Chapter 2 Exercises

(1) Suppose that $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{8}$

a. What is the value of $P(A|B)$ if $A$ and $B$ are mutually exclusive?

b. What is the answer to (a) if $A$ and $B$ are independent?

(2) Let $A$ and $B$ be two events such that $P(A) = 0.3$, $P(B) = 0.5$, and $P(A \cup B) = 0.7$. Find:

a. $P(A \cap B)$

b. $P(\overline{A} \cup B)$

c. $P(\overline{A} \cap B)$

(3) Consider the two events, $A = \{\text{The person is an accountant}\}$ and $B = \{\text{The person has green hair}\}$. If it is known that $P(A \cap B) = 0$, what can we say about accountants and people with green hair?

(4) A large lake is stocked with three species of trout: brown, rainbow and brook. It is known that brown and rainbow trout together account for 80% and female brook trout account for 11%. Furthermore, among the brown and rain trout, 48% are female. What is the probability of finding a female trout in the lake?

(5) If a football team has a 10% chance of winning this Saturday’s game, a 30% chance of winning the next Saturday’s game, and a 65% chance of losing both games, what are their chances of winning exactly once in the next two games?

(6) Suppose $P(A \cap \overline{B}) = 0.3$, $P(\overline{(A \cup B)}) = 0.2$, and $P(A \cap B) = 0.1$, find $P(A|B)$.

(7) A fair coin is a coin with $P(\text{Heads}) = \frac{1}{2}$. Suppose a fair coin is tossed twice and two heads are obtained. If outcomes of all tosses are assumed independent, what is the probability that

a. the next toss will be a head?

b. the next five tosses will all be heads?

(8) A student is applying to Oxford and Cambridge. She estimates that she has a probability of 0.3 of being accepted at Cambridge and 0.6 of being accepted at Oxford. She also thinks the probability that she will be accepted by both is 0.2.

a. What is the probability that she is accepted by Cambridge if she has been accepted by Oxford?

b. Are the events “accepted at Oxford” and “accepted at Cambridge” independent?

(9) Assume that $A$ and $B$ are two events with positive probabilities. Show that if $P(A|B) = P(A)$, then $P(B|A) = P(B)$.

(10) Suppose $A$ and $B$ independent events with probabilities $0 < P(A), P(B) < 1$, prove that the following pairs are also independent:

a. $A$ and $\overline{B}$

b. $\overline{A}$ and $B$

(11) Suppose $E$ and $F$ are events such that $P(E|F) = P(F|E)$ and $P(E \cup F) = 1$ and $P(E \cap F) > 0$. Prove that $P(E) > 1/2$.

(12) Suppose $A$, $B$, $C$ are events with positive probabilities, prove $P(A \cap B|C) = P(A|B \cap C)P(B|C)$.
The Capilano Suspension Bridge in Vancouver, Canada is a famous tourist destination in that city. The original bridge was built in 1889 and made of cedar planks and ropes. Assuming the bridge is 450 feet long and made of 1350 planks. If at anytime, when there are more than 100 people on the bridge, the probability that the bridge will break at any plank is 0.000001. Furthermore, assume the chance of a break at any plank is independent of breaks at other planks.

a. Let \( A = \{ \text{Break at plank No. 335} \} \), \( B = \{ \text{Break at plank No. 1034} \} \). Find \( P(A) \), \( P(AB) \), \( P(A|B) \).

b. Find the probability that the bridge will break when there are more than 100 visitors on the bridge.

The Capilano Suspension Bridge is about 230 feet above the Capilano River, over the history of the bridge, there have been some falls from the bridge. It was found that 80% of the victims from these falls were adults and the rest were children. The probability of survival from a fall is different for an adult than from a child, they are 0.1 and 0.4, respectively. A child went on a field trip on the bridge and in the afternoon, her mother heard on the radio that there was a fall but the victim survived. What is the chance that the victim was a child?

In Vancouver, half of the days have some rain. The weather forecaster is correct 3/4 of the time. When the forecast is rain, a parent gives her child an umbrella to school. When forecast is not rain, she gives it to her child with probability 1/4. Find (Hint: you can assume that the mother’s decision to give an umbrella depends on the forecast only, and not whether it may rain, i.e., the events “Mother gives umbrella” and “It rains” are independent if she knows the forecast. This type of independence is called **conditional independence**. In general if events \( A, B, \) and \( C \) are such that \( P(A|B, C) = P(A|C) \), then \( A \) and \( B \) are said to be conditionally independent given \( C \). However, conditional independence does not imply independence and vice versa)

(a) the probability that the child finds himself with no umbrella when it rains
(b) the probability that it doesn’t rain, given that he brings his umbrella

An investor is interested in oil exploration in Siberia, Russia. She was told that there are two types of oil fields, those with a small and a large oil reserve. It is known that 75% of the oil fields in Siberia have low reserve. The investor can pay a moderate sum of money to carry out test drills. Past experience tells her that if an area has a large oil reserve, then the probability that the test drills will yield oil is 0.7; on the other hand, if an area has a small oil reserve, then the probability that the test drills will yield oil is 0.2. Oil was found in test drills, determine the probability that the area has a large oil reserve.
Answers

(1) a. \( P(A\mid B) = 0 \)
b. \( P(A\mid B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A) = \frac{1}{4} \)

(2) a.
\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]
\[
\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B)
\]
\[
= 0.3 + 0.5 - 0.7 = 0.1.
\]
b. It is known that \( A \cap B \) is the complementary event of \( \bar{A} \cup \bar{B} \), so we have
\[
P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B) = 1 - 0.1 = 0.9.
\]
c. By the partition rule:
\[
P(B) = P(\bar{A} \cap B) + P(A \cap B)
\]
\[
\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)
\]
\[
= 0.5 - 0.1 = 0.4.
\]

(3) No accountants have green hair and vice versa.

(4) Let \( B=\{\text{Brown trout}\}, R=\{\text{Rainbow trout}\}, Br=\{\text{Brook trout}\}, F=\{\text{Female trout}\} \). Therefore,
\[
P(B \cup R) = 0.8, \quad P(F\mid B \cup R) = 0.48, \quad P(Br \cap F) = 0.11
\]
It is easiest to group the brown and rainbow trout (let’s called the group \( BR \)), so we have
\[
P(BR) = 0.8, \quad P(F\mid BR) = 0.48, \quad P(Br \cap F) = 0.11.
\]
We then use the partition rule,
\[
P(F) = P(F \cap BR) + P(F \cap Br)
\]
\[
= P(F\mid BR)P(BR) + P(F \cap Br)
\]
\[
= (0.8)(0.48) + 0.11
\]
\[
= 0.494
\]

(5) Let \( A \) be the event that the football team wins the game this Saturday, and \( B \) be the event that it wins the game next Saturday. So we have:
\[
P(A) = 0.1, \quad P(B) = 0.3, \quad P(\bar{A} \cap \bar{B}) = 0.65,
\]
and we are looking for exactly winning once, which equals to one minus the probability of losing both games or winning both games
\[
1 - P(\bar{A} \cap \bar{B}) - P(A \cap B).
\]
We know from above \( P(\bar{A} \cap \bar{B}) = 0.65 \) so it remains to determine \( P(A \cap B) \).
We know:
\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]
\[
\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B)
\]
\[
= 0.1 + 0.3 - 0.35 = 0.05
\]
Therefore, \( P(\text{exactly winning once}) = 1 - P(A \cap B) - P(A \cap B) = 1 - 0.65 - 0.05 = 0.3 \).

(6) By the partition rule:

\[
P(A \cap \bar{B}) = P(A) - P(A \cap B)
\]

\[
\Rightarrow P(A) = 0.3 + 0.1 = 0.4.
\]

Since \( P(A \cup B) = 0.8 \) (complementary event of \((A \cup B)^c\)):

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

\[
0.8 = 0.4 + P(B) - 0.1
\]

\[
\Rightarrow P(B) = 0.5.
\]

Using the multiplication rule:

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.5} = \frac{1}{5}.
\]

(7)

a. \( P(H|HH) = P(H) = \frac{1}{2} \) since all outcomes are independent.

b. \( P(HHHHHH) = P(H)P(H)\ldots P(H) = (\frac{1}{2})^5 \) since all outcomes are independent.

(8) a. Let \( O = \{\text{Accept at Oxford}\}; C = \{\text{Accept at Cambridge}\} \). We are given:

\[
P(O) = 0.6, \quad P(C) = 0.3, \quad P(O, C) = 0.2.
\]

\[
P(C|O) = \frac{P(O, C)}{P(O)} = \frac{0.2}{0.6} = \frac{1}{3}.
\]

b. Since \( P(C|O) \neq P(C) \) therefore, the events \( O \) and \( C \) are not independent.

(9)

\[
P(A|B) = P(A) \quad \Rightarrow \quad \frac{P(A, B)}{P(B)} = P(A)
\]

\[
\Rightarrow \quad \frac{P(A, B)}{P(A)} = P(B)
\]

\[
\Rightarrow \quad P(B|A) = P(B)
\]

(10)

a.

\[
P(A|B) = P(A)
\]

\[
\Rightarrow \quad \frac{P(A, B)}{P(B)} = P(A)
\]

\[
\Rightarrow \quad \frac{P(A) - P(A, \bar{B})}{1 - P(B)} = P(A)
\]

\[
\Rightarrow \quad P(A) - P(A, \bar{B}) = P(A) - P(A)P(\bar{B})
\]

\[
\Rightarrow \quad P(A, \bar{B}) = P(A)P(\bar{B})
\]

which means \( A \) and \( \bar{B} \) are independent.
b. Using the results from (10a), we can write

\[ \begin{align*}
P(A|\bar{B}) &= P(A) \\
\Rightarrow \frac{P(A,\bar{B})}{P(B)} &= P(A) \\
\Rightarrow \frac{P(\bar{B}) - P(A,\bar{B})}{P(B)} &= 1 - P(A) \\
\Rightarrow P(\bar{B}) - P(A,\bar{B}) &= P(\bar{B}) - P(B)P(\bar{A}) \\
\Rightarrow P(\bar{A},\bar{B}) &= P(\bar{A})P(\bar{B})
\end{align*} \]

which means \(\bar{A}\) and \(\bar{B}\) are independent.

(11) Using the first fact, we have

\[ \begin{align*}
\frac{P(E,F)}{P(E)} &= \frac{P(E,F)}{P(F)} \Rightarrow P(E) = P(F).
\end{align*} \]

Using the second fact, we have

\[ P(E \cup F) = P(E) + P(F) - P(E \cap F) = 1. \]

Combining the first two facts, we have

\[ \begin{align*}
2P(E) - P(E \cap F) &= 1 \\
P(E) &= \frac{1 + P(E \cap F)}{2} \Rightarrow P(E) > 0.5,
\end{align*} \]

since it is known that \(P(E \cap F) > 0.\)

(12)

\[ \begin{align*}
P(A \cap B|C) &= \frac{P(A \cap B \cap C)}{P(C)} \\
&= \frac{P(A \cap B \cap C)}{P(B \cap C)} \\
&= \left[ \frac{P(A \cap B \cap C)}{P(B \cap C)} \right] \left[ \frac{P(B \cap C)}{P(C)} \right] \\
&= P(A|B \cap C)P(B|C)
\end{align*} \]

(13)

a. \(P(A) = 0.000001.\)
\(P(A\,B) = P(A)P(B) = (0.000001)^2\) since all breaks are independent.
\(P(A|B) = P(A) = 0.000001\) since all breaks are independent.

b. Let \(D_i\) be the event that a break occurs at the \(i\)-th plank, \(i = 1,\ldots,1350.\)

\[ \begin{align*}
P(\text{Bridge breaks}) &= P(D_1 \cup D_2 \cup D_3 \cup \ldots \cup D_{1350}) \\
&= 1 - P(D_1 \cap D_2 \cap D_3 \cap \ldots \cap D_{1350}) \\
&= 1 - P(D_1)P(D_2)P(D_3)\ldotsP(D_{1350}) \\
&= 1 - (1 - 0.000001)^{1350} \approx 0.001349.
\end{align*} \]

Note that the transition from the first to the second expression is based on the argument that, breaking at one or more of the planks is the complementary event of not breaking at any of the
planks. Furthermore, the transition from the second to the third expression is because breaks are independent.

(14) Let \( A = \{ \text{Victim is an adult} \} \); \( S = \{ \text{Survived from a fall} \} \). We are given:

\[
P(A) = 0.8, \quad P(\bar{A}) = 0.2, \quad P(S|A) = 0.1, \quad P(S|\bar{A}) = 0.4.
\]

\[
P(\bar{A}|S) = \frac{P(S|\bar{A})P(\bar{A})}{P(S|A)P(A) + P(S|\bar{A})P(\bar{A})} = \frac{(0.4)(0.2)}{(0.4)(0.2) + (0.1)(0.8)} = \frac{1}{2}.
\]

(15) Let \( R = \{ \text{It rains} \} \); \( F = \{ \text{Forecast rain} \} \); \( U = \{ \text{Brings umbrella} \} \). We are given:

\[
P(R) = \frac{1}{2}, \quad P(R|F) = \frac{3}{4}, \quad P(\bar{R}|\bar{F}) = \frac{3}{4}, \quad P(U|F) = 1, \quad P(U|\bar{F}) = \frac{1}{4}.
\]

a. We will need to find \( P(F) \) for later use. Using the partition rule,

\[
P(R) = P(R,F) + P(R,\bar{F}) = P(R|F)P(F) + P(R|\bar{F})[1 - P(F)]
\]

\[
1/2 = (3/4)P(F) + (1/4)[1 - P(F)]
\]

\[
1/4 = 1/2P(F)
\]

\[
P(F) = 1/2.
\]

We want

\[
P(\bar{U}|R) = 1 - P(U|R) = 1 - \frac{P(U,R)}{P(R)}.
\]

Recall that \( U \) and \( R \) are conditionally independent given the forecast, hence

\[
P(U|R,F) = P(U|F) \quad \text{and} \quad P(U|R,\bar{F}) = P(U|\bar{F}).
\]

Using the partition rule, we can write

\[
P(U,R) = P(U,R,F) + P(U,R,\bar{F}) = P(U,R|F)P(F) + P(U,R|\bar{F})P(\bar{F})
\]

\[
= P(U|F)P(R|F)P(F) + P(U|R,\bar{F})P(\bar{F})
\]

\[
= 1(3/4)(1/2) + (1/4)(1/4)(1/2) = \frac{13}{32}.
\]

Finally,

\[
P(\bar{U}|R) = 1 - \frac{13/32}{1/2} = 3/16.
\]
Another way to solve this problem is to form the probability tree, as follows:

Based on the tree, we obtain

\[
P(\bar{U}|R) = \frac{P(R|\bar{U})P(\bar{U})}{P(R)} = \frac{P(R|\bar{U})P(U|R)P(F)}{P(R)} = \frac{(1/4)(3/4)(1/2)}{1/2} = \frac{3}{16}
\]

b.

\[
P(\bar{R}|U) = 1 - P(R|U) = 1 - \frac{P(U,R)}{P(U)} = 1 - \frac{P(U,R)}{P(U|F)P(F) + P(U|\bar{F})P(\bar{F})} = 1 - \frac{13/32}{1(1/2) + (1/4)(1/2)} = \frac{7}{20}
\]
Using probability tree:

\[
P(\bar{R}|U) = \frac{P(\bar{R}U)}{P(U)}
\]

\[
P(\bar{R}U) = P(\bar{R}U,F)P(U|F)P(F) + P(\bar{R}U,\bar{F})P(U|\bar{F})P(\bar{F})
\]

\[= \frac{1}{4}(1)(1/2) + \frac{3}{4}(1/4)(1/2)
\]

\[= \frac{7}{32}
\]

\[
P(U) = \frac{1}{4}(1)(1/2) + \frac{3}{4}(1/4)(1/2) + \frac{1}{2}(1/4)(1/4) + \frac{1}{2}(1)(3/4)
\]

\[= \frac{20}{32}
\]

\[
P(\bar{R}|U) = \frac{7}{20}
\]

(16) Let \(L=\{\text{low reserve}\}\); \(T=\{\text{test yields oil}\}\). We are given:

\[
P(L) = 0.75, \quad P(\bar{L}) = 0.25, \quad P(T|L) = 0.2, \quad P(T|\bar{L}) = 0.7.
\]

\[
P(\bar{L}|T) = \frac{P(T|\bar{L})P(\bar{L})}{P(T|L)P(L) + P(T|\bar{L})P(\bar{L})}
\]

\[= \frac{(0.7)(0.25)}{(0.7)(0.25) + (0.2)(0.75)} \approx 0.538.
\]