Frequency Modulation

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Lecture material was abstracted from "Communication Systems" by Simon Haykin.

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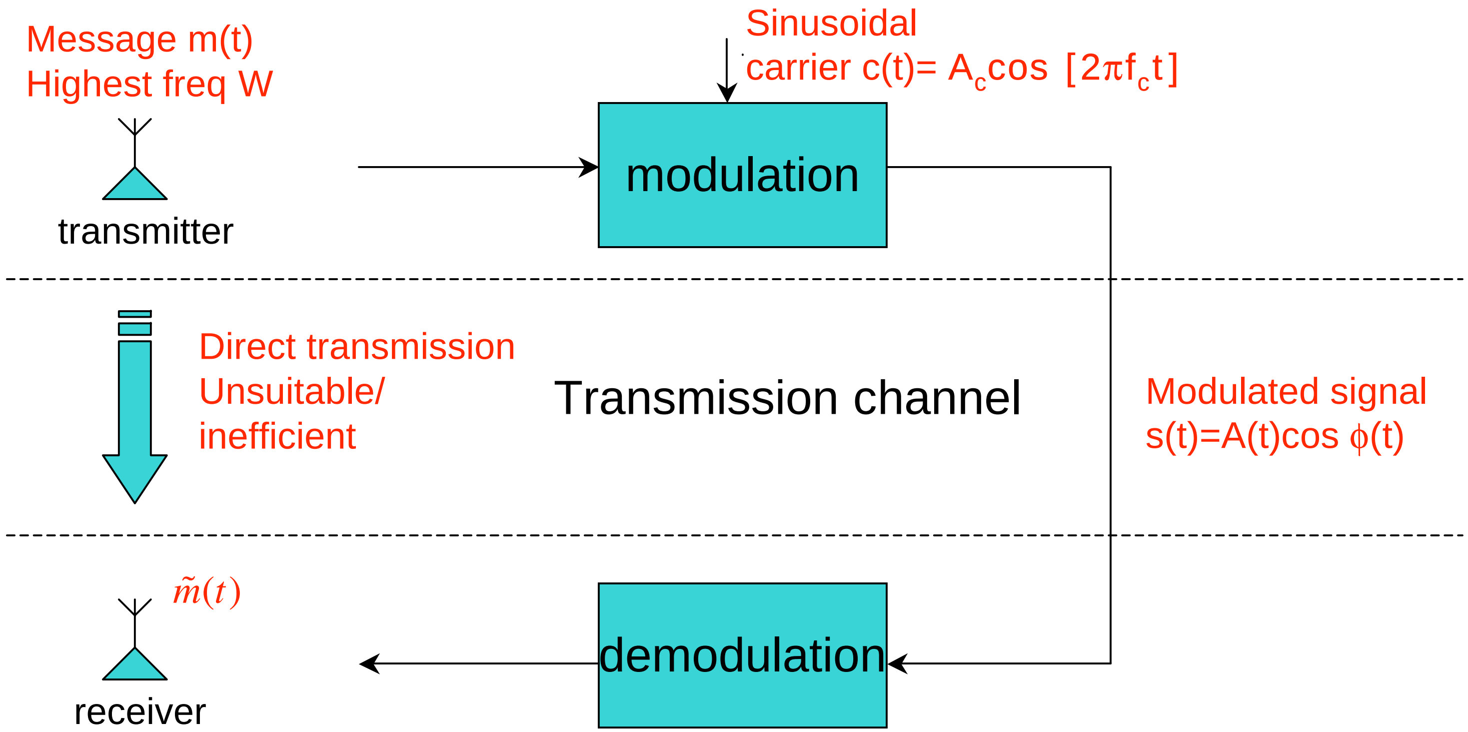
Outline

- Day 1
  - Angle Modulation
  - Frequency Modulation (FM)
    - Narrowband and Wideband FM
    - Transmission bandwidth
    - FM Stereo Mix
- Day 2
  - Phase-locked Loop (PLL)
  - Non-linear effects in FM receivers
  - Summary
- Day 3
  - Tutorial

Recall…What is modulation?



- s(t) obtained by varying *characteristic* of c(t) according to m(t)
  - *Amplitude* A(t)<-> Amplitude Modulation
  - *Angle* ϕ(t) <-> Angle Modulation

Recall…Amplitude Modulation

- $s(t) = A_c[1 + k_a m(t)]\cos 2\pi f_c t$
- Envelope of s(t) has *same* shape as m(t) provided:
  - $|k_a m(t)| < 1$
  - $f_c >> W$
- Easy and cheap to generate s(t) and reverse

Recall…Amplitude Modulation

- Drawbacks of AM
  - wasteful of power
    - transmission of carrier
  - wasteful of bandwidth
    - transmission bandwidth, $B_T = 2W$
- Improved resource utilization (power or bandwidth) traded-off with increased system complexity
- Angle modulation offers practical means of trading-off between power and bandwidth

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Angle modulation

- $s(t) = A_c \cos[\theta_i(t)], \quad f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$
  - Phase Modulation (PM)

$$\theta_i(t) = 2\pi f_c t + k_p m(t)$$

$$\Rightarrow \quad s(t) = A_c \cos[2\pi f_c t + k_p m(t)]$$

  - Frequency Modulation (FM)

$$f_i(t) = f_c + k_f m(t)$$

$$\Rightarrow \quad s(t) = A_c \cos[2\pi f_c t + 2\pi k_f \int_0^t m(t)dt]$$

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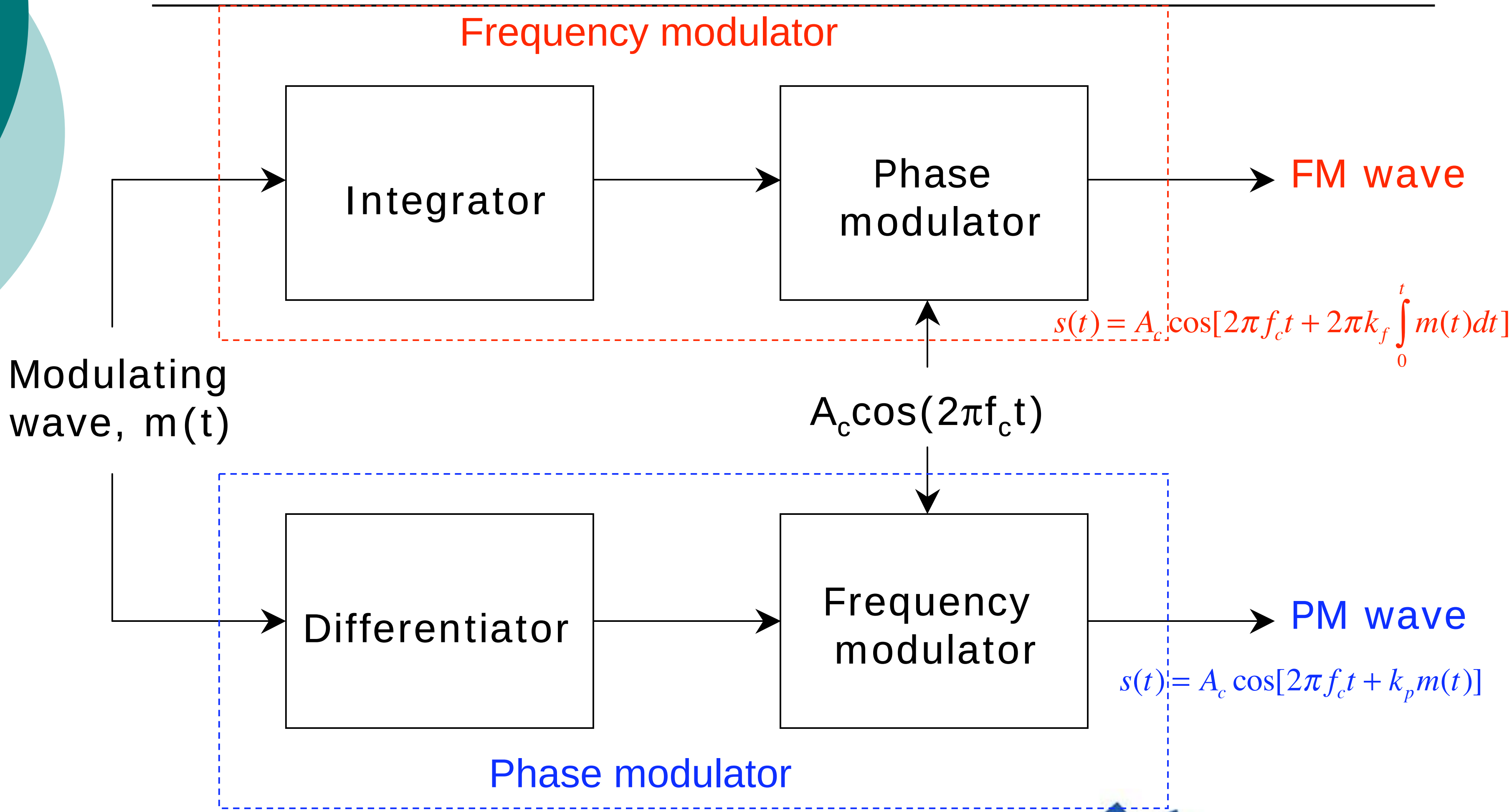
PM vs FM



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Frequency modulation

- FM signal: $s(t) = A_c \cos[2\pi f_c t + 2\pi k_f \int_0^t m(t)dt]$ Non-linear function of m(t)
- Consider single tone signal: $m(t) = A_m \cos 2\pi f_m t$
  - $$f_i(t) = f_c + k_f m(t)$$
    $$= f_c + k_f A_m \cos[2\pi f_m t]$$
    $$= f_c + \Delta f \cos[2\pi f_m t]$$ Frequency deviation
  - $$\theta_i(t) = 2\pi \int_0^t f_i(t)dt$$
    $$= 2\pi f_c t + \frac{\Delta f}{f_m} \sin[2\pi f_m t]$$
    $$= 2\pi f_c t + \beta \sin[2\pi f_m t]$$ Modulation index
  - $$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

Narrowband FM (β<<1)

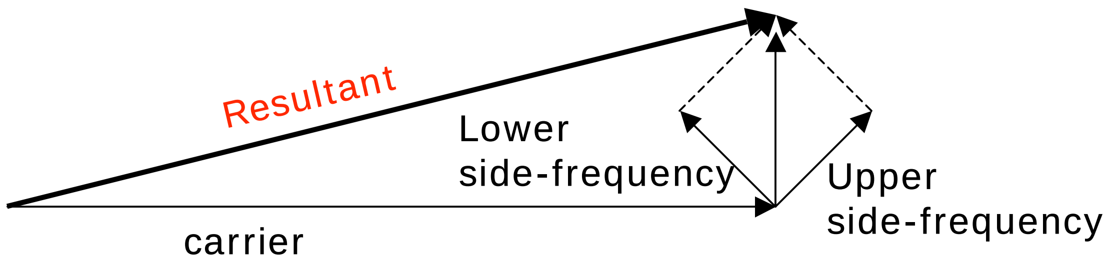
Expanding *s(t)*, we have:

$$s(t) = A_c \cos(2\pi f_c t)\cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t)\sin[\beta \sin(2\pi f_m t)]$$

≈ 1 $\quad\quad$ $\approx \beta \sin(2\pi f_m t)$

$$\text{If } \beta << 1, s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t)\sin(2\pi f_m t)$$

$$\approx A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c \{\cos[2\pi(f_c + f_m)t] - \cos[2\pi(f_c - f_m)t]\}$$



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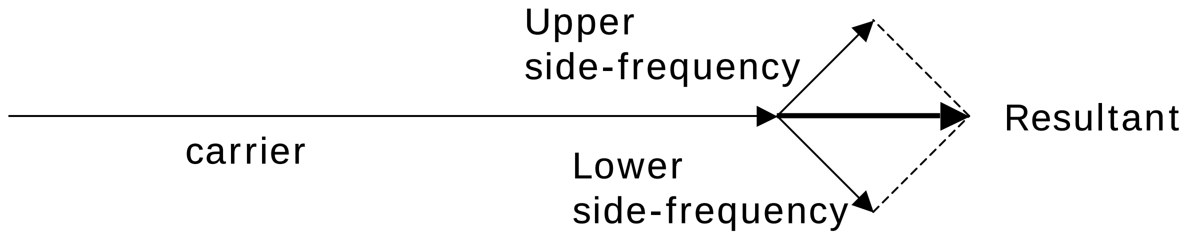
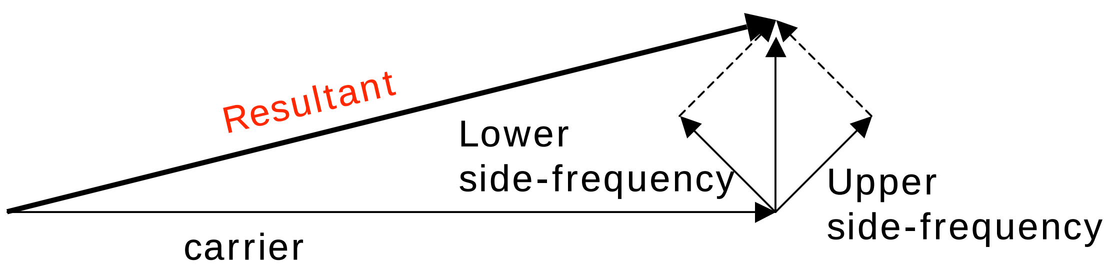
Comparison with AM

AM Signal



Narrow band FM Signal



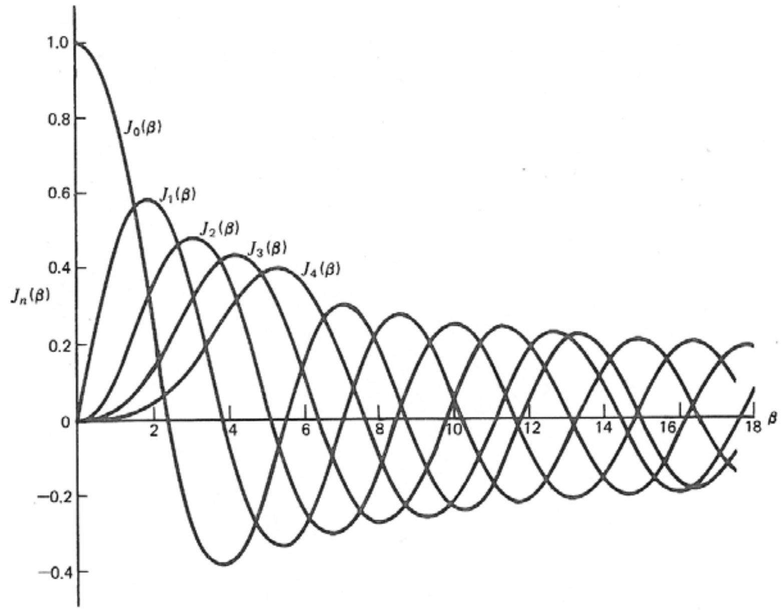
Wideband FM

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

$f_c >> f_m$

$$= \mathrm{Re}[A_c \exp(j2\pi f_c t + j\beta \sin(2\pi f_m t))]$$
$$= \mathrm{Re}[\tilde{s}(t)\exp(j2\pi f_c t)] \quad (*)$$

$$\tilde{s}(t) = A_c \exp[j\beta \sin(2\pi f_m t)]$$
$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\exp(j2\pi n f_m t)$$



nth order Bessel function of first kind

Subst. into (*), and applying FT:

$$S(f) = \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]$$

$$= A_c J_0(\beta)\delta(f - f_c) + \frac{A_c}{2} J_{\pm 1}(\beta)\delta(f - f_c \mp f_m) + \frac{A_c}{2} J_{\pm 2}(\beta)\delta(f - f_c \mp 2f_m) + \ldots$$

Carrier component

Side freq $f_c \pm f_m$

Side freq $f_c \pm 2f_m$

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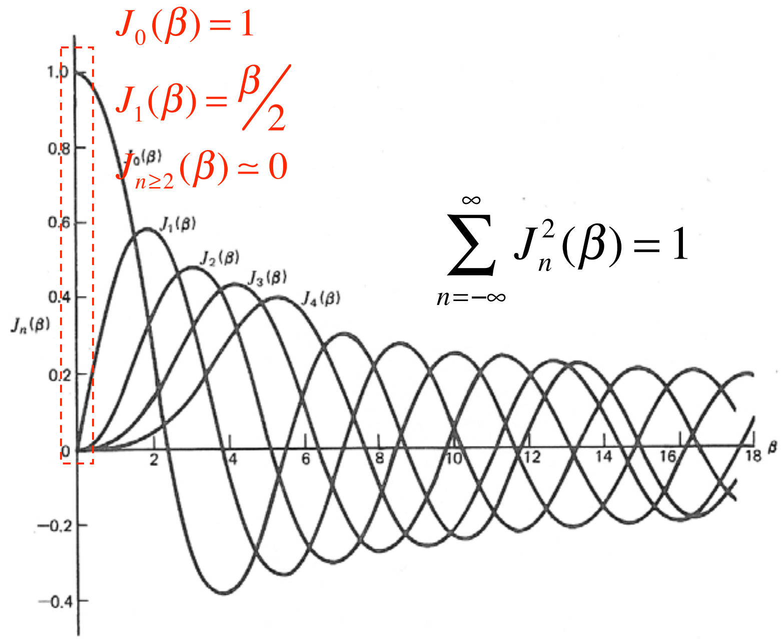
Observations

- Amplitude of carrier varies with $J_0(\beta)$
  - With AM, amplitude of carrier = A_c
- Special case: $\beta << 1$
  - Only $J_0(\beta)$, $J_1(\beta)$ ⇔ $f_c \pm f_m$ significant (narrowband FM)



nth order Bessel function of first kind

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]$$

$$= A_c J_0(\beta)\delta(f - f_c) + \frac{A_c}{2} J_{\pm 1}(\beta)\delta(f - f_c \mp f_m) + \frac{A_c}{2} J_{\pm 2}(\beta)\delta(f - f_c \mp 2f_m) + \ldots$$

Carrier component

Side freq $f_c \pm f_m$

Side freq $f_c \pm 2f_m$

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Example - fixed f_m, variable A_m

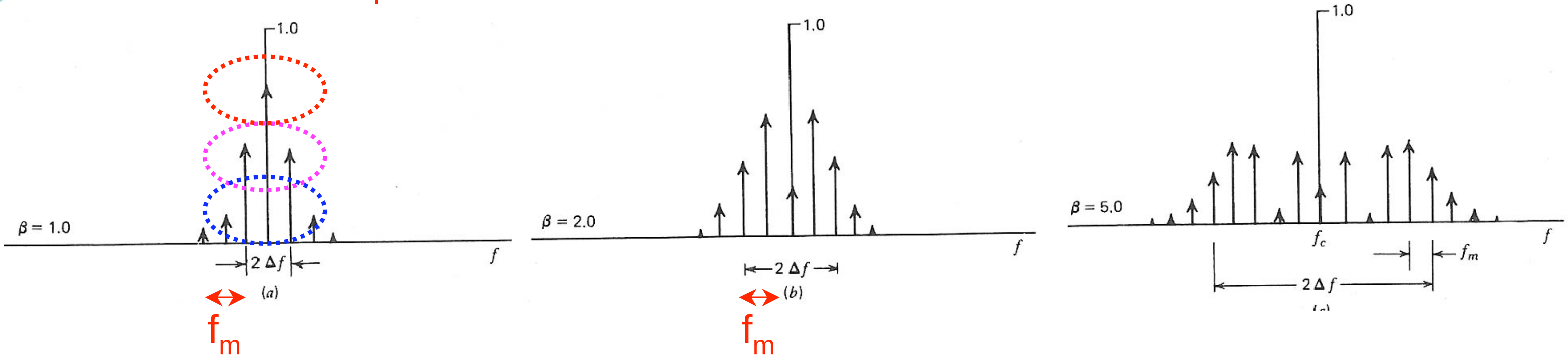
$$m(t) = A_m \cos[2\pi f_m t],\ \Delta f = k_f A_m,\ \beta = \frac{\Delta f}{f_m}$$

$$S(f) = A_c J_0(\beta)\delta(f - f_c) + \frac{A_c}{2} J_{\pm 1}(\beta)\delta(f - f_c \mp f_m) + \frac{A_c}{2} J_{\pm 2}(\beta)\delta(f - f_c \mp 2 f_m) + \ldots$$

Carrier component

Side freq $f_c \pm f_m$

Side freq $f_c \pm 2f_m$



$$\uparrow A_m \Leftrightarrow \uparrow \beta$$

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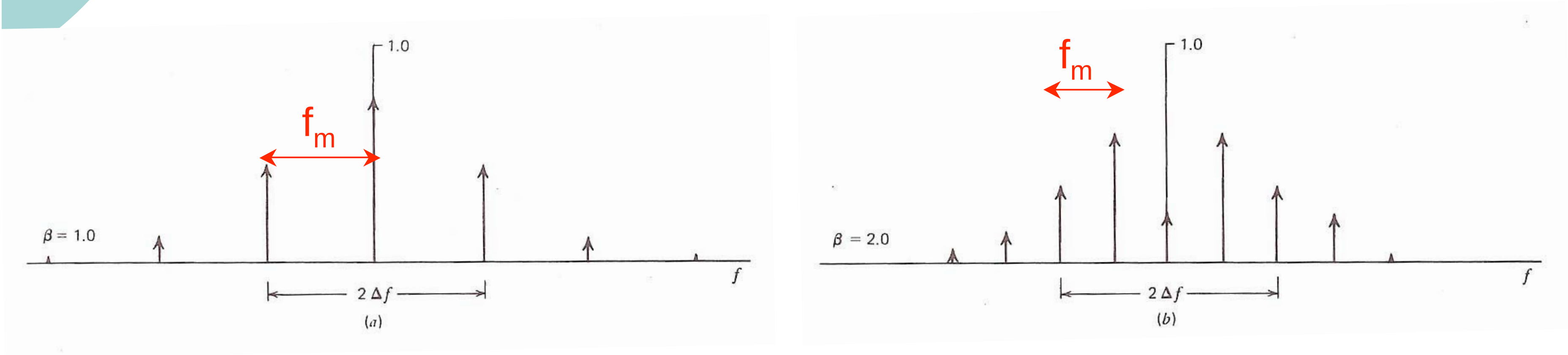
Example - fixed A_m, variable f_m

$$m(t) = A_m \cos[2\pi f_m t],\ \Delta f = k_f A_m,\ \beta = \frac{\Delta f}{f_m}$$

$$S(f) = A_c J_0(\beta)\delta(f - f_c) + \frac{A_c}{2} J_{\pm 1}(\beta)\delta(f - f_c \mp f_m) + \frac{A_c}{2} J_{\pm 2}(\beta)\delta(f - f_c \mp 2 f_m) + \ldots$$

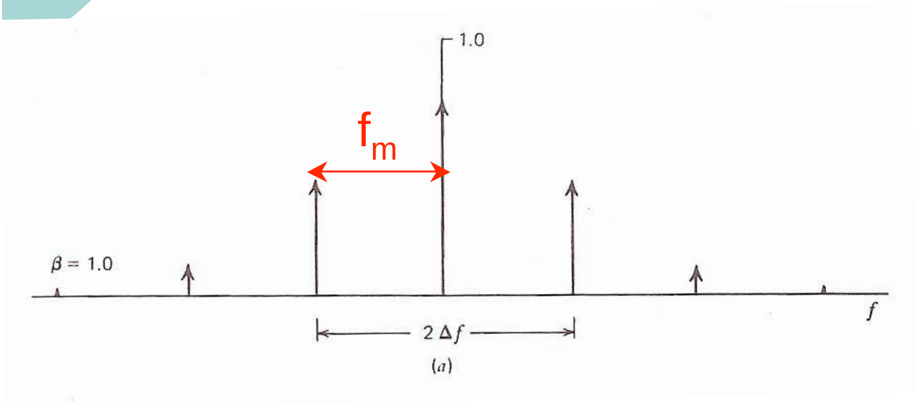


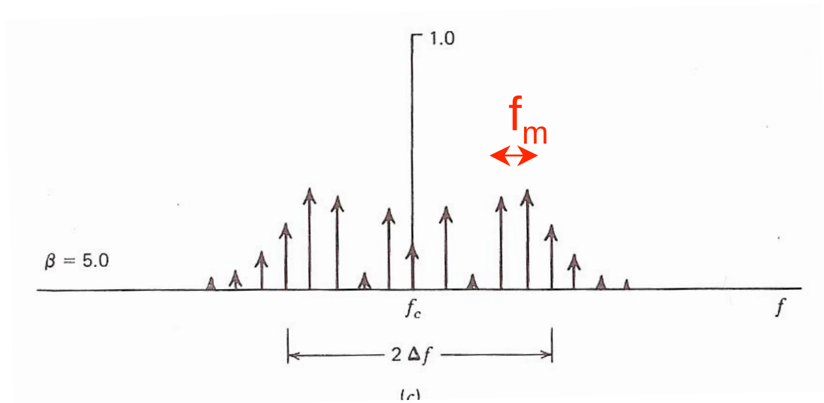
$\downarrow f_m \Leftrightarrow \uparrow \beta$

Example - fixed A_m, variable f_m

$$m(t) = A_m \cos[2\pi f_m t],\ \Delta f = k_f A_m,\ \beta = \frac{\Delta f}{f_m}$$

$$S(f) = A_c J_0(\beta)\delta(f - f_c) + \frac{A_c}{2} J_{\pm 1}(\beta)\delta(f - f_c \mp f_m) + \frac{A_c}{2} J_{\pm 2}(\beta)\delta(f - f_c \mp 2 f_m) + \dots$$





As $\beta \uparrow$, number of spectral lines within $f_c - \Delta f < |f| < f_c + \Delta f \uparrow$

As $\beta \rightarrow \infty$, the bandwidth of $s(t)$ approaches the limiting value of $2\Delta f$!!

[Note: For $\beta \ll 1$, bandwidth of $s(t) \simeq 2 f_m$ (As in AM)]

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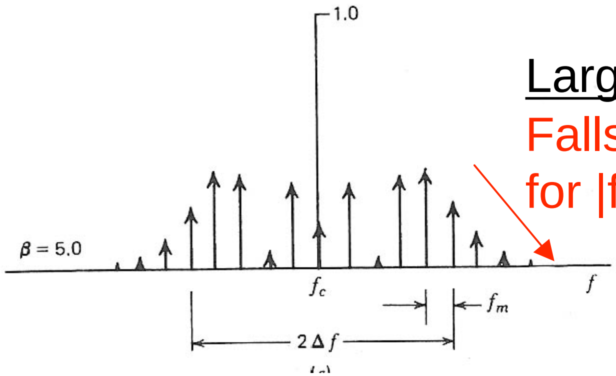
Transmission bandwidth

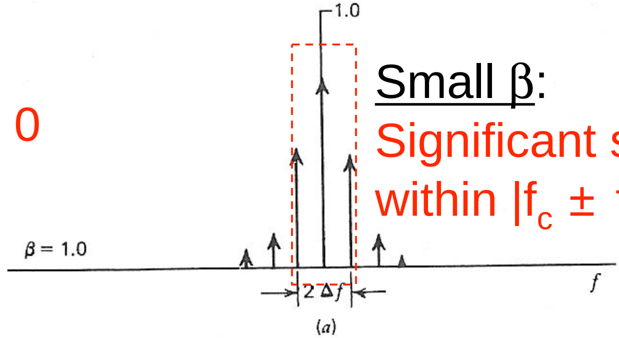
$$S(f)=\frac{A_c}{2}\sum_{n=-\infty}^{\infty}J_n(\beta)[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)]$$

$\Rightarrow$ transmission bandwidth = ∞!!!

But, effectively, *finite* number of side frequencies are significant



Large β:
Falls rapidly towards 0
for $|f-f_c|>2\Delta f$



Small β:
Significant sidebands
within $|f_c \pm f_m|$

$\Rightarrow$ Carson's rule: $B_{T,Carson} \simeq 2\Delta f(1+\frac{1}{\beta})$

Transmission bandwidth

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]$$

$\Rightarrow$ transmission bandwidth $= \infty$!!!

But, effectively, *finite* number of side frequencies are significant

$\Rightarrow$ retain up to $n_{\max}$ side frequencies s.t. $J_{n_{\max}}(\beta) \geq \varepsilon J_0(\beta)$

$\Rightarrow B_T = 2 n_{\max} f_m$

$B_{T,1\%} = 1\ \%$ bandwidth with $\varepsilon = 0.01$

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1 percent bandwidth of FM wave

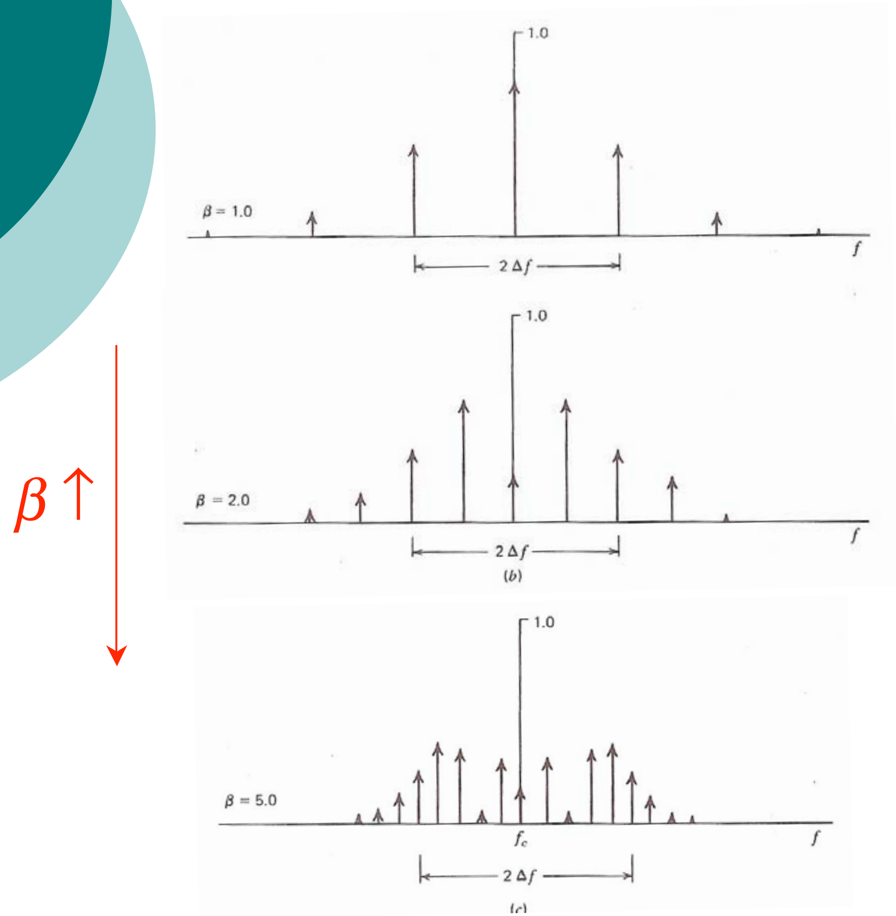

Table 3.1 Number of Significant Side Frequencies of a Wide-band FM Signal for Varying Modulation Index

Modulation Index β	Number of Significant Side Frequencies $2n_{max}$
0.1	2
0.3	4
0.5	4
1.0	6
2.0	8
5.0	16
10.0	28
20.0	50
30.0	70

As $\beta\uparrow, n_{max}\uparrow \Rightarrow B_{T,1\%}\uparrow$

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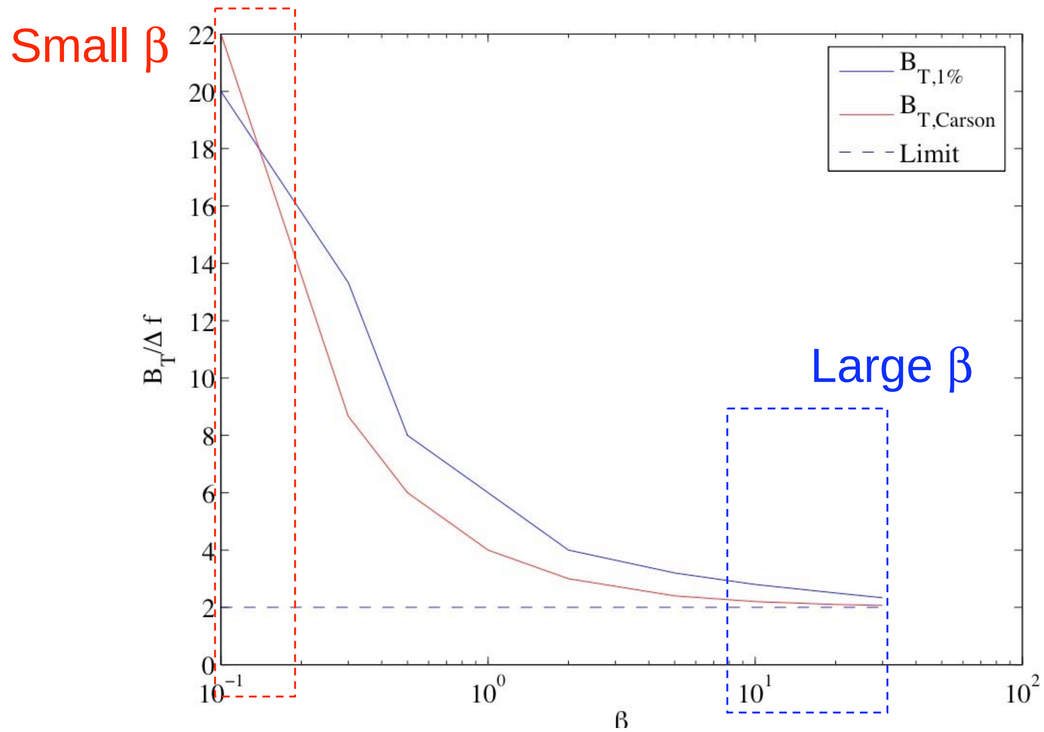
1 percent bandwidth of FM wave



Small values of β more extravagant in B_T than larger β!!

Practically, $B_{T,Carson} \leq B_T \leq B_{T,1\%}$

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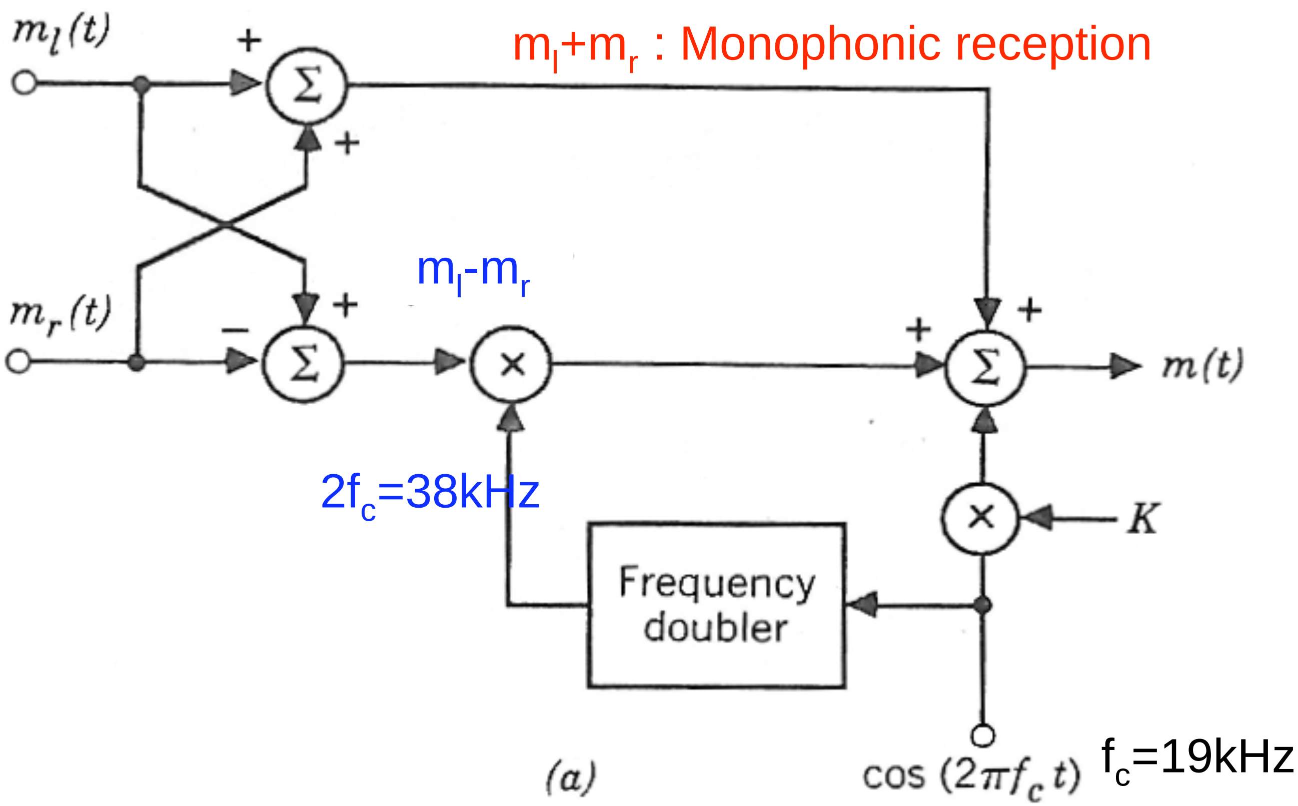
FM Stereo

- Transmit two separate signals via *same* carrier
  - 2 different sections of orchestra, e.g., vocalist and accompanist, to give spatial dimension to its perception
- Requirements
  - Must operate within allocated FM broadcast channels
  - Must be compatible with monophonic radio receivers

FM Stereo Mux



$$m(t) = [m_l(t) + m_r(t)] + [m_l(t) - m_r(t)]\cos(2\pi[2f_c]t) + K\cos(2\pi f_c t)$$

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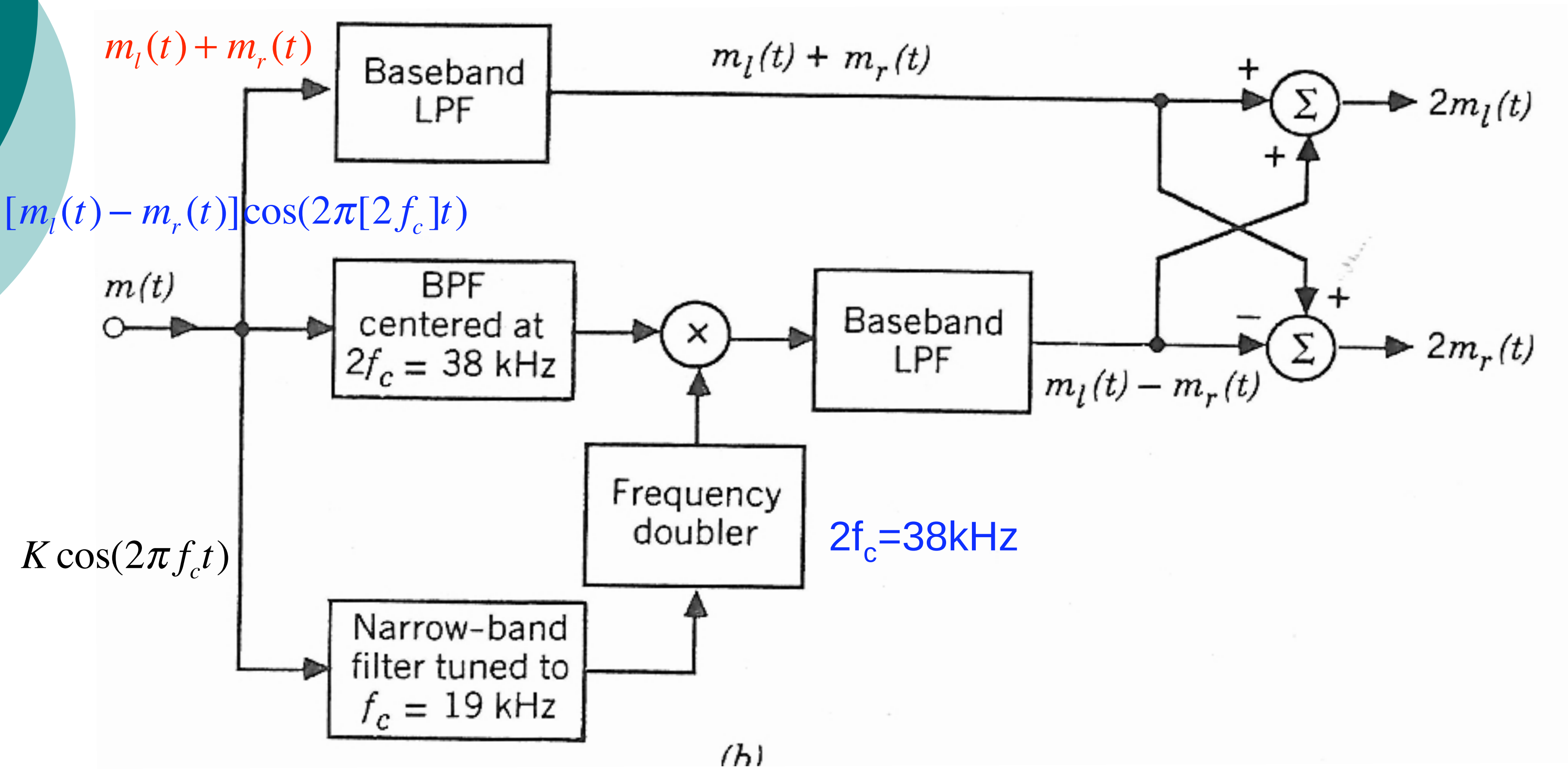
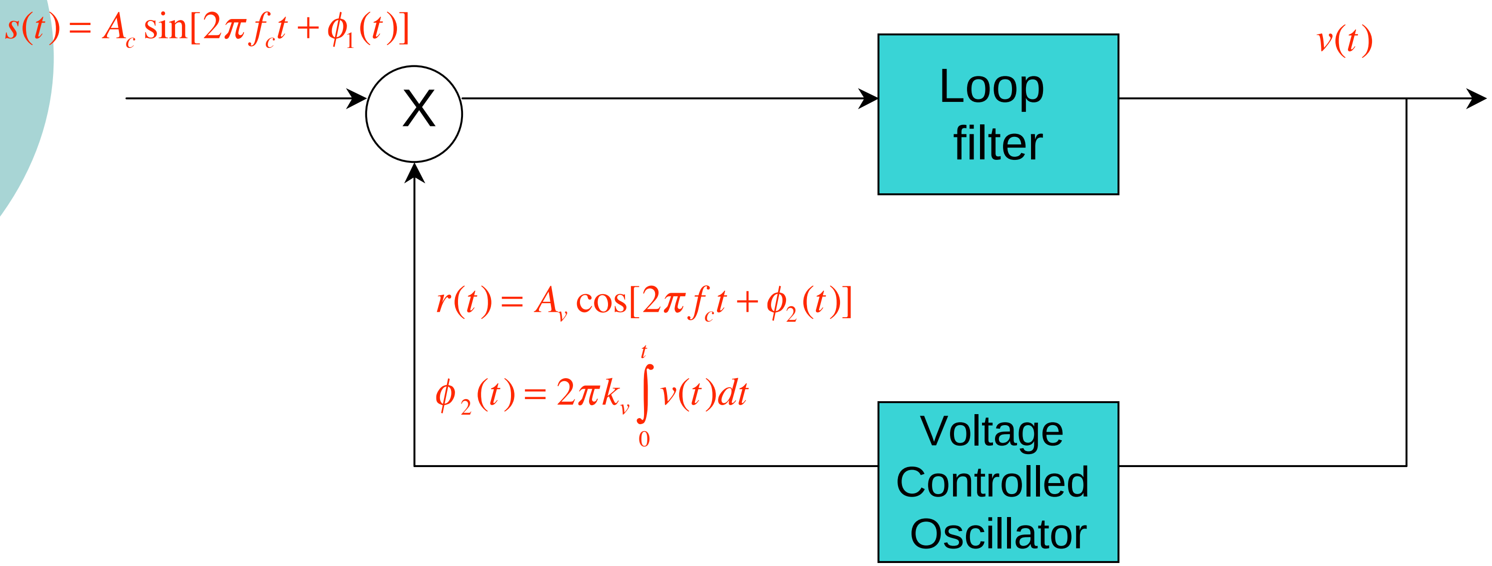
FM Stereo Demux



$$m(t) = [m_l(t)+m_r(t)] + [m_l(t)-m_r(t)]\cos(2\pi[2f_c]t) + K\cos(2\pi f_c t)$$

Phase Locked Loop (PLL)



- PLL for freq. demod
  - If s(t) is FM wave, obtain m(t) from v(t)
  - Require $\phi_1 \approx \phi_2 + 90^o$ <=> Phase lock!

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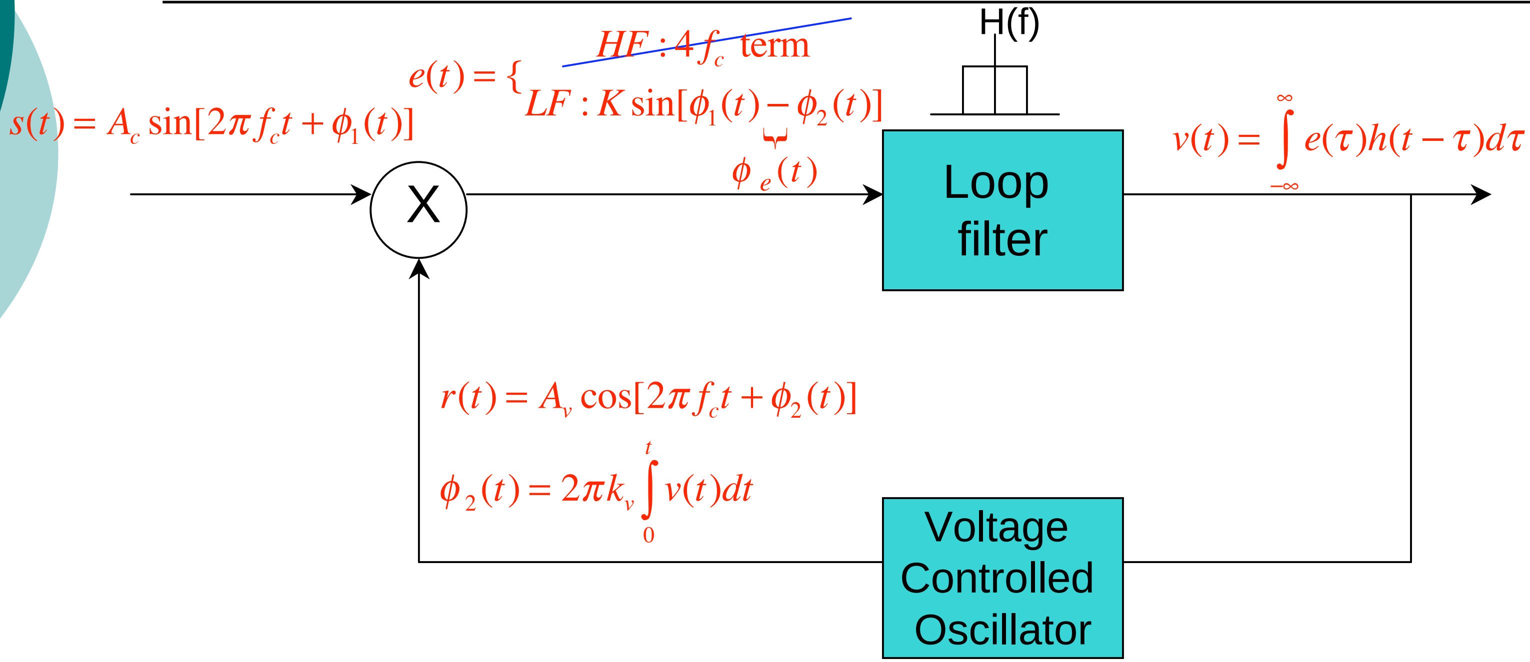
Phase Locked Loop (PLL)



- Dynamic behavior of PLL
  - $$\frac{d\phi_e(t)}{dt} = \frac{d\phi_1(t)}{dt} - 2\pi K_o \int_{-\infty}^{\infty} \sin[\phi_e(\tau)] h(t-\tau) d\tau,$$
    $$K_o = k_m k_v A_c A_v$$

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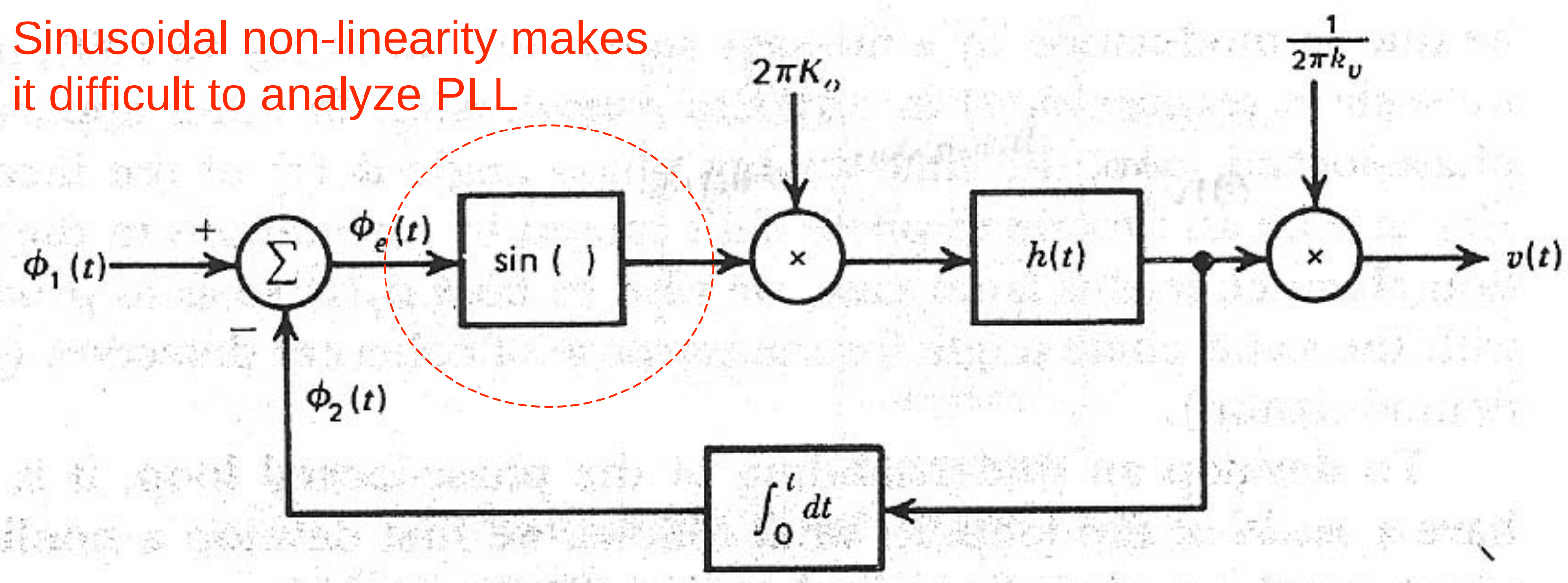
Non-linear PLL model

Sinusoidal non-linearity makes it difficult to analyze PLL



- Dynamic behavior of PLL
  - $$\frac{d\phi_e(t)}{dt} = \frac{d\phi_1(t)}{dt} - 2\pi K_o \int_{-\infty}^{\infty} \sin[\phi_e(\tau)]h(t-\tau)d\tau,$$

    $$K_o = k_m k_v A_c A_v$$

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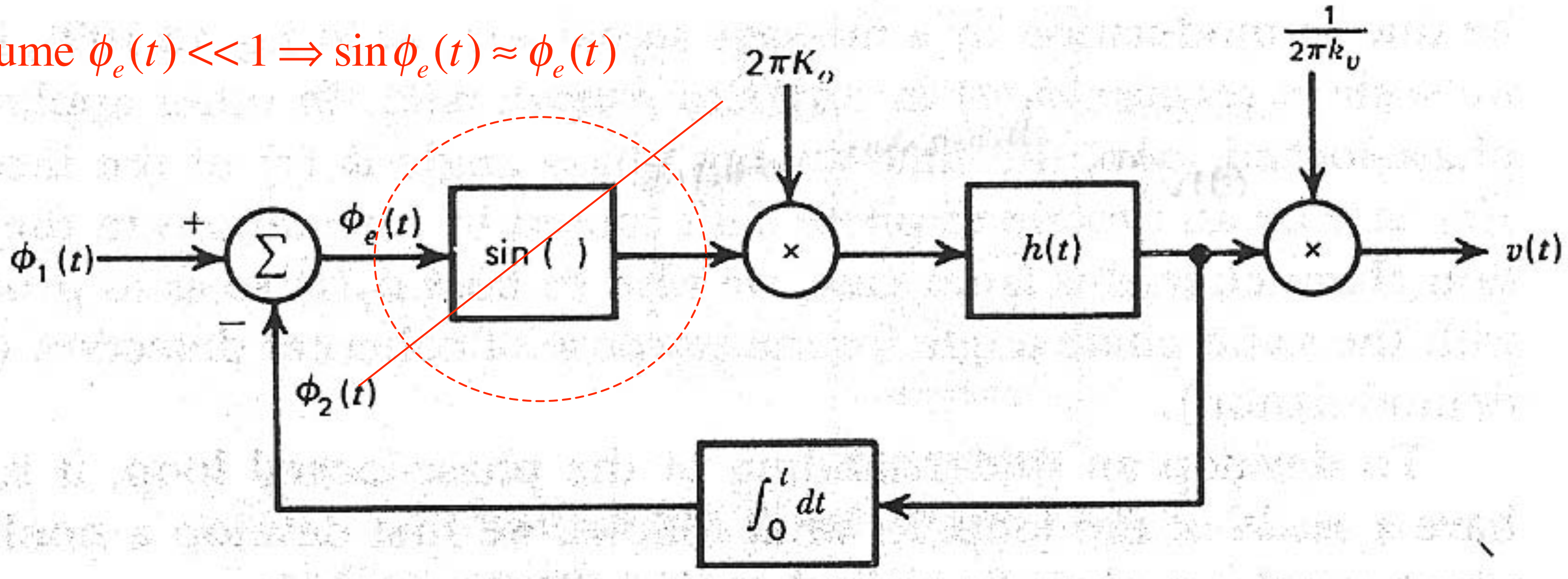
Non-linear PLL model

Assume $\phi_e(t) << 1 \Rightarrow \sin\phi_e(t) \approx \phi_e(t)$



- Linearized behavior of PLL
  - $$\frac{d\phi_e(t)}{dt} = \frac{d\phi_1(t)}{dt} - 2\pi K_o \int_{-\infty}^{\infty} \phi_e(\tau) h(t-\tau) d\tau,$$

    $$K_o = k_m k_v A_c A_v$$

FT can be applied!

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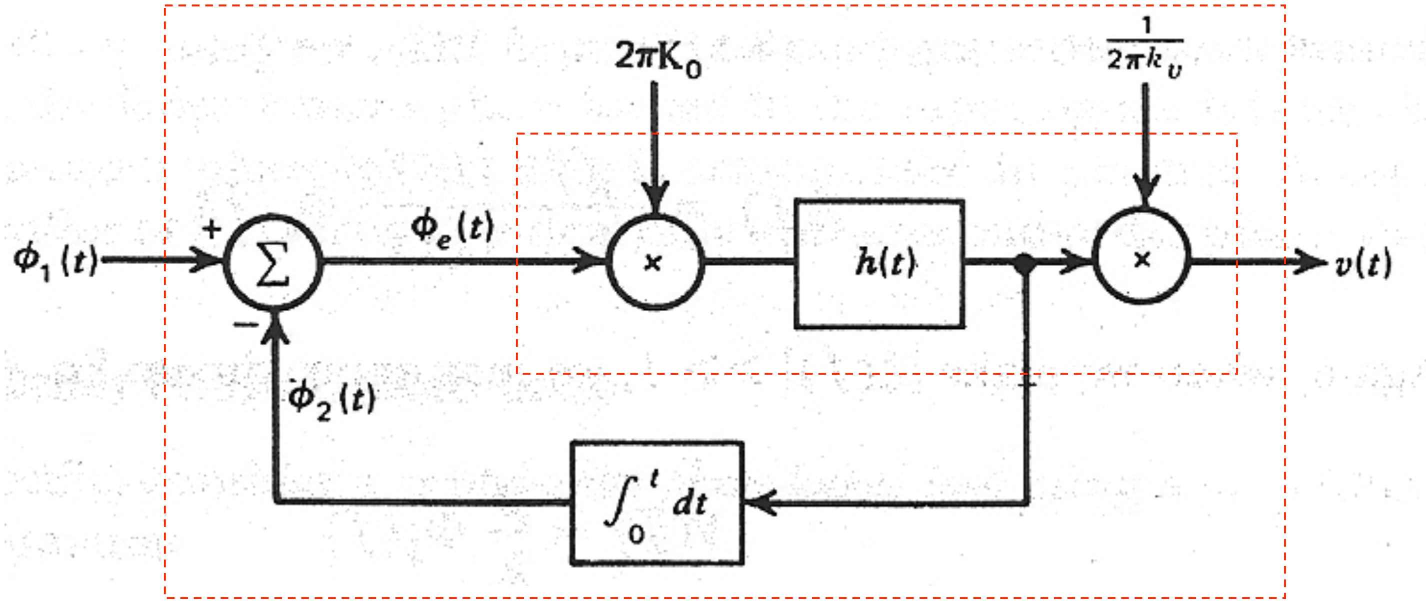
Linear PLL model



- $$\Phi_e(f) = \frac{1}{1+L(f)}\Phi_1(f),$$

  $$L(f) = K_o \frac{H(f)}{jf}$$ [Open-loop transfer function]

  $$V(f) = \frac{(jf/k_v)L(f)}{1+L(f)}\Phi_1(f)$$

- $$V(f) = \frac{K_o}{k_v}H(f)\Phi_e(f)$$

  $$= \frac{jf}{k_v}L(f)\Phi_e(f)$$

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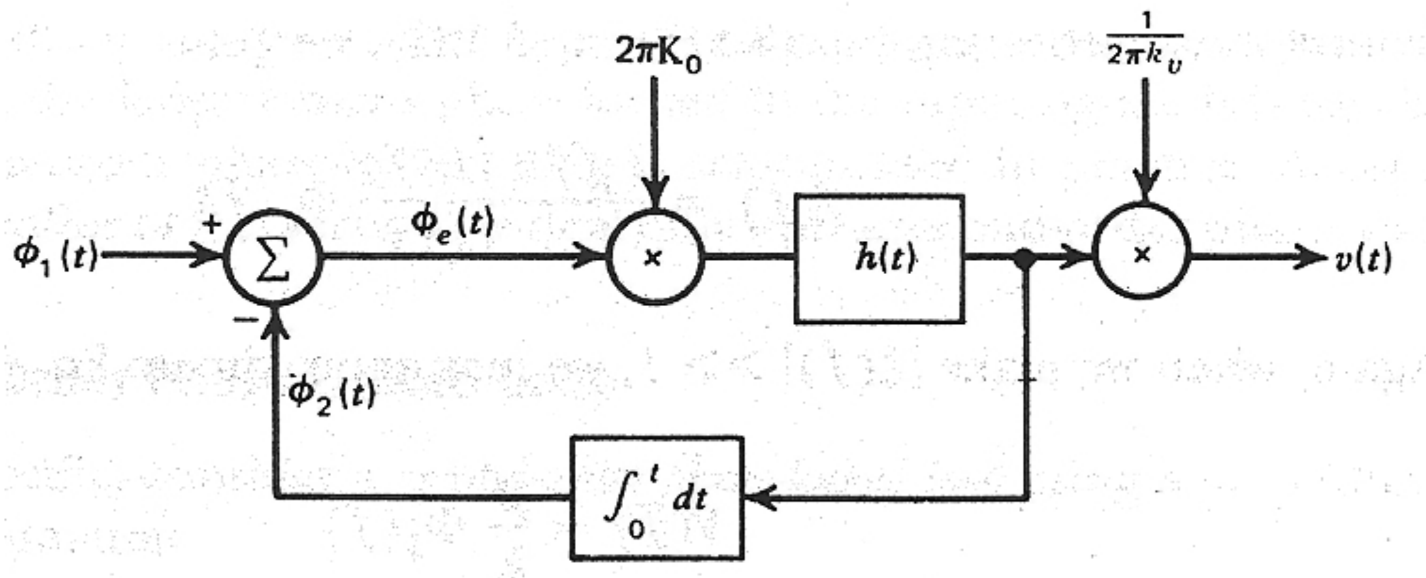
Phase-locked Linear PLL



$$V(f) = \frac{(jf / k_v)L(f)}{1 + L(f)}\Phi_1(f)$$

$$\Phi_e(f) = \frac{1}{1 + L(f)}\Phi_1(f)$$

$|L(f)| \gg 1$

$$V(f) \simeq \frac{jf}{k_v}\Phi_1(f)$$

$$\Rightarrow v(t) = \frac{1}{2\pi k_v}\frac{d\phi_1(t)}{dt}$$

$\Phi_e(f) \to 0 \Leftrightarrow$ Phase lock!!!

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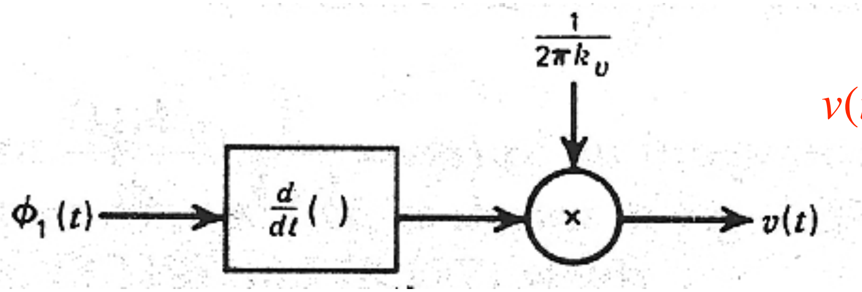
Phase locked linear PLL as frequency demodulator

If $s(t)$ is FM signal,

$$\Rightarrow \phi_1(t) = 2\pi k_f \int_0^t m(t)dt$$



$$v(t) = \frac{k_f}{k_v} m(t)$$

- If |L(f)|>>1
  - Linearized PLL model
  - Phase lock satisfied [$\phi_e \approx 0$]
  - $v(t) = \frac{1}{2\pi k_v} \frac{d\phi_1(t)}{dt}$
- Bandwidth of s(t) >> bandwidth of H(f) [m(t)]

Determines complexity of PLL

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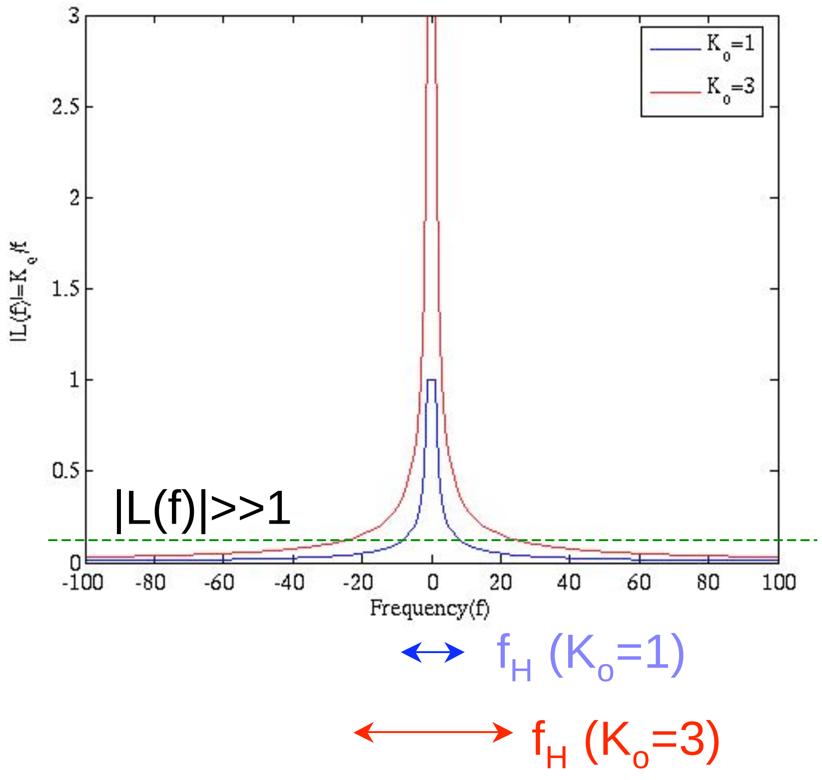
Design of H(f)

- First order
  - $H(f)=1$
  - $|L(f)|=K_o/f$

- Drawback
  - K_o controls both loop bandwidth and hold-in frequency range, f_H

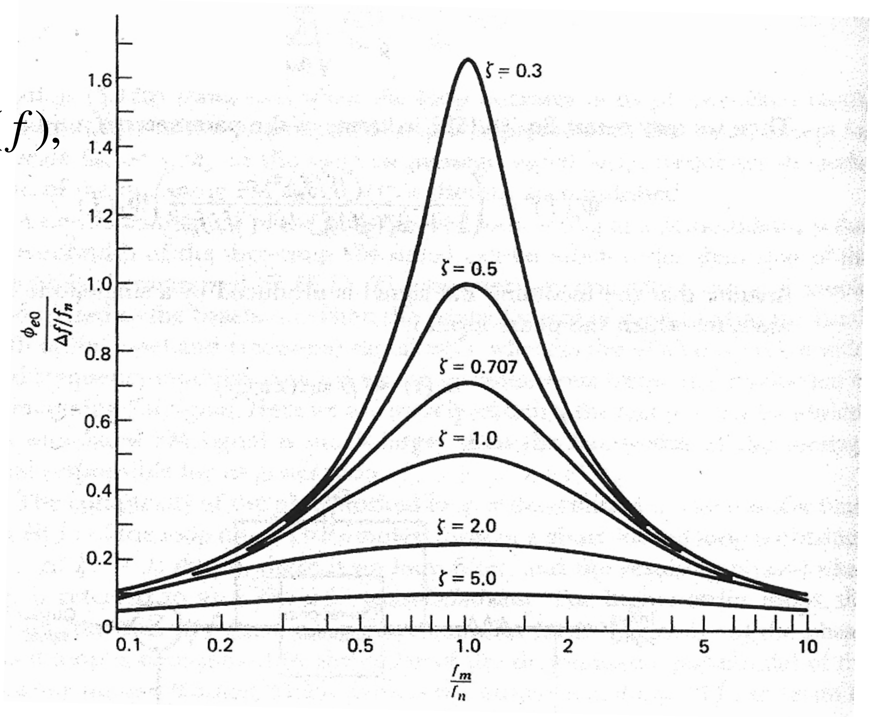


Second-order PLL

- $H(f) = 1 + \frac{a}{jf}$
- $\Phi_e(f) = \left(\frac{(jf/f_n)^2}{1 + 2\varsigma(jf/f_n) + (jf/f_n)^2} \right) \Phi_1(f),$

  $f_n = \sqrt{aK_o}$ Natural frequency

  $\varsigma = \sqrt{\frac{K_o}{4a}}$ Damping factor



- For $m(t) = A_m \cos 2\pi f_m t,$

  $\phi_1(t) = \beta \sin(2\pi f_m t)$

  $\Rightarrow \phi_e(t) = \phi_{e0} \cos(2\pi f_m t + \psi)$

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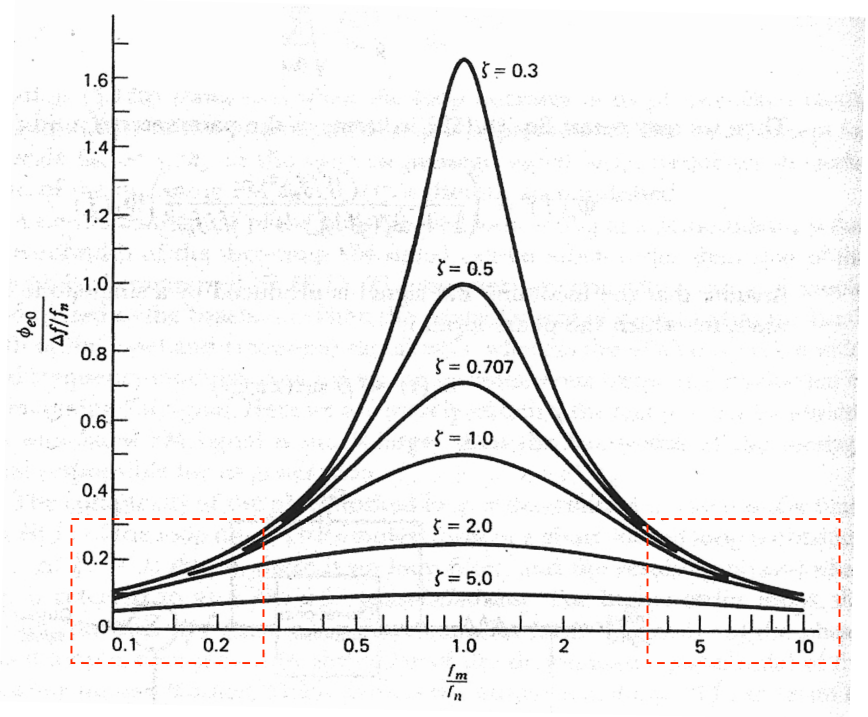
Second-order PLL

- With appropriate choice of (ζ, f_n), we can maintain ϕ_e small
  - Linear PLL model

- Rule of thumb:
  - Loop should remain locked if $\phi_{e0}(f_m = f_n) < 90^o$



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Non-linear effects in FM systems

$v_i(t)$ → memoryless communications channel → $v_o(t) = a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t)$

$$v_i(t) = A_c \cos[2\pi f_c t + \phi(t)],$$

$$\phi(t) = 2\pi k_f \int_0^t m(t)\,dt$$

$$v_o(t) = A_0$$
$$+ A_1 \cos[2\pi f_c t + \phi(t)]$$
$$+ A_2 \cos[4\pi f_c t + 2\phi(t)]$$
$$+ A_3 \cos[6\pi f_c t + 3\phi(t)]$$

- Assume $v_i(t)$ is FM signal
  - Δf = frequency deviation
  - W = highest freq. comp. of m(t)

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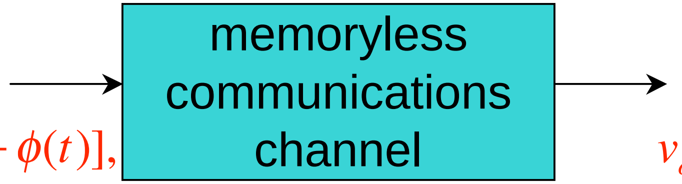
Non-linear effects in FM systems

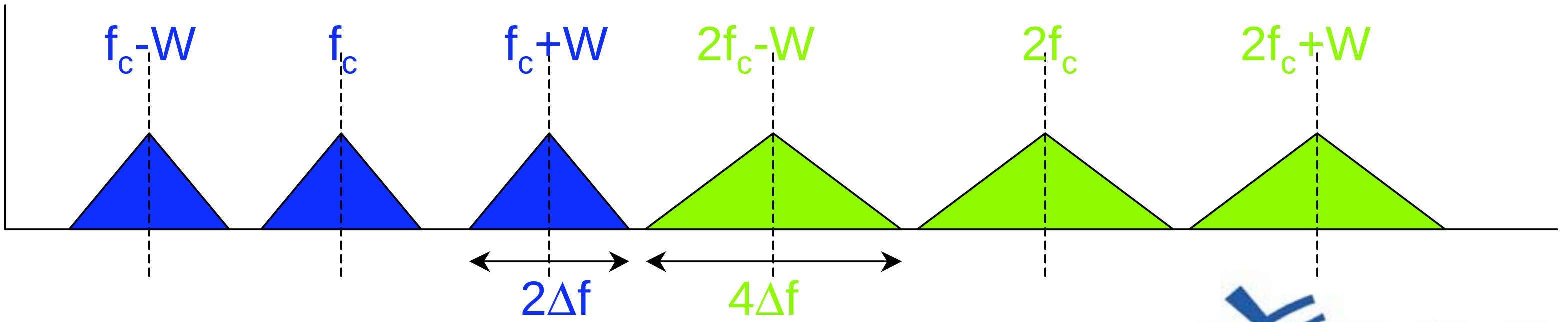


$$v_i(t) = A_c \cos[2\pi f_c t + \phi(t)],$$

$$\phi(t) = 2\pi k_f \int_0^t m(t)dt$$

$$v_o(t) = A_0 + A_1 \cos[2\pi f_c t + \phi(t)] + A_2 \cos[4\pi f_c t + 2\phi(t)] + A_3 \cos[6\pi f_c t + 3\phi(t)]$$

- Carson's Rule
  - $B_T \approx 2\Delta f$

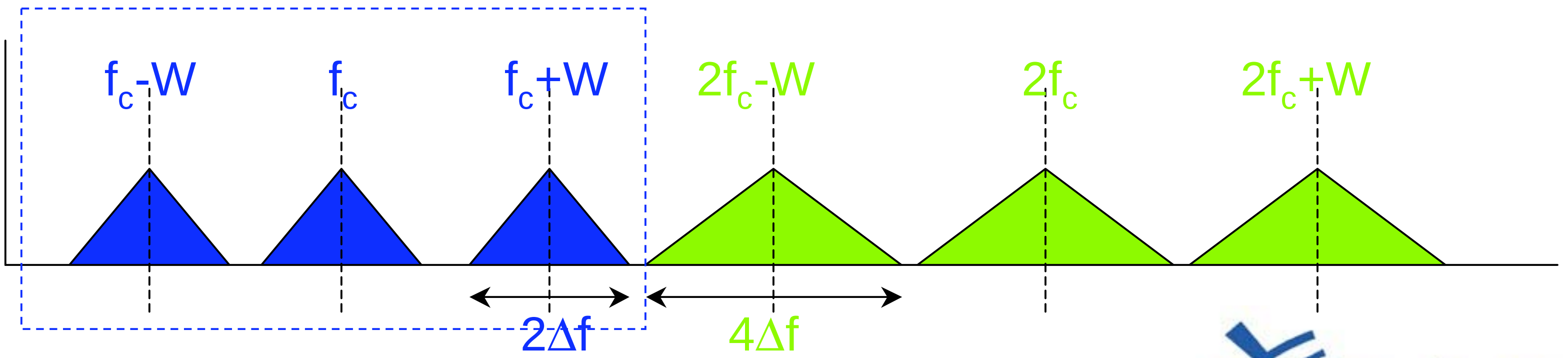


Non-linear effects in FM systems

- To separate out desired FM signal
  - $f_c + \Delta f + W < 2f_c - W - 2\Delta f$

  $\Rightarrow f_c > 3\Delta f + 2W$
  - Apply bandpass filter $[f_c - \Delta f - W,\ f_c + \Delta f + W]$

- $$v_o'(t) = (a_1 + \frac{3}{4} a_3 A_c^{\,2}) v_i(t)$$
  - Unlike AM, FM *not* affected by distortion due to channel with amp. non-linearities



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Summary

- Unlike AM, FM is non-linear modulation process
  - Spectral analysis is more difficult
  - Developed insight by studying single-tone FM
- Carson's rule for transmission bandwidth
  - $B_T = 2\Delta f(1+1/\beta)$
- Phase-Locked Loop for frequency demodulation