Outline

- **Day 1**
  - Angle Modulation
  - Frequency Modulation (FM)
    - Narrowband and Wideband FM
    - Transmission bandwidth
    - FM Stereo Mix
- **Day 2**
  - Phase-locked Loop (PLL)
  - Non-linear effects in FM receivers
  - Summary
- **Day 3**
  - Tutorial
Recall...What is modulation?

- Message $m(t)$
  - Highest freq $W$
  - Transmitter

- Sinusoidal carrier $c(t) = A_c \cos [2\pi f_c t]$
  - Modulation

- Direct transmission
  - Unsuitable/inefficient
  - Transmission channel

- Modulated signal $s(t) = A(t) \cos\phi(t)$
  - Receiver

- $s(t)$ obtained by varying *characteristic* of $c(t)$ according to $m(t)$
  - *Amplitude* $A(t) \leftrightarrow$ Amplitude Modulation
  - *Angle* $\phi(t) \leftrightarrow$ Angle Modulation
Recall... Amplitude Modulation

- $s(t) = A_c[1+k_a m(t)]\cos 2\pi f_c t$

- Envelope of $s(t)$ has *same* shape as $m(t)$ provided:
  - $|k_a m(t)| < 1$
  - $f_c \gg W$

- Easy and cheap to generate $s(t)$ and reverse
Recall…Amplitude Modulation

- **Drawbacks of AM**
  - wasteful of **power**
    - transmission of carrier
  - wasteful of **bandwidth**
    - transmission bandwidth, $B_T = 2W$

- Improved resource utilization (power or bandwidth) traded-off with increased system complexity

- Angle modulation offers practical means of trading-off between power and bandwidth
Angle modulation

- \( s(t) = A_c \cos[\theta_i(t)] \), \( f_i(t) = \frac{1}{2\pi} \frac{d\theta_i(t)}{dt} \)

  - **Phase Modulation (PM)**
    \[ \theta_i(t) = 2\pi f_c t + k_p m(t) \]
    \[ \Rightarrow s(t) = A_c \cos[2\pi f_c t + k_p m(t)] \]

  - **Frequency Modulation (FM)**
    \[ f_i(t) = f_c + k_f m(t) \]
    \[ \Rightarrow s(t) = A_c \cos[2\pi f_c t + 2\pi k_f \int_0^t m(t)dt] \]
PM vs FM

Frequency modulator

Modulating wave, \( m(t) \)

Integrator

Phase modulator

\[ s(t) = A_c \cos(2\pi f_c t + \int_0^t m(t) dt) \]

FM wave

\( A_c \cos(2\pi f_c t) \)

Differentiator

Frequency modulator

\[ s(t) = A_c \cos(2\pi f_c t + k_p m(t)) \]

PM wave

Phase modulator

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Frequency modulation

- FM signal: $s(t) = A_c \cos[2\pi f_c t + 2\pi k_f \int_0^t m(t)dt]$.

- Consider single tone signal: $m(t) = A_m \cos 2\pi f_m t$
  
  \[ f_i(t) = f_c + k_f m(t) \]
  \[ = f_c + k_f A_m \cos[2\pi f_m t] \]
  \[ = f_c + \Delta f \cos[2\pi f_m t] \]  
  Frequency deviation

  \[ \theta_i(t) = 2\pi \int_0^t f_i(t)dt \]
  \[ = 2\pi f_c t + \frac{\Delta f}{f_m} \sin[2\pi f_m t] \]
  \[ = 2\pi f_c t + \beta \sin[2\pi f_m t] \]  
  Modulation index

- $s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$
Narrowband FM (β<<1)

Expanding \( s(t) \), we have:

\[
s(t) = A_c \cos(2\pi f_c t) \cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t) \sin[\beta \sin(2\pi f_m t)]
\]

If \( \beta << 1 \), \( s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t) \)

\[
\approx A_c \cos(2\pi f_c t) + \frac{1}{2} \beta A_c \{ \cos[2\pi (f_c + f_m) t] - \cos[2\pi (f_c - f_m) t] \}
\]
Comparison with AM

AM Signal

Narrow band FM Signal
Wideband FM

\( s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)] \)

\[ f_c \gg f_m \]

\( = \text{Re}[A_c \exp(j2\pi f_c t + j\beta \sin(2\pi f_m t))] \]

\( = \text{Re}[\tilde{s}(t) \exp(j2\pi f_c t)] \quad (\ast) \)

\( \tilde{s}(t) = A_c \exp[j\beta \sin(2\pi f_m t)] \)

\( = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j2\pi nf_m t) \)

Subst. into \( (\ast) \), and applying FT:

\( S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) [\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)] \)

\( = A_c J_0(\beta) \delta(f - f_c) + \frac{A_c}{2} J_{\pm 1}(\beta) \delta(f - f_c \mp f_m) + \frac{A_c}{2} J_{\pm 2}(\beta) \delta(f - f_c \mp 2f_m) + \ldots \)

nth order Bessel function of first kind

\( \text{Carrier component} \quad \text{Side freq } f_c \pm f_m \quad \text{Side freq } f_c \pm 2f_m \)
Observations

○ Amplitude of carrier varies with $J_0(\beta)$
  - With AM, amplitude of carrier = $A_c$

○ Special case: $\beta << 1$
  - Only $J_0(\beta)$, $J_1(\beta) \Leftrightarrow f_c \pm f_m$ significant (narrowband FM)

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]$$

$$= A_c J_0(\beta) \delta(f - f_c) + \frac{A_c}{2} J_{\pm 1}(\beta) \delta(f - f_c \mp f_m) + \frac{A_c}{2} J_{\pm 2}(\beta) \delta(f - f_c \mp 2f_m) + \ldots$$

nth order Bessel function of first kind
Example - fixed $f_m$, variable $A_m$

\[ m(t) = A_m \cos(2\pi f_m t), \quad \Delta f = k_f A_m, \quad \beta = \frac{\Delta f}{f_m} \]

\[ S(f) = A_c J_0(\beta) \delta(f - f_c) + \frac{A_c}{2} J_{\pm 1}(\beta) \delta(f - f_c \mp f_m) + \frac{A_c}{2} J_{\pm 2}(\beta) \delta(f - f_c \mp 2f_m) + \ldots \]
Example - fixed $A_m$, variable $f_m$

$$m(t) = A_m \cos[2\pi f_m t], \quad \Delta f = k_f A_m, \quad \beta = \frac{\Delta f}{f_m}$$

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As $\beta \uparrow$, number of spectral lines within $f_c - \Delta f < |f| < f_c + \Delta f$ \uparrow

As $\beta \to \infty$, the bandwidth of $s(t)$ approaches the limiting value of $2\Delta f$!!

[Note: For $\beta \ll 1$, bandwidth of $s(t) = 2f_m$ (As in AM)]
Transmission bandwidth

\[ S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)] \]

⇒ transmission bandwidth = \( \infty \)!!!

But, effectively, *finite* number of side frequencies are significant

\[ \Rightarrow \text{Carson's rule: } B_{T,\text{Carson}} \approx 2\Delta f \left(1 + \frac{1}{\beta}\right) \]
Transmission bandwidth

\[ S(f) = \frac{A_c}{2} \sum_{n=\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)] \]

\[ \Rightarrow \text{transmission bandwidth} = \infty!!! \]

But, effectively, finite number of side frequencies are significant

\[ \Rightarrow \text{retain up to } n_{\text{max}} \text{ side frequencies s.t. } J_{n_{\text{max}}}(\beta) \geq \epsilon J_0(\beta) \]

\[ \Rightarrow B_T = 2n_{\text{max}}f_m \]

\[ B_{T,1\%} = 1 \% \text{ bandwidth with } \epsilon = 0.01 \]
As $\beta \uparrow$, $n_{\text{max}} \uparrow \Rightarrow B_{T,1\%} \uparrow$
Small values of $\beta$ more extravagant in $B_T$ than larger $\beta$!!

Practically, $B_{T,Carson} \leq B_T \leq B_{T,1\%}$
FM Stereo

- Transmit two separate signals via same carrier
  - 2 different sections of orchestra, e.g., vocalist and accompanist, to give spatial dimension to its perception

- Requirements
  - Must operate within allocated FM broadcast channels
  - Must be compatible with monophonic radio receivers
FM Stereo Mux

\[ m(t) = [m_l(t) + m_r(t)] + [m_l(t) - m_r(t)] \cos(2\pi[2f_c]t) + K \cos(2\pi f_c t) \]

\[ 2f_c = 38 \text{kHz} \]

\[ f_c = 19 \text{kHz} \]

\[ m_l + m_r : \text{Monophonic reception} \]
\[ m(t) = [m_l(t) + m_r(t)] + [m_l(t) - m_r(t)] \cos(2\pi[2f_c]t) + K \cos(2\pi f_c t) \]
Phase Locked Loop (PLL)

- **PLL for freq. demod**
  - If \( s(t) \) is FM wave, obtain \( m(t) \) from \( v(t) \)
  - Require \( \phi_1 \approx \phi_2 + 90^\circ \) \( \iff \) Phase lock!

\[
\begin{align*}
s(t) &= A_v \sin[2\pi f_c t + \phi_1(t)] \\
r(t) &= A_v \cos[2\pi f_c t + \phi_2(t)] \\
\phi_2(t) &= 2\pi k_v \int_0^t v(t)dt
\end{align*}
\]
Phase Locked Loop (PLL)

\[ s(t) = A_c \sin[2\pi f_c t + \phi_1(t)] \]

\[ e(t) = \begin{cases} 
    HF : 4f_c \text{ term} \\ 
    LF : K \sin[\phi_1(t) - \phi_2(t)] \\ 
    \phi_e(t) 
\end{cases} \]

\[ v(t) = \int_{-\infty}^{\infty} e(\tau)h(t - \tau)d\tau \]

\[ r(t) = A_v \cos[2\pi f_c t + \phi_2(t)] \]

\[ \phi_2(t) = 2\pi k_v \int_{0}^{t} v(t)dt \]

\[ \frac{d\phi_e(t)}{dt} = \frac{d\phi_1(t)}{dt} - 2\pi K_o \int_{-\infty}^{\infty} \sin[\phi_e(\tau)]h(t - \tau)d\tau, \]

\[ K_o = k_m k_v A_c A_v \]

Dynamic behavior of PLL
Non-linear PLL model

Sinusoidal non-linearity makes it difficult to analyze PLL

Dynamic behavior of PLL

\[ \frac{d\phi_e(t)}{dt} = \frac{d\phi_1(t)}{dt} - 2\pi K_o \int_{-\infty}^{\infty} \sin[\phi_e(\tau)]h(t - \tau)d\tau, \]

\[ K_o = k_m k_v A_c A_v \]
Non-linear PLL model

Assume $\phi_e(t) \ll 1 \Rightarrow \sin \phi_e(t) \approx \phi_e(t)$

$$\frac{d\phi_e(t)}{dt} = \frac{d\phi_1(t)}{dt} - 2\pi K_o \int_{-\infty}^{\infty} \phi_e(\tau) h(t - \tau) d\tau,$$

$K_o = k_m k_v A_c A_v$

- Linearized behavior of PLL
  - FT can be applied!
Linear PLL model

\[ \Phi_e(f) = \frac{1}{1 + L(f)} \Phi_1(f), \]

\[ L(f) = K_o \frac{H(f)}{jf} \quad \text{[Open-loop transfer function]} \]

\[ V(f) = \frac{(j \frac{f}{k_v})L(f)}{1 + L(f)} \Phi_1(f) \]

\[ V(f) = \frac{K_o}{k_v} H(f) \Phi_e(f) \]

\[ = \frac{jf}{k_v} L(f) \Phi_e(f) \]
Phase-locked Linear PLL

\[ V(f) = \frac{jf}{k_v} L(f) \Phi_1(f) \]

\[ \Phi_e(f) = \frac{1}{1 + L(f)} \Phi_1(f) \]

\[ |L(f)| \gg 1 \]

\[ V(f) = \frac{jf}{k_v} \Phi_1(f) \]

\[ v(t) = \frac{1}{2\pi k_v} \frac{d\phi_1(t)}{dt} \]

\[ \Phi_e(f) \rightarrow 0 \Leftrightarrow \text{Phase lock!!!} \]
Phase locked linear PLL as frequency demodulator

If $s(t)$ is FM signal,

$$\Rightarrow \phi_1(t) = 2\pi k_f \int_0^t m(t) dt$$

$$v(t) = \frac{k_f}{k_v} m(t)$$

- If $|L(f)| >> 1$
  - Linearized PLL model
  - Phase lock satisfied [$\phi_e \approx 0$]
  - $v(t) = \frac{1}{2\pi k_v} \frac{d\phi_1(t)}{dt}$

- Bandwidth of $s(t) >>$ bandwidth of $H(f)$ [$m(t)$]

Determines complexity of PLL
Design of $H(f)$

- **First order**
  - $H(f) = 1$
  - $|L(f)| = K_o/f$

- **Drawback**
  - $K_o$ controls both loop bandwidth and hold-in frequency range, $f_H$

\[ |L(f)| >> 1 \]
Second-order PLL

- $H(f) = 1 + \frac{a}{jf}$
- $\Phi_e(f) = \left(\frac{\left(jf / f_n\right)^2}{1 + 2\zeta(jf / f_n) + (jf / f_n)^2}\right) \Phi_1(f)$
- $f_n = \sqrt{a K_o}$ Natural frequency
- $\zeta = \sqrt{\frac{K_o}{4a}}$ Damping factor

- For $m(t) = A_m \cos 2\pi f_m t$
  - $\phi_1(t) = \beta \sin(2\pi f_m t)$
  - $\Rightarrow \phi_e(t) = \phi_{e0} \cos(2\pi f_m t + \psi)$
Second-order PLL

- With appropriate choice of \((\xi, f_n)\), we can maintain \(\phi_e\) small
  - Linear PLL model

- Rule of thumb:
  - Loop should remain locked if \(\phi_{e0}(f_m=f_n)<90^\circ\)
Non-linear effects in FM systems

- Assume $v_i(t)$ is FM signal
  - $\Delta f = $ frequency deviation
  - $W = $ highest freq. comp. of $m(t)$

$$v_i(t) = A_c \cos[2\pi f_c t + \phi(t)]$$
$$\phi(t) = 2\pi k_f \int_0^t m(t)dt$$

$$v_o(t) = a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t)$$

$$v_o(t) = A_0 + A_1 \cos[2\pi f_c t + \phi(t)] + A_2 \cos[4\pi f_c t + 2\phi(t)] + A_3 \cos[6\pi f_c t + 3\phi(t)]$$
Non-linear effects in FM systems

\[ v_i(t) = A_c \cos[2\pi f_c t + \phi(t)], \]
\[ \phi(t) = 2\pi k_f \int_0^t m(t) dt \]
\[ v_o(t) = A_0 + A_1 \cos[2\pi f_c t + \phi(t)] + A_2 \cos[4\pi f_c t + 2\phi(t)] + A_3 \cos[6\pi f_c t + 3\phi(t)] \]

- Carson’s Rule
  - \( B_T \approx 2\Delta f \)
To separate out desired FM signal:
- \( f_c + \Delta f + W < 2f_c - W - 2\Delta f \)
- \( \Rightarrow f_c > 3\Delta f + 2W \)
- Apply bandpass filter \([f_c - \Delta f - W, f_c + \Delta f + W]\)

Unlike AM, FM *not* affected by distortion due to channel with amp. non-linearities.

\[
v_o'(t) = (a_1 + \frac{3}{4}a_3A_c^2)v_i(t)
\]
Summary

- Unlike AM, FM is **non-linear** modulation process
  - Spectral analysis is more difficult
  - Developed insight by studying single-tone FM

- Carson’s rule for transmission bandwidth
  - $B_T = 2\Delta f(1+1/\beta)$

- Phase-Locked Loop for frequency demodulation