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A periodic-review inventory model with application to the continuous-review obsolescence problem

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Abstract

In this paper we consider a stochastic-demand periodic-review inventory model with sudden obsolescence. We characterize the structure of the optimal policy and propose a dynamic programming algorithm for computing its parameters. We then utilize this algorithm to approximate the solution to the continuous-review sudden obsolescence problem with general obsolescence distribution. We prove convergence of our approximation scheme, and demonstrate it numerically against known closed-form solutions of special cases.

Keywords: Inventory model; Sudden obsolescence; Continuous-review; Periodic-review

1. Introduction

The decision to hold inventory always incurs some kind of investment and the lost of money such as the inventory holding cost and the sudden obsolescence cost etc. The sudden obsolescence of the commodity makes the held inventory useless. Brown et al. (1964) point that "obsolescence may occur for a particular item, because the function served by that item is no longer required, because units are replaced by a suitable item which performs similar or identical functions or because of a program of systematic replacement by a substitute item." Obsolescence will result in a partial or total loss of value of the inventory on hand. In this paper we only treat the case of sudden obsolescence, which means that the items held in inventory lose all value at once and there is no salvage value.

In the literature there are many examples about the obsolescence phenomenon such as the spare parts management for military aircraft in Hadley and Whitin (1963), the fate of Swiss watches, music records, and breast implants in Joglekar and Lee (1993). Recently, some researchers study some new examples about

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the obsolescence issue in the inventory management such as the one in David et al. (1997), a central mapping unit that stocks military maps may face obsolescence when unforeseeable changes such as the redeployment of missiles occurs. Since the maps are expensive to manufacture and stock, it is desirable to determine optimal lot sizes to minimize the expected average total cost. Products which are subject to rapid changes in fashion or competition due to technological changes are particularly vulnerable to a sudden collapse of demand. For example, successive technological changes such as the computer market are frequent nowadays and might render existing stocks obsolete.

Song and Zipkin (1996) use an inventory-control model to address the obsolescence effect in the inventory management. Their model incorporates a Markovian sub-model to describe the uncertain events leading to obsolescence. Optimal stocking decisions must consider the effect of obsolescence. The traditional approach to the obsolescence is to add a cost component to the holding cost in the lot-sizing model. This approach is suggested by Naddor (1966), Tersine (1988), Masters (1991), Silver et al. (1998), and Cobbaert and van Oudheusden (1996), among others. On the other hand, Hadley and Whitin (1963) do not agree on this approach. They argue that it is much more unrealistic to attempt to include obsolescence costs in the carrying charge. As obsolescence costs incurs only at a single point in time and the holding cost incurs over a certain time interval, these two kinds of costs are quite different from each other. When obsolescence is an important consideration in the inventory management, an exact mathematical model should be used.

Recently, some research works have been done to handle the obsolescence cost exactly such as Joglekar and Lee (1993), David and Mehrez (1995), and David et al. (1997) for a continuous-review model under the average cost criterion. Masters (1991) studies the EOQ model with an exponential obsolescent distribution, he finds that the traditional approach of adding a component cost to the holding cost is appropriate. Both Joglekar and Lee (1993) and David and Mehrez (1995) provide an exact formulation for the same model. Joglekar and Lee (1993) find that the estimation error of the approximation approach of Masters (1991) is significant in some situations. Van Delft and Vial (1996) study the same model by using the discounted cost criterion. But they only can solve some special cases of the obsolescent distributions. To our knowledge, there is no efficient computational approach for solving the continuous-review model under a general obsolescent distribution.

There are some periodic-review models with obsolescence in the literature, such as Hadley and Whitin (1962), Brown et al. (1964), and Pierskalla (1969). All these research works except Pierskalla (1969) only present periodic-review models. Pierskalla (1969) considers both the loss of excess demand and the backlogging of excess demand in the case of the zero fixed setup cost for each replenishment. The author shows that the base stock policy is optimal and some characteristics of the model are provided. In the case of positive fixed setup cost, the author merely mentions that the optimal policy is (s, S) policy without any proof. No efficient algorithm is provided for the computation of the parameters of the optimal (s, S) policy.

In this paper, we first characterize the structure of the optimal policy for a stochastic periodic-review model with sudden obsolescence. Then a dynamic programming algorithm is developed for computing the parameters of this policy. Next, we utilize this algorithm to provide an approximate solution-scheme to the deterministic demand, EOQ-like, continuous-review sudden obsolescence problem with general obsolescence distribution, and prove its convergence. As a benchmark to our approach, we compare it numerically against special cases of closed-form solutions of the deterministic-demand problem that are available in the above sources.

The remainder of this paper is organized as follows. In Section 2, we present a stochastic periodic-review inventory model with sudden obsolescence. Then, we characterize the structure of the optimal policy and a dynamic programming algorithm is provided for computing the parameters of this policy. In Section 3, we present a discrete-time approximation scheme for a continuous-review model. Some numerical results based on this approach are reported in Section 4. Section 5 concludes the paper.

2. The periodic-review model and a solution algorithm

In the following, we first present a stochastic periodic-review inventory model with sudden obsolescence. Then a dynamic programming algorithm is provided for computing the optimal policy.

Suppose that the planning horizon $[0, H]$ is finite and it is equally divided into T periods. The discrete demand which happens at the beginning of period j is a random variable ξ_j. We assume that all ξ_j $(1 \leqslant j \leqslant T)$ are i.i.d. and bounded by a constant integer D. We denote $\Pr(\xi_j = d)$ by p_d for $0 \leqslant d \leqslant D$.

The obsolescence point is at the end point of some period. The probability of obsolescence at t_j, the end point of period j, is denoted by q_j $(1 \leqslant j \leqslant T)$, hence $\sum_{j=1}^{T} q_j = 1$. Without loss of generality, we assume $t_0 = 0$, $q_0 = 0$, and $q_T > 0$. Let $Q(j) = \sum_{l=0}^{j} q_l$ be the cumulative distribution of obsolescence and $\overline{Q}(j) = 1 - Q(j)$ for $1 \leqslant j \leqslant T$. For any $\tau = l\frac{H}{T}$, an integral multiple of one period length, let $\overline{Q}(l|j) = \frac{\overline{Q}(j+l)}{\overline{Q}(j)}$ be the conditional probability of obsolescence given that it does not occur at the end of period j and survives another l periods. For convenience, we define $\overline{Q}(l|T) = 0$ for any $l > 0$.

The purchasing cost for each replenishment with $x(> 0)$ units consists of a fixed cost $K(\geqslant 0)$ and a variable cost ax where $a(\geqslant 0)$ is the unit purchasing cost. Zero lead time is assumed. The excess demand in each period will be backlogged. The holding and backlogging costs will be charged at the end of the period after the period's demand has occurred, but prior to delivery of stock for the next period. The per unit per period holding cost and the per unit per period backlogging cost are denoted as h and b, respectively.

Let $L(y)$ be the expected holding and shortage cost in each period given the inventory level y after ordering but before demand occurs. The formulation of $L(y)$ can be expressed as

$$L(y) = \begin{cases} \sum_{d=0}^{y} h(y-d)p_d + \sum_{d=y+1}^{D} b(d-y)p_d, & y \geqslant 0, \\ \sum_{d=0}^{D} b(d-y)p_d, & y < 0. \end{cases}$$

Let $f_j(x)$ be the optimal cumulative expected cost from period j until period T given the initial inventory level x at the beginning of period j and obsolescence has not occurred in prior periods. If the obsolescence occurs at the end of period j, the holding and backlogging cost in period j is $L(y)$. Otherwise, the total cost is $L(y) + \sum_{d=0}^{D} f_{j+1}(y-d)p_d$ if $j < T$. Thus, the expected total cost from period j to period T is

$$g_j(y) = L(y) + \overline{Q}(1|j-1) \sum_{d=0}^{D} f_{j+1}(y-d)p_d. \tag{1}$$

Let $\delta(x) = 1$ if $x > 0$ and $\delta(0) = 0$. For convenience, we assume $f_{T+1}(x) = 0$ for any x. We have the following optimality recursive equations:

$$f_j(x) = \min_{y \in [x, +\infty)} \{K\delta(y-x) + a(y-x) + g_j(y)\} \tag{2}$$

for $j = 1, 2, \ldots, T$.

By using the result in Chan and Song (2003), we know that $f_j(x)$ is K-convex on $(-\infty, +\infty)$ and the optimal ordering policy in period j is (s^j, S^j) policy for any $1 \leqslant j \leqslant T$. Thus, the optimal policy for $f_j(x)$ is characterized by a pair of critical numbers s^j and S^j where $s^j < S^j$. If $x \leqslant s^j$, order $(S^j - x)$, and if $x > s^j$, do not order.

In order to design an efficient algorithm to compute the parameters of the optimal policy, we find that some results in the literature are very useful. One result is about the $X - Y$ band structure of the optimal policy for a periodic-review model. It is characterized by Chen and Lambrecht (1996) and reproduced here. Let x_L and S^T be the minimizers of $L(y)$ and $U(y) = ay + L(y)$, and $s^T(\leqslant S^T)$ be the maximum point at which $U(s^T) \geqslant U(S^T) + K$. For any initial inventory level x at the beginning of period j, the optimal ordering quantity $O_j(x)$ satisfies $O_j(x) = 0$ if $x \geqslant x_L$ and $O_j(x) = S^j - x$ if $x \leqslant s_T$. The second result is about the upper bound of S^j. If S^j is not unique, we choose the smallest one. Igelhart (1963) show that $S^j \leqslant M$ for all $1 \leqslant j \leqslant T$ where $M = \min_{y \geqslant x_L} \{y | L(y) \geqslant L(x_L) + K\}$.

Remark. Although the above results are proved under the assumption of the constant discount factor $0 < \alpha \leqslant 1$ for all $1 \leqslant j \leqslant T$, here $\alpha = \overline{Q}(1|j-1)$ and it is period dependent (not a constant), but all the proofs are valid.

Let $A = s^T$ and $B = M$. By using the above results we are ready to design a dynamic programming algorithm for the computation of s^j and S^j, $1 \leqslant j < T$.

Dynamic Programming (DP) Algorithm:
Step 1. Find s^T, S^T, x_L, and M.
Step 2. For $j = T-1$ to $j = 1$ (computing backwardly): Compute the values of $g_j(y)$ on $[A, B]$, find the smallest minimizer S^j of $g_j(y)$ on $[A, B]$, and find the maximal s^j on $[A, S^j]$ such that $g_j(s^j) \geqslant K + g_j(S^j)$.

In Step 1, the definitions of the parameters s_T, S_T, x_L, and M are described above. They can be obtained by using the functions $L(y)$ and $U(y)$. In Step 2, we need to compute the values of $g_j(y)$ on $[A, B]$ by using Eq. (1). After that, the minimizer S_j of $g_j(y)$ and the the maximal s^j on $[A, S^j]$ such that $g_j(s^j) \geqslant K + g_j(S^j)$ can be found on $[A, B]$. For any given y, suppose that the computational effort of evaluating $L(y)$ is $O(1)$, then the computational complexity in Step 1 is $O(B - A)$. The computational complexity in Step 2 is $O(T)$. Hence, the overall computational complexity is $O(T)$.

3. Approximating the EOQ model with general obsolescence distribution

In this section, we first present a continuous-review inventory model with sudden obsolescence, then we propose a discrete approximation approach for the computation of a good policy. Furthermore, we show that this discrete approximation scheme is convergent as the number of periods tends to infinity.

3.1. The EOQ model with sudden obsolescence

The well known EOQ model with sudden obsolescence can be described as follows. The cost parameters and demand process are the same as in the EOQ model, combined with an exogenous obsolescent process with a known distribution $G(t)$, $0 \leqslant t \leqslant H$. We assume that $G(t)$ is continuous. Let $\overline{G}(t) = 1 - G(t) > 0$ for any $0 \leqslant t < H$ and $\overline{G}(\tau|t) = \frac{\overline{G}(t+\tau)}{\overline{G}(t)}$, the conditional probability of living an extra time τ at t. The replenishment lead time is zero and shortage is not allowed.

Let μ be the deterministic continuous demand rate. The cost components for each replenishment are a fixed cost $K > 0$, a purchasing cost per unit $a > 0$, and the holding cost rate $h > 0$. Therefore the decision epochs occur only at the beginning of the planning horizon or when inventory hits zero.

David et al. (1997) is the latest research work for the EOQ model with sudden obsolescence. Let $V(t)$ be the minimal expected cumulative cost starting from the time t with zero stock given that there is no obsolescence until time $t (\geqslant 0)$. The determination of $V(t)$ also specifies the optimal decision at t. David et al. (1997) show that

$$V(t) = \min_{\tau > 0} \left\{ K + a\mu\tau + \overline{G}(\tau|t)\left[\frac{h\mu\tau^2}{2} + V(t+\tau)\right] + \int_{\theta=0}^{\tau} h\mu\left(\tau\theta - \frac{\theta^2}{2}\right)\frac{\mathrm{d}G(t+\theta)}{\overline{G}(t)} \right\}. \tag{3}$$

In Eq. (3), the time parameter τ is the decision variable. The total purchasing cost is $K + a\mu\tau$. If no obsolescence occurs during the planned horizon τ, the holding cost is $\frac{h\mu\tau^2}{2}$ and the remaining total optimal cost is $V(t+\tau)$. If the obsolescence occurs at $t+\theta$, $0 \leqslant \theta \leqslant \tau$, the holding cost is $h\mu(\tau\theta - \frac{\theta^2}{2})$. Therefore the expected holding cost is $\int_{\theta=0}^{\tau} h\mu(\tau\theta - \frac{\theta^2}{2})\frac{\mathrm{d}G(t+\theta)}{\overline{G}(t)}$. If H is finite, let $r = H - t$. Then Eq. (3) can be rewritten as

$$V(r) = \min_{0<\tau\leqslant r}\left\{K + a\mu\tau + \overline{G}(\tau|H-r)\left[\frac{h\mu\tau^2}{2} + V(r-\tau)\right] + \int_{\theta=0}^{\tau} h\mu\left(\tau\theta - \frac{\theta^2}{2}\right)\frac{\mathrm{d}G(H-r+\theta)}{\overline{G}(H-r)}\right\}. \quad (4)$$

David et al. (1997) obtain an optimal ordering policy for some special cases such as the cases of deterministic, exponential, and uniform obsolescence. Since the above equation of $V(t)$ is continuously recursive, it is difficult to solve and obtain some closed-form results for general obsolescent distribution.

3.2. The discrete-time approximation scheme

The discrete-time approximation scheme can be described as follows. First we equally partition one unit time into n periods and hence the planning horizon $[0, H]$ is divided into $T = nH$ periods. Then, by adjusting the unit holding cost per period to $\frac{\mu}{n^2}h$, we may regard the demand at the beginning of each period to be one unit and, accordingly, the per unit purchasing cost is also adjusted to $\frac{\mu}{n}a$. Finally, we accumulate the obsolescent probability in each period to the end point of this period, the obsolescent probability q_j at the end of period j is $q_j = G(\frac{j}{n}) - G(\frac{j-1}{n})$, $1 \leqslant j \leqslant nH$. Thus, we have generated a corresponding periodic-review model from the EOQ model with sudden obsolescence.

Next, we apply the DP algorithm to find the optimal policy (s^j, S^j), $1 \leqslant j \leqslant nH$, of this periodic-review model, and apply this policy to the original EOQ model with sudden obsolescence. The approximation cost of this policy via Eq. (3) can be computed easily. As this policy is just one of the feasible policies, this approximation cost will be larger than or equal to the optimal cost of the EOQ model with sudden obsolescence.

An Example: In order to illustrate the above approximation scheme, we choose $\mu = 1$, $H = 9$, $K = 20$, $a = 6$, $h = 0$, and the obsolescent distribution is uniform on $[0, H]$ in the EOQ model with sudden obsolescence. By taking $n = 1$, each unit time is considered as one period in the discrete approximation. Hence we have 9 periods in the periodic-review model. Then the obsolescent distribution in this periodic-review model is given by: $q_0 = 0$ and $q_j = \frac{1}{9}$ for $1 \leqslant j \leqslant 9$. The per period demand is taken to be 1. The per period per unit holding cost and per unit purchasing cost are 0 and 6, respectively. The backlogging is not allowed (or the backlogging cost is infinity). Applying the DP algorithm for this periodic-review model, we obtain s^j and S^j for all j $(1 \leqslant j \leqslant T)$ which are given in Table 1. Then we use this (s^j, S^j) policy to compute the approximation cost of the EOQ model with sudden obsolescence.

In Table 1, AppC^j is the approximation cost over $[j, H]$ obtained by applying the optimal (s, S) policy of the approximated periodic-review model to the EOQ model. OptC^j is the optimal cost of the EOQ model over $[j, H]$, which can be computed by using the results in David et al. (1997).

Table 1
An illustration of the discrete approximation

	j								
	9	8	7	6	5	4	3	2	1
(s_j, S_j)	$(0,1)$	$(0,2)$	$(0,3)$	$(0,4)$	$(0,4)$	$(0,5)$	$(0,5)$	$(0,6)$	$(0,6)$
AppC^j	26	32	38	44	49.2	54.3333	59.1429	64	68.6667
OptC^j	26	32	38	43.8333	49.1667	54.2222	59.119	63.9167	68.6481

3.3. Convergence of the discrete-time approximation scheme

In this subsection, we will show that the approximated periodic-review model will converge to the corresponding EOQ model with sudden obsolescence as the number of periods tends to infinity.

For any given $t = (i_0 - 1)\frac{H}{2^{k_0}}$ $(0 \leqslant t \leqslant H)$ for some nonnegative integer k_0 and i_0 $(1 \leqslant i_0 \leqslant 2^{k_0})$, suppose that the obsolescence has not occurred until t. If $[0, H]$ is equally divided into 2^{k_0} periods, $[t, H]$ will be equally divided into $m_0 = 2^{k_0} - (i_0 - 1)$ periods. For any $k \geqslant k_0$, if we equally divide $[0, H]$ into 2^k periods, $[t, H]$ will be equally divided into $m = m_0 2^{k-k_0}$ periods and t can also be expressed as $(i-1)\frac{H}{2^k}$ where $i - 1 = (i_0 - 1)2^{k-k_0}$. For simplicity, from now on we call period $(i_0 - 1)2^{k-k_0} + j$ as period j. Let $t_j = t + j\frac{H}{2^k}$ be the end point of period j, $q_j = G(t_j) - G(t_{j-1})$, $Q(j) = \sum_{l=0}^{j} q_j$ and $\overline{Q}(j) = 1 - Q(j)$ for $1 \leqslant j \leqslant m$.

Let $\tau = l\frac{H}{2^k}$ for some integer $l \geqslant 1$ be the length of the interval covered by an order quantity at t_j $(j + l \leqslant m)$ given that the initial inventory level at t_j is zero and $[0, H]$ is equally divided into 2^k periods. Then, $A^d(t_j, t_{j'}, \tau)$, the expected total holding cost incurred on $[t_j, t_{j'}]$ in the approximated periodic-review model, can be expressed as

$$A^d(t_j, t_{j'}, \tau) = \frac{\overline{Q}(j')}{\overline{Q}(j)} + \frac{(l-1)(l-(j'-j))}{2}(j'-j)\left(\frac{H}{2^k}\right)^2 \mu h \tag{5}$$

$$+ \frac{q_{j+1}}{\overline{Q}(j)}(l-1)\left(\frac{H}{2^k}\right)^2 \mu h + \frac{q_{j+2}}{\overline{Q}(j)}((l-1)+(l-2))\left(\frac{H}{2^k}\right)^2 \mu h + \cdots \tag{6}$$

$$+ \frac{q_{j'}}{\overline{Q}(j)} \frac{(l-1)+(l-(j'-j))}{2}(j'-j)\left(\frac{H}{2^k}\right)^2 \mu h \tag{7}$$

by applying (2) recursively over periods $(j+1), (j+2), \ldots$, and j'. For the corresponding EOQ model with sudden obsolescence, let $A^c(t_j, t_j + \delta, \tau)$ be the expected total holding cost incurred on $[t_j, t_j + \delta]$ given that the initial inventory at t_j is zero. This can be expressed as

$$A^c(t_j, t_j + \delta, \tau) = \overline{G}(\delta | t_j)\delta \frac{\tau + (\tau - \delta)}{2}\mu h + \int_{\theta=0}^{\delta} \theta \frac{\tau + (\tau - \theta)}{2}\mu h \frac{\mathrm{d}G(t_j + \theta)}{\overline{G}(t_j)}. \tag{8}$$

We can see that $A^d(t_j, t_{j'}, \tau)$ is the discretization of $A^c(t_j, t_j + \delta, \tau)$ and

$$A^d(t_j, t_{j'}, \tau) \leqslant A^c(t_j, t_j + \delta, \tau), \tag{9}$$

where $\delta = \frac{H}{2^k}(j' - j)$ and $\tau \geqslant \delta$. By using Eqs. (5)–(8), we get

$$A^c(t_j, t_j + \delta, \tau) \leqslant A^d(t_j, t_{j'}, \tau) + \left\{ \frac{q_{j+1}}{\overline{Q}(j)} + 2\frac{q_{j+2}}{\overline{Q}(j)} + \cdots + (j'-j)\frac{q_{j'}}{\overline{Q}(j)} + (j'-j)\frac{\overline{Q}(j')}{\overline{Q}(j)} \right\} \left(\frac{H}{2^k}\right)^2 \mu h$$

$$\leqslant A^d(t_j, t_{j'}, \tau) + (j'-j)\left(\frac{H}{2^k}\right)^2 \mu h.$$

Thus, we get

$$\lim_{k \to +\infty} A^d(t_j, t_{j'}, \tau) = A^c(t_j, t_j + \delta, \tau). \tag{10}$$

In the following we first provide some properties about $\tau^*(t_j) = l^* \frac{H}{2^k}$ for some $l^* \geqslant 1$, the optimal length of interval covered by the order quantity at t_j in the approximated periodic-review model given that the initial inventory at t_j is zero.

Lemma 1. *There exist an integer $k_1 \geqslant k_0$ and two small real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that for any $k \geqslant k_1$, we have $\tau^*(t_j) \geqslant \delta_1$ if $t_j \in [t, H - \delta^2)$ and $\tau^*(t_j) = H - t_j$ if $t_j \in [H - \delta_2, H]$ for all $0 \leqslant j \leqslant m$.*

The proof is provided in Appendix A. From Lemma 1, we know that the total number of replenishments is bounded by a constant if the number of periods in the approximated periodic-review model is large enough.

Let $F_{1,m}(t)$ and $f_{j,m}(t)$ be the corresponding optimal cumulative expected costs from period j to period m given that the initial inventory at t is zero if the demand in each period is satisfied at the end of the period and the beginning of the period, respectively. We have the following property and the proof can be found in Appendix A.

Lemma 2. *For any* $l \geqslant 1$, $F_{1,2l}(t) \leqslant F_{1,l}(t)$ *and* $f_{1,2l}(t) \geqslant f_{1,l}(t)$.

By using the result in Lemma 2, we shall show that both $\{f_{1,m}(t)\}$ and $\{F_{1,m}(t)\}$ are monotone and bounded. Hence, they are convergent. The proof of the following result is provided in Appendix A.

Lemma 3. *There exists a constant integer* M_0 *(not dependent on* t*) such that*

$$f_{1,m}(t) \leqslant V(t) \leqslant F_{1,m}(t) \leqslant f_{1,m}(t) + \frac{M_0}{2^k}. \tag{11}$$

Hence, we get

$$\lim_{k\to+\infty} f_{1,m}(t) = \lim_{k\to+\infty} F_{1,m}(t) = V(t). \tag{12}$$

From this result, we can see that for any $t = (i_0 - 1)\frac{H}{2^{k_0}}$ for some nonnegative integer k_0 and i_0 $(1 \leqslant i_0 \leqslant m_0)$, $\lim_{k\to+\infty} f_{1,m}(t) = V(t)$. Note that the set $\{t | t = (i_0 - 1)\frac{H}{2^{k_0}}$ for all nonnegative integers i_0 and $k_0\}$ is dense in $[0, H]$ and $V(t)$ is continuous, we have the following result.

Theorem 1. *The discrete-time approximation scheme for the EOQ model with sudden obsolescence is convergent when the number of periods in the approximated periodic-review model tends to infinity.*

4. Numerical results

The following two instances of the EOQ model with sudden obsolescence are treated by David et al. (1997) by using (4). As the optimal solutions for these two instances can be found, we can verify our discrete approximation solutions with the optimal ones.

Example 1 (*Deterministic obsolescence*). In this problem, $G(x)$ is an one-point distribution, namely $\Pr(X = H) = 1$. At each time t, let $r = H - t$ be the residual time, David et al. (1997) proved that the optimal order quantity at t is to cover demands up to $t + \tau^*(r)$, where

$$\tau^*(r) = \frac{r}{l^*(r)}, \qquad l^*(r) = \min\left\{ l | \frac{h\mu r^2}{2l(l+1)} \leqslant K, l \geqslant 1 \right\}.$$

The optimal value function is given by

$$V(r) = n^*(r)K + a\mu r + \frac{h\mu r^2}{2l^*(r)}.$$

We choose four sets, I_u $(1 \leqslant u \leqslant 4)$, of interval ranges for parameters H, K, a, and h to generate the parameters values as described in Table 2. For each I_u, 20 sets of parameters are randomly generated. Each set of parameters has four values representing the failure horizon H, the setup cost K, the per unit purchasing cost a and the holding cost rate h, respectively.

Table 2
Interval ranges for parameters

	I_1	I_2	I_3	I_4
H	$[1,10]$	$[1,10]$	$[5,6]$	$[1,20]$
K	$[5,20]$	$[30,50]$	$[30,80]$	$[30,100]$
a	$[1,9]$	$[1,9]$	$[1,9]$	$[1,100]$
h	$[1,9]$	$[1,9]$	$[1,9]$	$[1,100]$

Table 3
Mean approximation cost for Example 1

	I_1	I_2	I_3	I_4
$n = 1$	76.2865	122.191	144.754	1172.82
$n = 2$	75.7987	122.052	144.481	1159.78
$n = 5$	75.6004	122.03	144.492	1157.87
$n = 10$	75.5803	122.025	144.481	1157.29

For the generated H in each set of parameters and each n ($n = 1$, 2, 5, or 10), we partition H into nH periods to generate a periodic-review model. Then we apply the DP algorithm to determine s^j and S^j, $1 \leqslant j \leqslant nH$, of this periodic-review approximation model. Then we apply (s^j, S^j) policy to compute the approximation cost of the EOQ model with sudden obsolescence. Every value in Table 3 is the mean of the 20 approximation costs corresponding to the 20 sets of parameters. It shows that the approximation cost decreases as n increase in each interval range I_u, $1 \leqslant u \leqslant 4$.

For comparison, we define the deviation ratio of the discrete approximation cost from the optimal cost of the EOQ model by

$$\frac{\text{AppCost} - \text{OptCost}}{\text{OptCost}}.$$

Table 4 gives the mean deviation of the approximation costs from the optimal costs. The values show that the approximation costs are very close to the optimal costs of the EOQ model even if one unit time is treated as one period in the approximation, namely $n = 1$.

Example 2 (*Uniform obsolescence*). In this problem, $G(x)$ is a uniform distribution on $[0, H]$ and $h = 0$. Let $r = H - t$ be the residual time and $w = \frac{K}{a\mu}$. Suppose that $V(r)$ is measured in units of $a\mu$. David et al. (1997) obtained the following results:

(1) $\tau^*(r)$, for $0 \leqslant r \leqslant H$, is piecewise linear concave

$$\tau^*(r) = \frac{nw}{2} + \frac{r}{l+1}$$

for $r_l \leqslant r \leqslant r_{l+1}$, $r_0 = 0$ and $r_l = \frac{wl(l+1)}{2}$.

Table 4
Mean deviation ratio for Example 1

	I_1	I_2	I_3	I_4
$n = 1$	0.0074344	0.000822437	0.00184307	0.0140725
$n = 2$	0.00177371	0.000125374	0.00000	0.00203012
$n = 5$	0.000260697	3.29142e−05	7.37412e−05	0.000667373
$n = 10$	3.1471e−05	5.00839e−06	2.84217e−15	0.000144421

Table 5
Mean approximation cost for Example 2

	I_1	I_2	I_3	I_4
$n = 1$	35.4249	65.7362	83.9	437.252
$n = 2$	35.3784	65.7066	83.8717	436.444
$n = 5$	35.3496	65.7071	83.871	436.086
$n = 10$	35.3462	65.7055	83.871	436.01

Table 6
Mean deviation ratio for Example 2

	I_1	I_2	I_3	I_4
$n = 1$	0.00192911	0.000307615	0.000416294	0.00404961
$n = 2$	0.000920541	1.50136e−05	1.22738e−05	0.00167107
$n = 5$	0.000101368	1.94422e−05	2.05314e−06	0.000485199
$n = 10$	1.65986e−05	1.94422e−05	2.05314e−06	3.89082e−05

(2) The optimal value function (the expected remaining cost till obsolescence) is given by

$$V(r) = \frac{(l+2)r}{2(l+1)} + \frac{(l+2)w}{2} - \frac{l(l+1)(l+2)w^2}{24r}$$

for $r_l \leqslant r \leqslant r_{l+1}$.

Except choosing $h = 0$ in the model, the values of all other parameters in the experiment for Example 2 are the same as in Table 2. In the traditional lot-sizing inventory model, zero holding cost, $h = 0$, is very meaningless. But in the above Example 2, even if $h = 0$, we need to balance the purchasing cost and the obsolescence cost (no any value of the remaining inventory at the obsolescent time point). This is not trivial.

Similarly as in Table 3, every value in Table 5 is the mean of the 20 approximation costs corresponding to the 20 sets of parameters for Example 2. It shows that the approximation cost decreases as n increase in each interval range I_u, $1 \leqslant u \leqslant 4$. Table 6 gives the mean deviation ratios of the approximation costs from the optimal costs for Example 2. The values show that the approximation costs are very close to the optimal costs of the EOQ model even if the number of periods n in the discrete-time approximation model is very small ($n \leqslant 10$).

5. Conclusion

We present a stochastic periodic-review inventory model with sudden obsolescence which allows backlogging and stochastic demand in every period. We first characterize the structure of the optimal policy and then propose a DP algorithm for the computation of the optimal policy.

As there is no any efficient computational approach for the EOQ model with sudden obsolescence, as an application, we propose a discrete approximation approach for the EOQ model with sudden obsolescence. From both the numerical experiments and the analytical results, we show that the discrete approximation costs are very close to the corresponding optimal costs of the EOQ model with sudden obsolescence obtained by David et al. (1997).

The discrete approximation approach in this paper can be applied to the stochastic continuous-review models with sudden obsolescence such as the one in David and Mehrez (1995). As the structure of the optimal policy of this model is unknown, we cannot verify our approximation scheme in a numerical way.

We guess that our discrete-time approximation scheme is convergent even for this stochastic model but the proof may be complicated.

Further investigation and extension of the theory and applications of the stochastic periodic-review model with sudden obsolescence, such as the case where there is an ordering capacity constraint for each replenishment, should be of interest to many researchers and practitioners.

Appendix A

Proof of Lemma 1. We first show that there exists $\delta_2 > 0$ such that for any $k \geqslant k_0$, we have $\tau^*(t_j) = H - t_j$ if there exists $t_j \in [H - \delta_2, H]$.

It is obvious that there exists $\delta_2 > 0$ such that $A^c(H-\tau, H, \tau) < K$ if $\tau \leqslant \delta_2$. As $A^d(t_j, H, H-t_j) \leqslant A^c(t_j, H, H-t_j)$ if there exists any $t_j \in [H-\delta_2, H]$, we have $A^d(t_j, H, H-t_j) < K$. From this, we can see that we should order up to the end of the planning horizon if $H - t_j \leqslant \delta_2$.

Secondly, we shall show that there exist an integer $k_1 \geqslant k_0$ and $\delta_1 > 0$ such that for all $k \geqslant k_1$ we have $\tau^*(t_j) \geqslant \delta_1$ if $t_j \in [t, H-\delta_2)$.

For $\tau = \frac{H}{2^k}$, there exists $\delta_0 > 0$ such that $A^c(t_j, t_j+\tau, \tau+\mu(H-t-\tau)) < K$ for any $t_j \in [t, H-\delta_2]$ and $\tau \leqslant \delta_0$. As $A^d(t_j, t_j+\tau, \tau+\mu(H-t-\tau)) \leqslant A^c(t_j, t_j+\tau, \tau+\mu(H-t-\tau))$, there exists $k_1 \geqslant k_0$ such that for any $k \geqslant k_1$ $A^d(t_j, t_j+\tau, \tau+\mu(H-t-\tau)) < K$ and $\tau = \frac{H}{2^k} \leqslant \frac{H}{2^{k_1}} < \delta_0$. From this, we can see that $\tau^*(t_j) \geqslant \delta_1 = \frac{H}{2^{k_1}}$ for all $t_j \in [t, H-\delta_2)$. □

Proof of Lemma 2. As the proofs for $f_{1,m}(t)$ and $F_{1,m}(t)$ are similar, we only prove this lemma for $F_{1,m}(t)$. We prove this by induction on l, the number of periods over $[t, H]$ given that the obsolescence does not occur until t.

If $l = 1$, $[t, H]$ is treated as one period. In our approximated periodic-review model, the only obsolescent point is H and $F_{1,1}(t) = K + a\mu(H-t) + \mu(H-t)^2 h$. If $l = 2$, $[t, H]$ is equally divided into two periods, there are two possible obsolescent points, $t_1 = t + \frac{H-t}{2}$ and $t_2 = H$, with $q_1 = G(t_1) - G(t)$ and $q_2 = G(t_2) - G(t_1)$, respectively.

$$F_{1,2}(t) = K + a\mu(H-t) + \frac{q_1}{q_1+q_2}\frac{(H-t)^2}{2}h + \frac{q_2}{q_1+q_2}\frac{3(H-t)^2}{4}h.$$

It is obvious that $F_{1,2}(t) \leqslant F_{1,1}(t)$. Therefore, it is true for $l = 1$.

Suppose that it is true for any $l = n \geqslant 1$, i.e., $F_{1,2n}(t) \leqslant F_{1,n}(t)$. For $l = n+1$, $[t, H]$ is equally divided into $n+1$ periods. We call $[t, t+\frac{H-t}{n+1}]$, the first period, as interval 1 and $[t+\frac{H-t}{n+1}, H]$, from period 2 until period $n+1$, as interval 2. Suppose that the optimal order up to quantity at t is y^* in the approximated periodic-review model given that the initial inventory level at t is zero. The optimal expected total cost $F_{1,n+1}(t)$ includes two parts: one occurs on interval 1 and the other part occurs on interval 2. If we divided $[t, H]$ into $2(n+1)$ periods, we still order up to y^* units at t given that the initial inventory at t is zero. The optimal expected total cost $F_{1,2(n+1)}(t)$ is also includes two parts: one occurs on interval 1 (the first two periods) and the other parts occurs on interval 2 (the last $2n$ periods). Applying the argument for $l = 1$ on interval 1 and the assumption for $l = n$ on interval 2, we obtain that $F_{1,2(n+1)}(t) \leqslant F_{1,n+1}(t)$. Thus, it is also true for $l = n+1$. □

Proof of Lemma 3. From the definitions of $f_{1,m}(t)$, $F_{1,m}(t)$, and $V(t)$, we can see that $f_{1,m}(t) \leqslant V(t) \leqslant F_{1,m}(t)$. For any $k \geqslant k_0$, by Lemma 2, $\lim_{k\to+\infty} F_{1,m}(t)$ and $\lim_{k\to+\infty} f_{1,m}(t)$ exist. By Lemma 1, for any $k \geqslant k_1$, the number of replenishments over $[t, H]$ is uniformly bounded by an integer l_0. From the definitions of $f_{1,m}(t)$ and $F_{1,m}(t)$, we get

$$F_{1,m}(t) \leqslant f_{1,m}(t) + \Delta,$$

where $\Delta = l_0 \frac{\mu H^2 h}{2^{2k}} \frac{1}{\overline{\overline{G}}(t)} \sum_{j=0}^{m} jq_j$. As $\lim_{k\to+\infty} l_0 \frac{\mu H^2 h}{2^{2k}} \frac{1}{\overline{\overline{G}}(t)} \sum_{j=0}^{m} jq_j = 0$, we complete the proof. □

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