

Appendix to “Subsidies for FDI: Implications from a Model with Heterogeneous Firms”

Davin Chor*

This appendix contains the detailed proofs of the propositions and results in the main article for the *Journal of International Economics*. The equation numbers follow those in the main article.

1 Details of Proof of Proposition 1

Proof that $W_{Ff} \rightarrow -\infty$ when $s_f \rightarrow 1^-$. Recall the expression for W_{Ff} from (17). Recall also that:

$$\begin{aligned}\Lambda_{Ff} &= \frac{\varepsilon - 1}{k - \varepsilon + 1} \left(\frac{\alpha}{a_U}\right)^k \left(\frac{1 - \alpha}{w_H}\right)^{\frac{k}{\varepsilon - 1}} \left[\frac{(f_X)^{\frac{k}{1-\varepsilon}+1} w_H}{(\tau w_H)^k} + \frac{((f_I - f_X)(1 - s_f))^{\frac{k}{1-\varepsilon}+1} w_H}{((w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon})^{\frac{k}{1-\varepsilon}}} \right] \\ \frac{\partial \Lambda_{Ff}}{\partial s_f} &= \left(\frac{\alpha}{a_U}\right)^k \left(\frac{1 - \alpha}{w_H}\right)^{\frac{k}{\varepsilon - 1}} \frac{(f_I - f_X)^{\frac{k}{1-\varepsilon}+1} w_H}{((w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon})^{\frac{k}{1-\varepsilon}}} (1 - s_f)^{\frac{k}{1-\varepsilon}}\end{aligned}$$

Note that $\lim_{s_f \rightarrow 1^-} \Lambda_{Ff} = +\infty$, since $1 - \varepsilon < 0$. Thus, $\lim_{s_f \rightarrow 1^-} \tilde{\Lambda}_{Ff} = +\infty$, since $\tilde{\Lambda}_{Ff}$ is Λ_{Ff} raised to a positive power ($\frac{\mu\alpha}{\mu\alpha - k(\mu - \alpha)} > 0$). Moreover, $\frac{1}{\Lambda_{Ff}} \frac{\partial \Lambda_{Ff}}{\partial s_f} = \frac{k - \varepsilon + 1}{\varepsilon - 1} \frac{1}{1 - s_f} g(s_f)$, where:

$$g(s_f) = \left(\frac{((f_I - f_X)(1 - s_f))^{\frac{k}{1-\varepsilon}+1}}{((w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon})^{\frac{k}{1-\varepsilon}}} \right) \bigg/ \left(\frac{(f_X)^{\frac{k}{1-\varepsilon}+1}}{(\tau w_H)^k} + \frac{((f_I - f_X)(1 - s_f))^{\frac{k}{1-\varepsilon}+1}}{((w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon})^{\frac{k}{1-\varepsilon}}} \right)$$

Clearly, $\lim_{s_f \rightarrow 1^-} g(s_f) = 1$, which implies that $\lim_{s_f \rightarrow 1^-} \frac{1}{\Lambda_{Ff}} \frac{\partial \Lambda_{Ff}}{\partial s_f} = +\infty$. Hence, the limit of the term in the square brackets in (17) as s_f approaches 1 is $-\infty$. Together with the fact that $\lim_{s_f \rightarrow 1^-} \tilde{\Lambda}_{Ff} = +\infty$, we have $W_{Ff} \rightarrow -\infty$ when $s_f \rightarrow 1^-$ as claimed. ■

Proof that $\frac{\partial W_{Ff}}{\partial s_f} > 0$ for all $s_f < 0$. Since $\frac{\partial^2 \Lambda_{Ff}}{\partial s_f^2} / \frac{\partial \Lambda_{Ff}}{\partial s_f} = \frac{k}{\varepsilon - 1} \frac{1}{1 - s_f}$, the derivative in (18) can be written as:

$$\begin{aligned}\frac{\partial W_{Ff}}{\partial s_f} &= \left(N \frac{k}{\alpha} \right)^{\frac{\mu\alpha}{\mu\alpha - k(\mu - \alpha)}} \frac{\tilde{M}_F \tilde{\Lambda}_{Ff}}{M_F \Lambda_{Ff}} \frac{\partial \Lambda_{Ff}}{\partial s_f} \frac{\alpha}{k} \left[\frac{k(1 - \alpha) - \mu\alpha}{\mu\alpha - k(\mu - \alpha)} \dots \right. \\ &\quad \left. \dots + s_f \frac{k(\alpha - \mu)}{\mu\alpha - k(\mu - \alpha)} \frac{1}{\Lambda_{Ff}} \frac{\partial \Lambda_{Ff}}{\partial s_f} - \frac{s_f}{1 - s_f} \frac{k}{\varepsilon - 1} \right] \quad (26)\end{aligned}$$

*School of Economics, Singapore Management University, 90 Stamford Rd, Singapore 178903. Tel: +(65) 6828-0876. Fax: +(65) 6828-0833. E-mail: davinchor@smu.edu.sg

The first summand on the right-hand side is positive, since $k(1 - \alpha) - \mu\alpha > 0$ follows from $k > \mu(\varepsilon - 1)$. Some algebraic substitution shows that the last two summands are equal to:

$$\begin{aligned} \frac{s_f}{1 - s_f} \frac{k}{\varepsilon - 1} \left[\frac{(k - \varepsilon + 1)(\alpha - \mu)}{\mu\alpha - k(\mu - \alpha)} g(s_f) - 1 \right] &> \frac{s_f}{1 - s_f} \frac{k}{\varepsilon - 1} \left[\frac{(k - \varepsilon + 1)(\alpha - \mu)}{\mu\alpha - k(\mu - \alpha)} - 1 \right] \\ &= \frac{s_f}{1 - s_f} \frac{k}{\varepsilon - 1} \frac{(\alpha - \mu)(1 - \varepsilon) - \mu\alpha}{\mu\alpha - k(\mu - \alpha)} \\ &> 0 \end{aligned}$$

for $s_f < 0$; we rely on the conditions: (i) $\varepsilon > 1$; and (ii) $\mu < \alpha$, to derive this last inequality. It follows that the expression in the square brackets in (18) is positive, and hence that $\frac{\partial W_{Ff}}{\partial s_f} > 0$ when $s_f < 0$. ■

Proof that $2(\varepsilon - 1) > k$ is a sufficient condition for W_{Ff} to have a unique turning point.

Setting (26) equal to zero, any turning point of the welfare function must satisfy:

$$\tilde{g}(s_f) \equiv \frac{k(1 - \alpha) - \mu\alpha}{\mu\alpha - k(\mu - \alpha)} \frac{1 - s_f}{s_f} + \frac{k}{\varepsilon - 1} \left(\frac{(k - \varepsilon + 1)(\alpha - \mu)}{\mu\alpha - k(\mu - \alpha)} g(s_f) - 1 \right) = 0$$

Note first that $\frac{1 - s_f}{s_f}$ is decreasing in s_f , while $g(s_f)$ is increasing in s_f , so $\tilde{g}(s_f)$ need not be monotonic. Nevertheless, $\lim_{s_f \rightarrow 0^+} \tilde{g}(s_f) = +\infty$, while $\lim_{s_f \rightarrow 1^-} \tilde{g}(s_f) = \frac{k}{\varepsilon - 1} \left(\frac{(k - \varepsilon + 1)(\alpha - \mu)}{\mu\alpha - k(\mu - \alpha)} - 1 \right) < 0$. For $\tilde{g}(s_f)$ to have a unique zero in $(0, 1)$, it therefore suffices that $\tilde{g}(s_f)$ be strictly convex in this interval. Now, $\frac{1 - s_f}{s_f}$ is strictly convex when $s_f > 0$, so I explore a sufficient condition for $g(s_f)$ to be strictly convex. Some differentiation and algebraic substitution yields:

$$g''(s_f) = \frac{k - \varepsilon + 1}{\varepsilon - 1} \frac{g(s_f)(1 - g(s_f))}{(1 - s_f)^2} \left[\frac{k}{\varepsilon - 1} - 2 \frac{k - \varepsilon + 1}{\varepsilon - 1} g(s) \right]$$

Since $g(s_f) \in (0, 1)$ for $s_f \in (0, 1)$, we have strict convexity if and only if $g(s_f) < \frac{k}{2(k - \varepsilon + 1)}$. A sufficient condition is therefore: $1 < \frac{k}{2(k - \varepsilon + 1)}$, or equivalently $2(\varepsilon - 1) > k$. ■

2 Details of Proof of Proposition 2

Proof. It suffices to examine the terms $W_{Fv} \equiv (1 - t_v)w_F + \frac{1 - \mu}{\mu}(X_F^H)^\mu$ in the indirect utility function. Use the budget constraint (19) to re-write t_v in terms of s_v . Substituting for t_v and using the expression for $(X_F)^\mu$ from (15), one obtains:

$$W_{Fv} = w_F + \left(N \frac{k}{\alpha} \right)^{\frac{\mu\alpha}{\mu\alpha - k(\mu - \alpha)}} \frac{\tilde{M}_F}{M_F} \left[\frac{1 - \mu}{\mu} \tilde{\Lambda}_{Fv} - s_v \left(\frac{\alpha}{k} \right) \frac{\tilde{\Lambda}_{Fv}}{\Lambda_{Fv}} \frac{\partial \Lambda_{Fv}}{\partial s_v} \right] \quad (27)$$

where:

$$\begin{aligned} \Lambda_{Fv} &= \frac{\varepsilon - 1}{k - \varepsilon + 1} \left(\frac{\alpha}{a_U} \right)^k \left(\frac{1 - \alpha}{w_H} \right)^{\frac{k}{\varepsilon - 1}} \left[\frac{(f_X)^{\frac{k}{1 - \varepsilon} + 1} w_H}{(\tau w_H)^k} + \frac{(f_I - f_X)^{\frac{k}{1 - \varepsilon} + 1} w_H}{(((1 - s_v)w_F)^{1 - \varepsilon} - (\tau w_H)^{1 - \varepsilon})^{\frac{k}{1 - \varepsilon}}} \right] \\ \frac{\partial \Lambda_{Fv}}{\partial s_v} &= \frac{\varepsilon - 1}{k - \varepsilon + 1} \left(\frac{\alpha}{a_U} \right)^k \left(\frac{1 - \alpha}{w_H} \right)^{\frac{k}{\varepsilon - 1}} \frac{(f_I - f_X)^{\frac{k}{1 - \varepsilon} + 1} w_H}{(((1 - s_v)w_F)^{1 - \varepsilon} - (\tau w_H)^{1 - \varepsilon})^{\frac{k}{1 - \varepsilon} + 1}} k w_F^{1 - \varepsilon} (1 - s_v)^{-\varepsilon} \end{aligned}$$

(Recall that $\tilde{\Lambda}_{Fv} = (\Lambda_{Fv})^{\frac{\mu\alpha}{\mu\alpha - k(\mu - \alpha)}}$.) The welfare function to be maximized (W_{Fv}) is thus analogous to that from the fixed cost subsidy case (W_{Ff}). It is straightforward to check that the proof that $\frac{\partial W_{Fv}}{\partial s_v} > 0$ at $s_v = 0$ follows essentially the same algebraic steps as the proof of Proposition 1. ■

Proof that $W_{Fv} \rightarrow -\infty$ when $s_v \rightarrow 1^-$. Observe that: $\lim_{s_v \rightarrow 1^-} \Lambda_{Fv} = \lim_{s_v \rightarrow 1^-} \tilde{\Lambda}_{Fv} = +\infty$. Moreover, some algebraic work shows that $\frac{1}{\Lambda_{Fv}} \frac{\partial \Lambda_{Fv}}{\partial s_v} = \frac{k}{1-s_v} \frac{((1-s_v)w_F)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} h(s_v)$, where:

$$h(s_v) = \left(\frac{(f_I - f_X)^{\frac{k}{1-\varepsilon} + 1}}{(((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon})^{\frac{k}{1-\varepsilon}}} \right) \bigg/ \left(\frac{(f_X)^{\frac{k}{1-\varepsilon} + 1}}{(\tau w_H)^k} + \frac{(f_I - f_X)^{\frac{k}{1-\varepsilon} + 1}}{(((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon})^{\frac{k}{1-\varepsilon}}} \right)$$

Now, $\lim_{s_v \rightarrow 1^-} h(s_v) = 1$, while $\lim_{s_v \rightarrow 1^-} \frac{((1-s_v)w_F)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} = 1 > 0$. Together, these imply that $\lim_{s_v \rightarrow 1^-} \frac{1}{\Lambda_{Fv}} \frac{\partial \Lambda_{Fv}}{\partial s_v} = +\infty$. The limit of the term in the square brackets in (27) as $s_v \rightarrow 1^-$ is therefore $-\infty$, so that we have $\lim_{s_v \rightarrow 1^-} W_{Ff} = -\infty$ as desired. ■

Proof that $\frac{\partial W_{Fv}}{\partial s_v} > 0$ for all $s_v < 0$. The derivative of (27) is:

$$\begin{aligned} \frac{\partial W_{Fv}}{\partial s_v} &= \left(N \frac{k}{\alpha} \right)^{\frac{\mu\alpha}{\mu\alpha - k(\mu - \alpha)}} \frac{\tilde{M}_F \tilde{\Lambda}_{Fv}}{M_F \Lambda_{Fv}} \frac{\partial \Lambda_{Fv}}{\partial s_v} \frac{\alpha}{k} \left[\frac{k(1-\alpha) - \mu\alpha}{\mu\alpha - k(\mu - \alpha)} + s_v \frac{k(\alpha - \mu)}{\mu\alpha - k(\mu - \alpha)} \frac{1}{\Lambda_{Fv}} \frac{\partial \Lambda_{Fv}}{\partial s_v} \dots \right. \\ &\quad \left. \dots - \frac{s_v}{1-s_v} \frac{(k+1)((1-s_v)w_F)^{1-\varepsilon} - \varepsilon(\tau w_H)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \right] \end{aligned}$$

Since $\frac{k(1-\alpha) - \mu\alpha}{\mu\alpha - k(\mu - \alpha)} > 0$, it suffices to show that the last two summands on the right-hand side add up to a positive quantity. Observe first that $(a_X)^{1-\varepsilon} < (a_I)^{1-\varepsilon}$ implies $\frac{f_X}{(\tau w_H)^{1-\varepsilon}} < \frac{f_I - f_X}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}}$. Using this inequality to replace f_X in the denominator of $h(s_v)$ and simplifying, one obtains: $h(s_v) < \frac{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon}}$. Algebraic substitution now shows that the last two summands are equal to:

$$\begin{aligned} &\frac{s_v}{1-s_v} \left[\frac{k^2(\alpha - \mu)}{\mu\alpha - k(\mu - \alpha)} \frac{((1-s_v)w_F)^{1-\varepsilon} h(s_v)}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} - \frac{(k+1)((1-s_v)w_F)^{1-\varepsilon} - \varepsilon(\tau w_H)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \right] \\ &> \frac{s_v}{1-s_v} \left[\frac{k^2(\alpha - \mu)}{\mu\alpha - k(\mu - \alpha)} - \frac{(k+1)((1-s_v)w_F)^{1-\varepsilon} - \varepsilon(\tau w_H)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \right] \end{aligned}$$

for $s_v < 0$. For this last expression to be positive when $s_v < 0$, it suffices to show that:

$$\frac{(k+1)((1-s_v)w_F)^{1-\varepsilon} - \varepsilon(\tau w_H)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} > k \quad (28)$$

since $\frac{k^2(\alpha - \mu)}{\mu\alpha - k(\mu - \alpha)} - k = \frac{-k\mu\alpha}{\mu\alpha - k(\mu - \alpha)} < 0$. This will then ensure that $\frac{\partial W_{Fv}}{\partial s_v} > 0$ whenever $s_v < 0$. Bearing in mind that $((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon} > 0$, rearranging (28) yields: $((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon} + (k - \varepsilon + 1)(\tau w_H)^{1-\varepsilon} > 0$, which holds since $k - \varepsilon + 1 > 0$. ■

3 Proofs from Section 3.4 (N endogenous)

Proof of Lemma 1. Log-differentiating (25), for all $s_f < 1$, one has:

$$\frac{\partial N_{Ff}}{\partial s_f} = N_{Ff} \frac{\mu\alpha}{k(\alpha - \mu)} \frac{\tilde{M}_F \tilde{\Lambda}_{Ff}}{\tilde{M}_H \tilde{\Lambda}_H + \tilde{M}_F \tilde{\Lambda}_{Ff}} \frac{1}{\Lambda_{Ff}} \frac{\partial \Lambda_{Ff}}{\partial s_f} > 0$$

We have an analogous expression for $\frac{\partial N_{Fv}}{\partial s_v}$ with the subscript f replaced by v throughout. ■

Sketch of proof of Proposition 3. I illustrate this for a fixed cost subsidy (the argument for a variable cost subsidy is identical). Using the expression for W_{Ff} from (17), we have shown in Appendix 7.1 that $\tilde{\Lambda}_{Ff} \left[\frac{1-\mu}{\mu} - s_f \left(\frac{\alpha}{k} \right) \frac{1}{\Lambda_{Ff}} \frac{\partial \Lambda_{Ff}}{\partial s_f} \right]$ is a positive and increasing function in s_f when $s_f < 0$. Since N_{Ff} is also a positive and increasing function in s_f for all $s_f < 1$, this implies that $\frac{\partial W_{Ff}}{\partial s_f} > 0$ when $s_f < 0$. Also, observe that $\lim_{s_f \rightarrow 1^-} N_{Ff} = +\infty$. Since $\tilde{\Lambda}_{Ff} \left[\frac{1-\mu}{\mu} - s_f \left(\frac{\alpha}{k} \right) \frac{1}{\Lambda_{Ff}} \frac{\partial \Lambda_{Ff}}{\partial s_f} \right]$ tends to $-\infty$ when s_f approaches 1, this implies that $\lim_{s_f \rightarrow 1^-} W_{Ff} = -\infty$.

Finally, the expression for $\frac{\partial W_{Ff}}{\partial s_f}$ when N is endogenous is given by (26) plus an extra term (from the product rule) to reflect the effect of s_f on N . This extra term is:

$$\frac{\mu\alpha}{\mu\alpha - k(\mu - \alpha)} \left(\frac{k}{\alpha} \right)^{\frac{\mu\alpha}{\mu\alpha - k(\mu - \alpha)}} (N_{Ff})^{\frac{k(\mu - \alpha)}{\mu\alpha - k(\mu - \alpha)}} \frac{\partial N_{Ff}}{\partial s_f} \frac{\tilde{M}_F}{M_F} \tilde{\Lambda}_{Ff} \left[\frac{1-\mu}{\mu} - s_f \left(\frac{\alpha}{k} \right) \frac{1}{\Lambda_{Ff}} \frac{\partial \Lambda_{Ff}}{\partial s_f} \right]$$

This is clearly positive at $s_f = 0$, and hence the slope of W_{Ff} at $s_f = 0$ is larger when N is endogenous when compared to the baseline case where N is fixed. ■

4 Proof of Proposition 4

Proof. We pursue a proof by contradiction. Suppose the total subsidy bills from s_f and s_v are equal, where $s_f, s_v \in (0, 1)$. From (17) and (27), this implies: $s_f \frac{\tilde{\Lambda}_{Ff}}{\Lambda_{Ff}} \frac{\partial \Lambda_{Ff}}{\partial s_f} = s_v \frac{\tilde{\Lambda}_{Fv}}{\Lambda_{Fv}} \frac{\partial \Lambda_{Fv}}{\partial s_v}$, or namely that:

$$\frac{s_f}{1-s_f} \tilde{\Lambda}_{Ff} \frac{k-\varepsilon+1}{\varepsilon-1} g(s_f) = \frac{s_v}{1-s_v} \tilde{\Lambda}_{Fv} \frac{k((1-s_v)w_F)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} h(s_v) \quad (29)$$

Suppose to the contrary that the consumption gains from the fixed cost subsidy are larger. Pulling out the relevant terms from the welfare functions, this means that $\tilde{\Lambda}_{Ff} \geq \tilde{\Lambda}_{Fv}$, which simplifies to:

$$\frac{((f_I - f_X)(1-s_f))^{\frac{k}{1-\varepsilon}+1}}{((w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon})^{\frac{k}{1-\varepsilon}}} \geq \frac{(f_I - f_X)^{\frac{k}{1-\varepsilon}+1}}{(((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon})^{\frac{k}{1-\varepsilon}}} \quad (30)$$

Observe that (30) implies that $g(s_f) \geq h(s_v)$, and hence that $\tilde{\Lambda}_{Ff} g(s_f) \geq \tilde{\Lambda}_{Fv} h(s_v)$. Looking back at (29), we must therefore have:

$$\frac{s_f}{1-s_f} \frac{k-\varepsilon+1}{\varepsilon-1} \leq \frac{s_v}{1-s_v} \frac{k((1-s_v)w_F)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \quad (31)$$

Moreover, (30) simplifies directly to $s_f \geq 1 - \left[\frac{(w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \right]^{\frac{k}{k-\varepsilon+1}}$, which implies that:

$$\frac{s_f}{1-s_f} \geq \left[\frac{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}}{(w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \right]^{\frac{k}{k-\varepsilon+1}} - 1 \quad (32)$$

Combining (31) and (32) to eliminate $\frac{s_f}{1-s_f}$, the following inequality needs to be satisfied:

$$\left[\frac{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}}{(w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \right]^{\frac{k}{k-\varepsilon+1}} - 1 - \frac{s_v}{1-s_v} \frac{k(\varepsilon-1)}{k-\varepsilon+1} \frac{((1-s_v)w_F)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \leq 0 \quad (33)$$

Define the function in s_v on the left-hand side of (33) to be $\psi(s_v)$. Note that $\psi(0) = 0$. I now show that if $\varepsilon > 2$, then $\psi'(s_v) > 0$ for all $s_v \in (0, 1)$, so that $\psi(s_v) > 0$ for all $s_v > 0$. This will yield the desired contradiction to (33). Now, $\psi'(s_v)$ is equal up to a positive multiplicative constant to:

$$\begin{aligned}
& \left[\frac{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}}{(w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \right]^{\frac{k}{k-\varepsilon+1}} - \frac{1}{1-s_v} + \frac{s_v}{1-s_v} \frac{(\varepsilon-1)(\tau w_H)^{1-\varepsilon}}{((1-s_v)w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \\
> & \left[1 + \frac{((1-s_v)^{1-\varepsilon} - 1)(w_F)^{1-\varepsilon}}{(w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon}} \right]^{\frac{k}{k-\varepsilon+1}} - \frac{1}{1-s_v} \\
> & (1 + ((1-s_v)^{1-\varepsilon} - 1))^{\frac{k(1-\varepsilon)}{k-\varepsilon+1}} - \frac{1}{1-s_v} \\
= & \frac{1}{1-s_v} \left[(1-s_v)^{\frac{k(1-\varepsilon)}{k-\varepsilon+1} + 1} - 1 \right]
\end{aligned}$$

This last expression is positive for all $s_v \in (0, 1)$ if and only if: $\frac{k(1-\varepsilon)}{k-\varepsilon+1} + 1 < 0$. This holds when $\varepsilon > \frac{2k+1}{k+1}$, and in particular when $\varepsilon > 2$. ■

5 Sketch of proof of Proposition 5 [Import Subsidies]

The proofs on the welfare implications of an import subsidy mirror closely those for an FDI subsidy. The exposition below is therefore brief, showing that the welfare function under an import subsidy has a positive slope when the subsidy level is less than or equal to zero, but asymptotes towards $-\infty$ as the subsidy level tends towards its maximum value. The optimal policy is thus a strictly positive subsidy. We focus on the case of a subsidy to the fixed cost, f_X ; the proof for a variable cost subsidy is similar.

Fixed cost subsidy to Home exporters. Consider a subsidy that reduces the fixed cost of exporting for each Home firm by the amount $s_f f_X w_H$, with $s_f < 1$. Suppose as before that this is financed by a tax on labor income equal to $t_f w_H$. The balanced-budget constraint for Foreign is:

$$t_f w_F M_F = s_f f_X w_H N (G^H(a_X) - G^H(a_I))$$

Substituting the implied value of t_f from this budget constraint into $W_{Ff} = (1 - t_f)w_F + \frac{1-\mu}{\mu}(X_F^H)^\mu$, one obtains the following expression for welfare in Foreign:

$$W_{Ff} = w_F + \left(N \frac{k}{\alpha} \right)^{\frac{\mu\alpha}{\mu\alpha - k(\mu-\alpha)}} \frac{\tilde{M}_F}{M_F} \left[\frac{1-\mu}{\mu} \tilde{\Phi}_{Ff} - s_f \left(\frac{\alpha}{k} \right) \frac{\tilde{\Phi}_{Ff}}{\Phi_{Ff}} \frac{\partial \Phi_{Ff}}{\partial s_f} \right] \quad (34)$$

where:

$$\begin{aligned}
\Phi_{Ff} &= \frac{\varepsilon-1}{k-\varepsilon+1} \left(\frac{\alpha}{a_U} \right)^k \left(\frac{1-\alpha}{w_H} \right)^{\frac{k}{\varepsilon-1}} \left[\frac{((1-s_f)f_X)^{\frac{k}{1-\varepsilon}+1} w_H}{(\tau w_H)^k} + \frac{(f_I - (1-s_f)f_X)^{\frac{k}{1-\varepsilon}+1} w_H}{((w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon})^{\frac{k}{1-\varepsilon}}} \right] \\
\frac{\partial \Phi_{Ff}}{\partial s_f} &= \left(\frac{\alpha}{a_U} \right)^k \left(\frac{1-\alpha}{w_H} \right)^{\frac{k}{\varepsilon-1}} \left[\frac{(1-s_f)^{\frac{k}{1-\varepsilon}} (f_X)^{\frac{k}{1-\varepsilon}+1} w_H}{(\tau w_H)^k} - \frac{(f_I - (1-s_f)f_X)^{\frac{k}{1-\varepsilon}} f_X w_H}{((w_F)^{1-\varepsilon} - (\tau w_H)^{1-\varepsilon})^{\frac{k}{1-\varepsilon}}} \right]
\end{aligned}$$

and $\tilde{\Phi}_{Ff} = (\Phi_{Ff})^{\frac{\mu\alpha}{\mu\alpha - k(\mu - \alpha)}}$. Note that Φ_{Ff} is precisely equal to Λ_F with f_X replaced by $(1 - s_f)f_X$. (The switch of notation to Φ is intended to avoid a clash with Λ , which has been used for the analysis of FDI subsidies.) The welfare function in (34) clearly parallels that in (17) for the case of a fixed cost FDI subsidy, except that the *ex ante* profits from sales in Foreign are now given by $\tilde{\Phi}_{Ff}$ instead of Λ_{Ff} . The expression for Φ_{Ff} also makes apparent the two opposing effects that an import subsidy has: The first summand in the square brackets captures how s_f lowers the $a_X^{1-\varepsilon}$ threshold for exporting, which results in consumption gains for Foreign, but the second summand captures how s_f raises the $a_I^{1-\varepsilon}$ cut-off for FDI, which cuts into these consumption gains.

Differentiating (34) with respect to s_f yields:

$$\begin{aligned} \frac{\partial W_{Ff}}{\partial s_f} = & \left(N \frac{k}{\alpha} \right)^{\frac{\mu\alpha}{\mu\alpha - k(\mu - \alpha)}} \frac{\tilde{M}_F \tilde{\Phi}_{Ff}}{M_F \Phi_{Ff}} \frac{\partial \Phi_{Ff}}{\partial s_f} \frac{\alpha}{k} \left[\frac{k(1 - \alpha) - \mu\alpha}{\mu\alpha - k(\mu - \alpha)} \dots \right. \\ & \left. \dots + s_f \frac{k(\alpha - \mu)}{\mu\alpha - k(\mu - \alpha)} \frac{1}{\Phi_{Ff}} \frac{\partial \Phi_{Ff}}{\partial s_f} - s_f \left(\frac{\partial^2 \Phi_{Ff}}{\partial s_f^2} / \frac{\partial \Phi_{Ff}}{\partial s_f} \right) \right] \end{aligned}$$

Evaluating s_f at 0, it is straightforward to check once again that $\frac{\partial W_{Ff}}{\partial s_f} > 0$, given that $k > \mu(\varepsilon - 1)$. Thus, a small subsidy to exporting firms from Home raises welfare in Foreign.

Moreover, as $s_f \rightarrow 1^-$, we have $\Phi_{Ff}, \tilde{\Phi}_{Ff}, \frac{1}{\Phi_{Ff}} \frac{\partial \Phi_{Ff}}{\partial s_f} \rightarrow +\infty$. This implies from (34) that W_{Ff} asymptotes to $-\infty$ as s_f tends towards its maximum value of 1. Last but not least, $\frac{\partial \Phi_{Ff}}{\partial s_f} > 0$ so long as $a_X^{1-\varepsilon} < a_I^{1-\varepsilon}$. One can also verify that: $\frac{1}{\Phi_{Ff}} \frac{\partial \Phi_{Ff}}{\partial s_f} < \frac{k-\varepsilon+1}{\varepsilon-1} \frac{1}{1-s_f}$ and $\frac{\partial^2 \Phi_{Ff}}{\partial s_f^2} / \frac{\partial \Phi_{Ff}}{\partial s_f} > \frac{k}{\varepsilon-1} \frac{1}{1-s_f}$. Substituting these two inequalities into the above expression for $\frac{\partial W_{Ff}}{\partial s_f}$, one can then show that when $s_f < 0$, we have $\frac{\partial W_{Ff}}{\partial s_f} > 0$.

It is straightforward to see that these properties of W_{Ff} continue to hold when N is endogenized, as the argument in Appendix 7.3 can be adapted with Φ_{Ff} replacing Λ_{Ff} throughout. ■

6 Robustness to specification of utility function

I establish the robustness of the welfare results on FDI subsidies under the following alternative utility function:

$$U_i = \left((x_i^0)^\rho + \left(\sum_{c=H,F} \frac{1}{\mu} (X_i^c)^\mu \right)^{\frac{\rho}{\mu}} \right)^{\frac{1}{\rho}}, \quad 0 < \rho < \mu < \alpha < 1 \quad (35)$$

under which the country- i demand for country- c differentiated products will now depend on labor income, as well as the prices of products from country $c' \neq c$. In particular, this introduces more general income effects into the demand for differentiated products. Also, the utility derived from the consumption of Home and Foreign products will no longer be additively separable, relaxing this feature of the baseline model. The exposition focuses on a fixed cost subsidy; the proof for a variable cost subsidy is similar. I

focus also on the exogenous N case, wherein the policy action by Foreign does not affect the number of varieties, N . The more general setting that endogenizes N is more algebraically involved, without adding substantially new insights.

Income effects. Suppose that the utility of a representative consumer from country i is given instead by (35). Maximizing this utility function subject to the budget constraint (2), one obtains a demand function for differentiated products of the familiar iso-elastic form, $x_i^c(a) = A_i^c p_i^c(a)^{-\varepsilon}$, $\varepsilon = \frac{1}{1-\alpha}$, where the level of aggregate demand is now given by:

$$A_i^c = M_i w_i \times (P_i^c)^{\frac{1}{1-\alpha} \frac{\alpha-\mu}{1-\mu}} \times \frac{\mu^{-\frac{1}{\mu} \frac{\rho}{1-\rho}} (\sum_{c=H,F} (P_i^c)^{-\frac{\mu}{1-\mu}})^{\frac{1}{\mu} \frac{\rho-\mu}{1-\rho}}}{1 + \mu^{-\frac{1}{\mu} \frac{\rho}{1-\rho}} (\sum_{c=H,F} (P_i^c)^{-\frac{\mu}{1-\mu}})^{\frac{1-\mu}{\mu} \frac{\rho}{1-\rho}}}$$

and $P_i^c = (\int_{\Omega_i^c} p_i^c(a)^{-\frac{\alpha}{1-\alpha}} dG^c(a))^{\frac{1-\alpha}{\alpha}}$ is the ideal price index of country- c differentiated products in the country i market. As claimed, the demand function for differentiated products now depends on labor income, as well as the prices of products from the other country. Substituting this demand function into the budget constraint, it is straightforward to derive the residual demand for the homogeneous good, and from there obtain an expression for the indirect utility function to serve as our welfare metric:

$$W_i = w_i \left[1 + \mu^{-\frac{1}{\mu} \frac{\rho}{1-\rho}} \left(\sum_{c=H,F} (P_i^c)^{-\frac{\mu}{1-\mu}} \right)^{\frac{1-\mu}{\mu} \frac{\rho}{1-\rho}} \right]^{\frac{1-\rho}{\rho}} \quad (36)$$

Our key exercise focuses on $i = F$, namely whether Foreign can raise W_F through an FDI subsidy.

Turning from the demand side to the industry equilibrium, the structure of the Home differentiated goods sector is identical to that in the baseline model with quasilinear utility in Section 2. In particular, we maintain the Pareto parametrization for the distribution of productivity draws in each country's differentiated goods sector. Therefore, from (13) and (14), the ideal price indices for differentiated goods purchased by Foreign are equal to:

$$P_F^c = \left[N^c \frac{k}{\alpha} \Lambda_{Ff}^c (A_F^c)^{\frac{k-\varepsilon+1}{\varepsilon-1}} \right]^{-\frac{1-\alpha}{\alpha}}, \quad c = H, F \quad (37)$$

where N^c is the measure of firms in the country- c differentiated goods sector, and Λ_{Ff}^c are the expected profits which firms from this country- c differentiated goods sector can earn from their sales in the Foreign market when the demand level in Foreign is normalized to 1. (The additional f subscript is a reminder that these expected profits are computed taking into account a fixed cost FDI subsidy, s_f .)

Consider now a subsidy by Foreign that reduces the fixed cost of FDI for Home firms by $s_f(f_I - f_X)w_H$. The balanced-budget constraint (16) still applies, and can be re-written (after some algebraic work) to obtain the following expression for the tax rate levied on Foreign workers to fund the subsidy program:

$$t_f = \frac{s_f N^H}{w_F M_F} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} (A_F^H)^{\frac{k}{\varepsilon-1}} \quad (38)$$

In the presence of this income tax, the expression for the level of Foreign demand for differentiated goods needs to be modified to:

$$A_F^c = M_F(1 - t_f)w_F \times (P_F^c)^{\frac{1}{1-\alpha} \frac{\alpha-\mu}{1-\mu}} \times \frac{\mu^{-\frac{1}{\mu} \frac{\rho}{1-\rho}} (\sum_{c=H,F} (P_i^c)^{-\frac{\mu}{1-\mu}})^{\frac{1}{\mu} \frac{\rho-\mu}{1-\rho}}}{1 + \mu^{-\frac{1}{\mu} \frac{\rho}{1-\rho}} (\sum_{c=H,F} (P_i^c)^{-\frac{\mu}{1-\mu}})^{\frac{1-\mu}{\mu} \frac{\rho}{1-\rho}}}, \quad c = H, F \quad (39)$$

Observe that (37), (38), and (39) form a system of five equations in the five variables, A_F^H , A_F^F , P_F^H , P_F^F , and t_f , in which s_f acts as a state variable. A complication that arises relative to the quasilinear utility case is that we cannot solve for P_F^c as a closed-form function of the model parameters only, so the behavior of the welfare function with respect to the subsidy rate, s_f , will have to be deduced by performing comparative statics on the above system of five equations.

Let W_{Ff} be equal to W_F with labor income w_F replaced by $(1 - t_f)w_F$ in (36). Log-differentiating with respect to s_f yields:

$$\frac{1}{W_{Ff}} \frac{dW_{Ff}}{ds_f} = -\frac{1}{1 - t_f} \frac{dt_f}{ds_f} - \phi_1 \left(\phi_2 \frac{1}{P_F^H} \frac{dP_F^H}{\partial s_f} + (1 - \phi_2) \frac{1}{P_F^F} \frac{dP_F^F}{\partial s_f} \right) \quad (40)$$

where we have defined for notational ease:

$$\begin{aligned} \phi_1 &= \frac{\mu^{-\frac{1}{\mu} \frac{\rho}{1-\rho}} (\sum_{c=H,F} (P_i^c)^{-\frac{\mu}{1-\mu}})^{\frac{1-\mu}{\mu} \frac{\rho}{1-\rho}}}{1 + \mu^{-\frac{1}{\mu} \frac{\rho}{1-\rho}} (\sum_{c=H,F} (P_i^c)^{-\frac{\mu}{1-\mu}})^{\frac{1-\mu}{\mu} \frac{\rho}{1-\rho}}} \in (0, 1) \\ \phi_2 &= \frac{(P_F^H)^{-\frac{\mu}{1-\mu}}}{\sum_{c=H,F} (P_i^c)^{-\frac{\mu}{1-\mu}}} \in (0, 1) \end{aligned}$$

To obtain an expression for $\frac{dt_f}{ds_f}$, totally differentiating (38) yields:

$$\begin{aligned} \frac{dt_f}{ds_f} &= \frac{t_f}{s_f} + t_f \left[\frac{\partial^2 \Lambda_{Ff}^H}{\partial s_f^2} / \frac{\partial \Lambda_{Ff}^H}{\partial s_f} + \frac{k}{\varepsilon - 1} \frac{1}{A_F^H} \frac{dA_F^H}{ds_f} \right] \\ &= \frac{\alpha}{k} \frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} (1 - t_f) \phi_1 \phi_2 + t_f \left[\frac{\partial^2 \Lambda_{Ff}^H}{\partial s_f^2} / \frac{\partial \Lambda_{Ff}^H}{\partial s_f} + \frac{k}{\varepsilon - 1} \frac{1}{A_F^H} \frac{dA_F^H}{ds_f} \right] \end{aligned} \quad (41)$$

Note that the expression $\frac{t_f}{s_f} = \frac{\alpha}{k} \frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} (1 - t_f) \phi_1 \phi_2$ used in this last step is derived from (38).

Similarly, (37) implies:

$$\frac{1}{P_F^H} \frac{dP_F^H}{ds_f} = -\frac{1 - \alpha}{\alpha} \frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} - \frac{1 - \alpha}{\alpha} \frac{k - \varepsilon + 1}{\varepsilon - 1} \frac{1}{A_F^H} \frac{dA_F^H}{ds_f} \quad (42)$$

$$\frac{1}{P_F^F} \frac{dP_F^F}{ds_f} = \frac{1}{P_F^H} \frac{dP_F^H}{ds_f} + \frac{\alpha(1 - \mu)}{\mu\alpha - k(\mu - \alpha)} \frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} \quad (43)$$

Totally differentiating (39) and using (43) to replace the terms involving $\frac{1}{P_F^F} \frac{dP_F^F}{ds_f}$ yields:

$$\begin{aligned} \frac{1}{A_F^H} \frac{dA_F^H}{ds_f} &= -\frac{1}{1 - t_f} \frac{dt_f}{ds_f} + \left(\frac{\alpha - \rho}{(1 - \alpha)(1 - \rho)} + \frac{\rho}{1 - \rho} \phi_1 \right) \frac{1}{P_F^H} \frac{dP_F^H}{ds_f} \dots \\ &\dots + \left(\frac{\mu - \rho}{(1 - \mu)(1 - \rho)} + \frac{\rho}{1 - \rho} \phi_1 \right) (1 - \phi_2) \frac{\alpha(1 - \mu)}{\mu\alpha - k(\mu - \alpha)} \frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} \end{aligned} \quad (44)$$

Now, (41), (42) and (44) can be solved simultaneously to obtain expressions for $-\frac{1}{1-t_f} \frac{dt_f}{ds_f}$ and $\frac{1}{P_F^H} \frac{dP_F^H}{ds_f}$ in terms of model parameters only. These can then be substituted into (40) to obtain:

$$\left(K_0 + \frac{t_f}{1-t_f} \frac{k}{\varepsilon-1} \right) \frac{1}{W_{Ff}} \frac{dW_{Ff}}{ds_f} = -K_1 \frac{t_f}{1-t_f} \left(\frac{\partial^2 \Lambda_{Ff}^H}{\partial s_f^2} / \frac{\partial \Lambda_{Ff}^H}{\partial s_f} \right) + K_2 \frac{t_f}{1-t_f} \frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} + K_3 \frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} \quad (45)$$

where the various constants are given by:

$$\begin{aligned} K_0 &= 1 + \frac{1-\alpha}{\alpha} \frac{k-\varepsilon+1}{\varepsilon-1} \left(\frac{\alpha-\rho}{(1-\alpha)(1-\rho)} + \frac{\rho}{1-\rho} \phi_1 \right) \\ K_1 &= 1 + \frac{1-\alpha}{\alpha} \frac{k-\varepsilon+1}{\varepsilon-1} \left(\frac{\alpha-\rho}{(1-\alpha)(1-\rho)} + \frac{1}{1-\rho} \phi_1 \right) \\ K_2 &= \frac{k}{\varepsilon-1} \frac{1}{1-\rho} \left[\frac{1-\alpha}{\alpha} \left(\phi_1 + \frac{\alpha-\rho}{1-\alpha} \right) - (1-\phi_2) \frac{\alpha(1-\mu)}{\mu\alpha-k(\mu-\alpha)} \left(\phi_1 + \frac{\mu-\rho}{1-\mu} \right) \right] \\ K_3 &= \frac{\phi_1 \phi_2}{k} \frac{1-\alpha}{1-\rho} \left[\frac{k}{\varepsilon-1} - \rho - \left(\frac{k}{\varepsilon-1} - 1 \right) \phi_1 \right] \end{aligned}$$

(45) readily allows us to analyze the behavior of the Foreign welfare function. At $s_f = 0$, the tax t_f is also zero. Since $\frac{\partial \Lambda_{Ff}^H}{\partial s_f} > 0$, this implies that $\text{sign}\left(\frac{dW_{Ff}}{ds_f}\right) = \text{sign}\left(\frac{K_3}{K_0}\right)$. Clearly, $K_0 > 0$ given that $0 < \rho < \alpha < 1$, $\varepsilon > 1$, and $k > \varepsilon - 1$. Moreover, $K_3 > 0$ since $\frac{k}{\varepsilon-1} - \rho > \frac{k}{\varepsilon-1} - 1 > \left(\frac{k}{\varepsilon-1} - 1\right)\phi_1$. Thus, $\frac{dW_{Ff}}{ds_f} > 0$ in the neighborhood of $s_f = 0$, and so a small subsidy improves welfare under this more general utility specification.

As $s_f \rightarrow 1^-$, observe that $\frac{t_f}{1-t_f} = \frac{\alpha}{k} \frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} \phi_1 \phi_2 \rightarrow +\infty$, where we recall from Appendix 7.1 that $\frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} \rightarrow +\infty$. Recall also that $\frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f} = \frac{k-\varepsilon+1}{\varepsilon-1} \frac{1}{1-s_f} g(s_f)$ and $\frac{\partial^2 \Lambda_{Ff}^H}{\partial s_f^2} / \frac{\partial \Lambda_{Ff}^H}{\partial s_f} = \frac{k}{\varepsilon-1} \frac{1}{1-s_f}$. Some algebraic simplification leads to:

$$\begin{aligned} \frac{1}{W_{Ff}} \frac{dW_{Ff}}{ds_f} &\rightarrow \frac{1}{1-s_f} \left\{ -\frac{k}{\varepsilon-1} K_1 + \frac{k-\varepsilon+1}{\varepsilon-1} g(s_f) K_2 \right\} / \frac{k}{\varepsilon-1} \\ &= \frac{1}{1-s_f} \left\{ -1 - \frac{k-\varepsilon+1}{\varepsilon-1} \frac{1}{1-\rho} \left[\frac{1-\alpha}{\alpha} \left(\phi_1 + \frac{\alpha-\rho}{1-\alpha} \right) (1-g(s_f)) \dots \right. \right. \\ &\quad \left. \left. \dots + (1-\phi_2) \frac{\alpha(1-\mu)}{\mu\alpha-k(\mu-\alpha)} \left(\phi_1 + \frac{\mu-\rho}{1-\mu} \right) g(s_f) \right] \right\} \\ &\rightarrow -\infty \end{aligned}$$

since the expression in the large curly braces in the penultimate step is negative (bearing in mind that $g(s_f) \in (0, 1)$). The welfare function therefore asymptotes to negative infinity as the subsidy is raised to its maximum possible level.

Finally, it remains to show that W_{Ff} is an increasing function in s_f when s_f and hence t_f are negative. Observe that the right-hand side of (45) is equal to: $\frac{1}{1-s_f} \frac{t_f}{1-t_f} \left\{ -\frac{k}{\varepsilon-1} K_1 + \frac{k-\varepsilon+1}{\varepsilon-1} g(s_f) K_2 \right\} + K_3 \frac{1}{\Lambda_{Ff}^H} \frac{\partial \Lambda_{Ff}^H}{\partial s_f}$. The second summand is clearly positive since $K_3 > 0$. Moreover, we have just seen that $\left\{ -\frac{k}{\varepsilon-1} K_1 + \frac{k-\varepsilon+1}{\varepsilon-1} g(s_f) K_2 \right\} < 0$, from which it follows that $\frac{1}{1-s_f} \frac{t_f}{1-t_f} \left\{ -\frac{k}{\varepsilon-1} K_1 + \frac{k-\varepsilon+1}{\varepsilon-1} g(s_f) K_2 \right\} > 0$ when $t_f < 0$. Thus, the right-hand side of (45) is positive when $s_f < 0$. As for the coefficient of $\frac{1}{W_{Ff}} \frac{dW_{Ff}}{ds_f}$ on the left-hand side, note that $\frac{t_f}{1-t_f} = \frac{\alpha}{k} \frac{s_f}{1-s_f} \frac{k-\varepsilon+1}{\varepsilon-1} g(s_f) \phi_1 \phi_2 > -\frac{\alpha}{k} \frac{k-\varepsilon+1}{\varepsilon-1} \phi_1 \phi_2$, since $\frac{s_f}{1-s_f} \in (-1, 0)$ when $s_f < 0$. Substituting this lower bound for $\frac{t_f}{1-t_f}$ and simplifying, we have:

$$\begin{aligned}
K_0 + \frac{t_f}{1-t_f} \frac{k}{\varepsilon-1} &> 1 - \frac{\alpha}{\varepsilon-1} \frac{k-\varepsilon+1}{\varepsilon-1} \phi_1 \phi_2 + \frac{1-\alpha}{\alpha} \frac{k-\varepsilon+1}{\varepsilon-1} \left(\frac{\alpha-\rho}{(1-\alpha)(1-\rho)} + \frac{\rho}{1-\rho} \phi_1 \right) \\
&> 1 - \frac{\alpha}{\varepsilon-1} \frac{k-\varepsilon+1}{\varepsilon-1} \phi_1 + \frac{1-\alpha}{\alpha} \frac{k-\varepsilon+1}{\varepsilon-1} \left(\frac{\alpha-\rho}{1-\alpha} + \rho \phi_1 \right) \\
&= 1 + \frac{k-\varepsilon+1}{(\varepsilon-1)^2} (\alpha-\rho) \left(\frac{1}{1-\alpha} - \phi_1 \right) \\
&> 0
\end{aligned}$$

Thus, $\frac{dW_{Ff}}{ds_f} > 0$ for all $s_f < 0$. ■