu_A}\right)^\rho = \bar{w}_{t+1} = \bar{w}_F = \mathbb{F}(1-\lambda)\underline{\epsilon}_K \tag{55}$$

$$\left(\frac{R}{\rho}\right)^\rho = w_A = \mathbb{F}\psi_A\underline{\epsilon}_A, \;\; \underline{\epsilon}_A = \left(\frac{\mu_A}{\lambda}\right)^{\frac{1}{1+\theta}}, \;\; \mu_A = \frac{\lambda}{1-\psi_A} \tag{56}$$

$$\Rightarrow (1-\lambda)\lambda^\rho = \left(\frac{1}{1-\psi_A} - \frac{1-\lambda}{\lambda}\right)^{\frac{1}{1+\theta}} \psi_A(1-\psi_A)^\rho. \tag{57}$$

  Let $\tilde{\psi}_{F,1}$ denote the solution to equation (57), which is a function of λ.

  Let us consider the limiting case of $\theta \to \infty$. Equation (57) becomes

$$(1-\lambda)\lambda^\rho = \psi_A(1-\psi_A)^\rho. \tag{58}$$

- Case 3.2: Multiple steady states arise if the concave part of the law of motion is at least tangent with the 45° line at point M, i.e., $w_{t+1} = w_t = w_M \in (w_A, \bar{w}_F)$, $\mathbb{J}_M \equiv \frac{\partial w_{t+1}}{\partial w_t}\|_{w_M} = 1$, and $r^* = r_A = \rho\mu_A$.[20] Let $\mathbb{D}_M \equiv 1 - \underline{\epsilon}_M^{-(1+\theta)}$ and $\mathbb{N} \equiv 1 - \psi_A$. Combine the three conditions with equations (27)-(28) to get

$$w_M \in (w_A, \bar{w}_F), \;\Rightarrow\; \frac{\psi_M\underline{\epsilon}_M}{\psi_A\underline{\epsilon}_A} = \frac{w_M}{w_A} = \left(\frac{\mu_M}{\mu_A}\right)^\rho = \left(\frac{1-\psi_A}{1-\psi_M}\right)^\rho, \tag{59}$$

$$r_M = r^* = \rho\mu_A, \;\Rightarrow\; \mathbb{D}_M = \frac{\mathbb{N}\psi_M}{1-\psi_M} > \psi_M, \tag{60}$$

$$\mathbb{J}_M = 1, \;\Rightarrow\; 1 - \frac{1}{\psi_M(\rho+1)} = \mathbb{S}_M = \frac{\mathbb{D}_M}{1+\theta(1-\mathbb{D}_M)}. \tag{61}$$

[20] The analysis is same as that for the boundary condition of region BC and C, except for $r_A = \rho\mu_A$.

Combine equations (60) and (61) to get

$$\left[1+\frac{1}{\rho(\theta+1)}\right]\mathbb{D}_M^2-\left[\frac{\mathbb{N}}{\rho(1+\frac{1}{\theta})}+1\right]\mathbb{D}_M+\frac{\mathbb{N}}{\rho}=0. \tag{62}$$

$\mathbb{D}_M$ is a root of equation (62).[21] Combine it with equation (60) to solve for ψ_M and $\underline{\epsilon}_M=(1-\mathbb{D}_M)^{-\frac{1}{1+\theta}}$. Plug them and $\underline{\epsilon}_A=(1-\psi_A)^{-\frac{1}{1+\theta}}$ in equation (59) and let $\tilde{\psi}_{F,2}$ denote the solution, which is independent of λ.

Let us consider the limiting case of $\theta\to\infty$. Equation (62) has two roots, i.e., $\mathbb{D}_M=1$ and $\mathbb{D}_M=\frac{1-\psi_A}{\rho}$. Combine $\mathbb{D}_M=\frac{1-\psi_A}{\rho}$ with equation (60) to get $\psi_M=1-\alpha$ and plug it back in equation (59) to get $\psi_A=1-\alpha$, which violates the condition of $\psi_A<\psi_M$. Thus, the true solution should be $\mathbb{D}_M=1$. Combine it with equation (60) to get $\psi_M=\frac{1}{2-\psi_A}$. Additionally, $\lim_{\theta\to\infty}\underline{\epsilon}_M=\lim_{\theta\to\infty}(1-\mathbb{D}_M)^{-\frac{1}{1+\theta}}=1$ and $\lim_{\theta\to\infty}\underline{\epsilon}_A=\lim_{\theta\to\infty}(1-\psi_A)^{-\frac{1}{1+\theta}}=1$. Inserting ψ_M, $\underline{\epsilon}_M$, and $\underline{\epsilon}_A$ in equation (59) to get the boundary condition for region AB and A

$$\psi_A^{1-\alpha}(2-\psi_A)=1. \tag{63}$$

The boundary between region AB and A is characterized by $\tilde{\psi}_F=\max\{\tilde{\psi}_{F,1},\tilde{\psi}_{F,2}\}$.

□

Proof of Proposition 3

Proof. Under IFA, the law of motion for wage in setting **N** is identical as that in the generalized model and so is that in setting **F** except for a kink at $w_t=\mathfrak{m}$. Thus, the proof follows exactly the proofs for proposition 1 and lemma 1. □

Proof of Lemma 3

Proof. In setting **N**, if $\tau\in(1-\lambda,1)$, the borrowing constraints are slack with $\psi_t=\tau\in(1-\lambda,1)$ and, according to equation (20), $\frac{\partial\ln r_t}{\partial\ln w_t}=-(1-\alpha)<0$; if $\tau\in(0,1-\lambda]$, the borrowing constraints are binding with $\psi_t=\tau\in(0,1-\lambda)$ and, according to equation (33), $\frac{\partial\ln r_t^i}{\partial\ln w_t^i}=-(1-\alpha)<0$.

In setting **N**, if $\psi_t\in(1-\lambda,1)$, the borrowing constraints are slack and, according to equation (20), $\frac{\partial\ln r_t}{\partial\ln w_t}=-(1-\alpha)<0$. If $\psi_t\in(0,1-\lambda)$, the borrowing constraints are binding $\psi_t=\frac{w_t}{\mathfrak{m}}\in(0,1-\lambda)$. Define $\tilde{\psi}_A\equiv\frac{1-\alpha}{2-\alpha}$. Rewrite equation (34)

$$\ln r_t^i=-(1-\alpha)\ln w_t^i-\ln(1-\psi_t)+\ln\lambda R^\alpha\rho^{1-\alpha}. \tag{64}$$

$$\Rightarrow\quad\frac{\partial\ln r_t^i}{\partial\ln w_t^i}=-(1-\alpha)+\frac{\psi_t}{1-\psi_t},\quad\text{and}\quad\frac{\partial\ln r_t^i}{\partial\ln w_t^i}\begin{cases}<0, & \text{iff } \psi_t\in(0,\tilde{\psi}_A);\\ >0, & \text{iff } \psi_t\in(\tilde{\psi}_A,1-\lambda).\end{cases} \tag{65}$$

□

Proof of Proposition 4

[21] According to equation (62), there are two roots for $\mathbb{D}_t$. However, only one root satisfies the condition of $\mathbb{D}_M>\psi_M$.

Proof. **Steady-State Property under FMG in Setting N**

In setting **N**, if the borrowing constraints are binding under FMG, combine equations (1)-(3) with $r_t^i = r^*$ to get the law of motion for wage,

$$r^*(m_t^i - w_t^i) = \lambda q_{t+1}^i R m_t^i \;\; \Rightarrow \;\; r^*\left[\frac{\rho}{R}(w_{t+1}^i)^{\frac{1}{\alpha}} - \tau w_t^i\right] = \lambda \rho w_{t+1}^i, \tag{66}$$

$$\frac{\partial w_{t+1}^i}{\partial w_t^i} = \frac{\tau}{\rho}\left[\frac{(w_{t+1}^i)^{\frac{1}{\rho}}}{\alpha R} - \frac{\lambda}{r^*}\right]^{-1} = \frac{\tau q_{t+1}^i R}{1 + \frac{\rho w_t^i}{m_t^i}} > 0, \tag{67}$$

$$\frac{\partial^2 w_{t+1}^i}{\partial (w_t^i)^2} = -\left(\frac{\partial w_{t+1}^i}{\partial w_t^i}\right)^3 \frac{(w_{t+1}^i)^{\frac{1}{\rho}-1}}{\tau \alpha R} < 0. \tag{68}$$

Equation (66) implies that, for $w_t^i \to 0$, the law of motion for wage has a positive intercept on the vertical axis at $w_{t+1}^i = \left(\frac{R\lambda}{r^*}\right)^{\rho}$. Define a threshold value $\bar{w}_F = \frac{\rho}{r^*}\frac{1-\lambda}{\tau}\left(\frac{R}{r^*}\right)^{\rho}$. For $w_t^i \in (0, \bar{w}_F)$, the borrowing constraints are binding and the law of motion for wage is increasing and concave, according to equations (67)-(68). For $w_t^i > \bar{w}_F$, aggregate saving and investment are so high that the marginal rate of return to investment is equal to the world interest rate, $Rq_{t+1}^i = r^*$ and the borrowing constraints are slack. In this case, any further increase in w_t^i leads to financial capital outflows, without affecting the domestic investment. The law of motion for wage is then flat at $\bar{w}_{t+1}^i = \left(\frac{R}{r^*}\right)^{\rho}$.

If $\tau \in (0, 1-\lambda)$, the borrowing constraints are slack in the autarkic steady state with $r_A = \frac{\lambda}{1-\tau}\rho < \rho$, implying that $\bar{w}_{t+1}^i < \bar{w}_F$. Thus, the kink point of the law of motion for wage is below the 45° line. Graphically, the law of motion for wage crosses the 45° line once and only once from the left, with the intersection point in its concave part, qualitatively the same as the left panel of figure 5.

If $\tau > 1-\lambda$, the borrowing constraints are slack in the autarkic steady state with $r_A = \rho$, implying that $\bar{w}_{t+1}^i > \bar{w}_F$. Thus, the kink point of the phase diagram is above the 45° line. Graphically, the law of motion for wage crosses the 45° line once and only once from the left, with the intersection point in its flat part, qualitatively the same as the right panel of figure 5.

To sum up, given $r^* = r_A$, the autarkic steady state is still the unique, stable steady state under FMG in setting **N**.

Steady-State Property under FMG in Setting F

In setting **F**, we first analyze the shape of the law of motion for wage and then describe the conditions for symmetry breaking.

If the borrowing constraints are binding, $Rq_{t+1}^i > r^*$ or equivalently $\psi_t^i < 1-\lambda$. Combine equations (1)-(3) with $r_t^i = r^*$ to get the law of motion for wage,

$$1 - \frac{w_t^i}{\mathfrak{m}} = \lambda \frac{q_{t+1}^i R}{r^*} = \frac{\lambda \rho}{r^*}\left(\frac{w_A}{w_{t+1}^i}\right)^{\frac{1}{\rho}}, \tag{69}$$

$$\mathbb{J} \equiv \frac{\partial w_{t+1}^i}{\partial w_t^i} = \frac{\rho}{\frac{1}{\psi_t^i} - 1}\frac{w_{t+1}^i}{w_t^i} > 0, \;\; \text{and} \;\; \mathbb{H} \equiv \frac{\partial^2 w_{t+1}^i}{\partial (w_t^i)^2} = \mathbb{J}^2 \frac{1}{\alpha w_{t+1}^i w_t^i} > 0. \tag{70}$$

Combine equation (69) with (1)-(2) and then compute the mass of entrepreneurs,

$$w^i_{t+1} = w_A \left[\frac{\lambda\rho}{r^*(1-\psi^i_t)}\right]^{\rho} \;\Rightarrow\; \tau^i_t = \frac{K^i_{t+1}}{R\mathfrak{m}} = \frac{\rho(w^i_{t+1})^{\frac{1}{\alpha}}}{R\mathfrak{m}} = \psi_A \left[\frac{\lambda\rho}{r^*(1-\psi^i_t)}\right]^{\frac{1}{1-\alpha}}. \tag{71}$$

The mass of entrepreneurs is bounded by the population size in each generation, $\tau^i_t \leq 1$.

According to equation (71), for $w^i_t \to 0$, $\psi^i_t \to 0$ and the law of motion for wage has a positive intercept on the vertical axis at $w^i_{t+1} = w_A \left(\frac{\lambda\rho}{r^*}\right)^{\rho}$. Similar as in setting **N**, the law of motion for wage in setting **F** under FMG consists of two subfunctions. The kink point depends on two factors, i.e., whether the borrowing constraints are binding or slack, and whether the mass of entrepreneurs is below or equal to unity. For $\psi_A \in (0, 1-\lambda]$, the borrowing constraints are binding in the autarkic steady state and, under FMG, $r^* = \frac{\lambda\rho}{1-\psi_A} < \rho$. In this case, according to equation (71), $\tau^i_t \leq 1$ implies that $\psi^i_t \leq \check{\psi}_F \equiv 1 - \psi_A^{1-\alpha}(1-\psi_A)$. For $\psi_A \in [1-\lambda, 1]$, the borrowing constraints are slack in the autarkic steady state and, under FMG, $r^* = \rho$. In this case, according to equation (71), $\tau^i_t \leq 1$ implies that $\psi^i_t \leq 1 - \lambda\psi_A^{1-\alpha}$.

In the following, I characterize the shape of the law of motion for wage in two cases:

- Case 1: if $\check{\psi}_F > 1-\lambda$,
  For $\psi^i_t \in (0, 1-\lambda)$, the borrowing constraints are binding, some agents become entrepreneurs, $\tau^i_t < 1$, and the law of motion for wage is convex, $w^i_{t+1} = w_A \left(\frac{1-\psi_A}{1-\frac{w^i_t}{\mathfrak{m}}}\right)^{\rho}$; for $\psi^i_t > 1-\lambda$, the borrowing constraints are slack, some agents become entrepreneurs, $\tau^i_t < 1$, and the law of motion for wage is flat at $w^i_{t+1} = w_A \left(\frac{1-\psi_A}{\lambda}\right)^{\rho}$.

- Case 2: if $\check{\psi}_F < 1-\lambda$,
  For $\psi^i_t \in (0, \check{\psi}_F)$, the borrowing constraints are binding, some agents become entrepreneurs, $\tau^i_t < 1$, and the law of motion for wage is convex, $w^i_{t+1} = w_A \left(\frac{1-\psi_A}{1-\frac{w^i_t}{\mathfrak{m}}}\right)^{\rho}$; for $\psi^i_t > \check{\psi}_F$, the borrowing constraints are binding, all agents become entrepreneurs, $\tau^i_t = 1$, and the law of motion for wage is flat at $w^i_{t+1} = \left(\frac{R\mathfrak{m}}{\rho}\right)^{\alpha} = \frac{w_A}{\psi_A^{\alpha}}$.

The convex part of the phase diagram creates the possibility of multiple steady states. Figure 10 shows the parameter configuration of five regions in the $\{\lambda, \psi_A\}$ space and in the $\{\lambda, Z\}$ space, respectively. Figure 11 shows the laws of motion for wage under FMG versus under IFA in five cases.

In the following, I derive the boundary conditions for the five regions in figure 10.

Case 1: Consider the upper-right triangle of figure 10, i.e., $\psi_A \in (1-\lambda, 1)$. Compare the upper-right and the lower-right panels of figure 11. Given $r^* = r_A = \rho$, the law of motion for wage under FMG is flat at the initial steady state (point S); the boundary between region BC and C is defined as the case where the convex part of the law of motion for wage is tangent with the 45° line at point M, i.e., $w^i_t = w^i_{t+1} = w_M < w_A$ and

$\mathbb{J}_M = 1$. Rewrite equations (69) and (70) at the tangent point,

$$1 - \frac{w_M}{\mathfrak{m}} = \lambda w_M^{-\frac{1}{\rho}} \frac{R}{\rho}, \ \Rightarrow \ \left(1 - \frac{w_M}{\mathfrak{m}}\right)\left(\frac{w_M}{\mathfrak{m}}\right)^{\frac{1}{\rho}} = \psi_A^{\frac{1}{\rho}} \lambda$$
$$\mathbb{J}_M = \frac{\rho \psi_M}{1 - \psi_M} = \frac{\rho\rho}{\lambda R \mathfrak{m}} (w_M)^{\frac{1}{\alpha}} = 1, \ \Rightarrow \ \left(\frac{w_M}{\mathfrak{m}}\right)^{\frac{1}{\alpha}} = \frac{\lambda}{\rho} \psi_A^{\frac{1}{\rho}}.$$

Combine them to get

$$\frac{w_M}{\mathfrak{m}} = 1 - \alpha \ \text{ and } \ \psi_A = (1-\alpha)\left(\frac{\alpha}{\lambda}\right)^{\rho}, \tag{72}$$
$$w_M < w_A \ \Rightarrow \ \frac{w_F}{\mathfrak{m}} < \psi_A \ \text{ and } \ \lambda < \alpha. \tag{73}$$

Equations (72)-(73) jointly define the boundary between region BC and C, the same as equations (51) in the limiting case of the generalized model with $\sigma = 1$ and $\theta \to \infty$.

Case 2: Consider the lower-left triangular of figure 10, i.e., $\psi_A \in (0, 1-\lambda)$. Given $r_t = r^* = r_A$ under FMG, case B arises if the law of motion for wage under FMG has a slope exceeds unity at the autarkic steady state, $\mathbb{J}_A = \frac{\rho}{\frac{1}{\psi_A} - 1} > 1$. The solution is

$$\psi_A \in (1-\alpha, 1) \ \text{ and } \ \psi_A \in (0, 1-\lambda) \tag{74}$$

which specifies the boundary between region B and AB, the same as equations (52) in the limiting case of the generalized model with $\sigma = 1$ and $\theta \to \infty$.

Case 3: consider the region with $\psi_A < \hat{\psi}_F$. Since $\mathbb{J}_A < 1$, the autarkic steady state is stable under FMG. Compare the upper-middle and the lower-left panel of figure 11. FMG may still generate multiple steady states if the kink point of the law of motion for wage is above the 45° line, i.e., $\bar{w}_{t+1} > \bar{w}_F$. There are two subcases.

- Case 3.1: if $\check{\psi}_F > 1 - \lambda$, the kink point is at $\bar{w}_F = (1-\lambda)\mathfrak{m}$ and $\bar{w}_{t+1} = w_A \left(\frac{1-\psi_A}{\lambda}\right)^{\rho}$. In the boundary case,

$$\bar{w}_{t+1} = \bar{w}_F, \ \Leftrightarrow \ (1-\psi_A)^{\rho}\psi_A = (1-\lambda)\lambda^{\rho}, \tag{75}$$

  which is the same as equations (58) in the limiting case of the generalized model with $\sigma = 1$ and $\theta \to \infty$.

  Let $\tilde{\psi}_{F,1}$ denote the solution to equation (75).

- Case 3.2: if $\check{\psi}_F < 1 - \lambda$, the kink point is at $\bar{w}_t^F = [1 - (1-\psi_A)\psi_A^{1-\alpha}]\mathfrak{m}$ and $\bar{w}_{t+1} = \frac{w_A}{\psi_A^{\alpha}}$. In the boundary case,

$$\bar{w}_{t+1} = \bar{w}_F, \ \Leftrightarrow \ \psi_A^{1-\alpha}(2 - \psi_A) = 1, \tag{76}$$

  which is the same as equations (63) in the limiting case of the generalized model with $\sigma = 1$ and $\theta \to \infty$

  Let $\tilde{\psi}_{F,2}$ denote the solution to equation (76).

The boundary between region AB and A is characterized by $\tilde{\psi}_F = \max\{\tilde{\psi}_{F,1}, \tilde{\psi}_{F,2}\}$.

□