Algorithmic Trading and the Market Price of Liquidity Risk

Quantitative Finance
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In the Past: High-Touch Brokers

Client → Broker → Market

Client → Broker → telephone → order to the floor

Client → Broker → telephone → confirmation

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Algorithmic Trading (AT)

**Definition**

- The use of programs (algorithms) and computers (automation) to generate and submit orders in electronic markets.

- Origin: U.S. institutional investors in late 1990s needed the tools to deal with major changes:
  - electronic markets: electronic communications networks
  - alternative trading systems, dark pools
  - decimalization (reduction) of tick size
  - reduction of commissions and exchange fees

- Today, brokers compete actively for the commission pool associated with algorithmic trading around the globe.
Characteristics of Algorithmic Trading (AT)

- **Non-discretionary**
  - Purely procedural or mechanical
  - Emotionless

- **Algorithmic**
  - Trading rules
  - Artificial intelligence, machine learning, cyborg

- **Data driven**
  - Tick-by-tick trades and quotes, news, fundamentals etc.
  - Live, real-time feeds absolutely necessary

- **Automatic**
  - Automation: major investment in high-tech IT infrastructure (ultra high-frequency, low latency, direct market access, co-location)
  - If the algorithm is not speed-sensitive, automation is not absolutely necessary.
Low-Touch Electronic Market

Client → Broker → Market

Client submits an order to the broker, who then places the order on the market. The client receives a confirmation after the order is executed.

Broker → Client

Broker communicates with the client regarding the execution of the order.
Electronic Market with API Gateway

- Algo orders thru API Gateway
- Algo execution
- leased line/internet
- confirmation
Direct Market Access (DMA) and Co-loc

- No touch
- Co-location: the trading algo sits in a server next to and connected to the exchange’s matching engine
- Low latency
Arrownet

Source: JPX
Market Fragmentation: Case Study of Japan

- **Exchanges**
  - **JPX Group**
    - First Section
    - Second Section
    - Jasdaq
    - Mothers
  - **Nagoya Stock Exchange**
  - **Sapporo Securities Exchange**
  - **Fukuoka Stock Exchange**

- **Proprietary trading system (PTS)**
  - **Chi-X Japan**
  - **SBI Japannext**

- **Exchanges versus PTS: More than one venue to transact the same stock.**
Adoption of AT in Asia

![Chart showing the adoption of Algorithmic Trading (AT) in Major Asian Cash Equities Markets between 2008 and 2018. The chart indicates increasing adoption rates for major Asian countries, with Japan leading and China following, while India and Australia show slower growth. The data is sourced from Aite Group.](chart.png)
Three Building Blocks of AT

- Order Routing Strategies
- Order Placement Strategies
- Algorithmic Trading Strategies
Two Major Categories of AT Strategies

Alpha Generation, Buy Side
- Location arbitrage
- Triangular arbitrage
- Statistical arbitrage
  - Momentum
  - Mean reversion
- Market making
Two Major Categories of AT Strategies (cont’d)

Brokerage Execution, Sell Side

- Services rendered by sell side to and for the buy side
- Time-weighted average price (TWAP)
- Volume-weighted average price (VWAP)
- Percentage of volume (POV)
- The Almgren-Chriss algorithm
- Liquidity-seeking
Institutional Brokerage Execution

- Since the 2008 financial crisis, “alpha” generation is strongly curtailed for major investment banks.

- Nevertheless, institutional clients need to trade large amounts of stocks, larger than what the market can absorb without impacting the price.

- To prevent slippage, a large (parent) order must be sliced into many smaller child orders.

- What is the algorithm to bring about an optimal solution?

- Providing trade execution services for institutional clients is a BIG business these days.
First-Generation AT for Brokerage Execution

- **TWAP**: Equal amount of shares or contracts in each time interval.

- But market’s trading volume typically exhibits a *U-shape pattern* from opening to closing.

- **VWAP**: Divide the trading session into 5- to 15-minute interval. Predict intra-day trading volume and price range for each interval by some analytics. In each time-interval, execute an amount proportional to the predicted volume for that interval.

- **POV**: Target a defined fraction of the actual volume for each time interval. The intention is to keep the trading activity in line with total volume. Trade at a constant percentage *participation rate*. 
Volume Weighted Average Price

Let $p_i, v_i, i = 1, 2, \ldots n$ be the prices and volumes, respectively for which $n$ child orders are filled.

Total volume traded is

$$v = \sum_{i=1}^{n} v_i.$$ 

With $w_i = \frac{v_i}{v}$ being the weight, The VWAP price is, by definition,

$$P_{vwap} := \sum_{i=1}^{n} w_i p_i = \sum_{i=1}^{n} \frac{v_i}{v} p_i = \frac{\sum_{i=1}^{n} p_i v_i}{v} = \frac{\text{Dollar Volume}}{\text{Volume}}$$

VWAP is the break-even price.
Objective Function and Approximation

- Objective of the algorithm is to minimize the expected error (see [Konishi, 2002]):

\[
\min_{v(t)} \mathbb{E} \left( (\text{VWAP} - \text{vwap})^2 \right)
\]

- Definition: Percentage of remaining volume

\[
X(t) := \frac{M(T) - M(t)}{M(T)}; \quad x(t) := \frac{v(T) - v(t)}{v(T)}
\]

- Change of objective to

\[
\min_{x(t)} \mathbb{E} \left( \left( \int_0^T (X(t) - x(t))dP(t) \right)^2 \right)
\]
[Konishi, 2002] shows that an optimal schedule is

$$x^*(t) = \mathbb{E}(X(t)) \pm \frac{1}{2v(T)}.$$

The optimal schedule is to overshoot $\mathbb{E}(X(t))$ by $\pm 1/(2v(T))$, which is a small amount if the number of shares to trade is large.

It is crucial to forecast the mean $X(t)$ at each time period $t$, which is the proxy for $\mathbb{E}(X(t))$.

Therefore, you need to forecast the total volume $M(T)$ of the market, and the dynamics of the market volume $M(t)$!
Second-Generation AT for Brokerage Execution

- [Almgren and Chriss, 2000]’s mathematical modelling

Consider liquidating $X$ shares. Let the number of shares yet to be liquidated be $x_t$, $t = 0, 1, \ldots, T$. So

$$x_0 = X, \quad x_T = 0,$$

and

$$x_s \leq x_t \quad \text{if } s > t.$$

Accordingly, $-(x_{t+\epsilon} - x_t)$ is the number of stocks sold in the time interval $(t + \epsilon, t)$. Hence

$$x_{t+\epsilon} - x_t \longrightarrow dx_t, \quad \text{as } \epsilon \rightarrow 0.$$
Almgren-Chriss' Assumptions

- The price $S_t$ is a drift-less (arithmetic) Brownian motion $B_t$:

$$dS_t = \sigma dB_t$$

with constant (intra-day) volatility $\sigma$.

- The price at which a sale takes place is not $S_t$ but $P_t$ given by

$$P_t = S_t + \eta v_t,$$

where the rate of trading is

$$v_t \equiv \dot{x}_t := \frac{dx_t}{dt},$$

i.e., $x_t$ is a (deterministically) differentiable function of $t$. 
Trading Risk

Trading risk is captured by

\[
\mathcal{V} := \nabla \left( \int_0^T x_t dS_t \right) = \sigma^2 \int_0^T x_t^2 \, dt.
\]

Main idea of [Almgren and Chriss, 2000] is to minimize the mean and variance with \( \lambda \) being the Lagrange multiplier:

\[
\mathcal{A} := \mathbb{E} \left( \int_0^T P_t \nu_t \, dt \right) + \lambda \nabla \left( \int_0^T x_t dS_t \right)
\]

\[
= \mathcal{K} + \lambda \sigma^2 \int_0^T x_t^2 \, dt
\]

\[
= \int_0^T \left[ \eta \left( \frac{dx_t}{dt} \right)^2 + \lambda \sigma^2 x_t^2 \right] \, dt
\]
The Euler-Lagrange equation (principle of least action) is

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_t} - \frac{\partial L}{\partial x_t} = 0. \]

For the [Almgren and Chriss, 2000] Lagrangian, i.e.,

\[ L = \eta \dot{x}_t^2 - \left( - \lambda \sigma^2 x_t^2 \right), \]

it is

\[ 2\eta \ddot{x}_t - 2\lambda \sigma^2 x_t = 0, \]

i.e., \( \ddot{x}_t = \kappa^2 x_t \), where

\[ \kappa^2 = \frac{\lambda \sigma^2}{\eta} \]

The solution that satisfies the boundary conditions is

\[ x_t^* = X \frac{\sinh \left( \kappa (T - t) \right)}{\sinh(\kappa T)} \]
Optimal Schedule for Different Urgency $\kappa$
Third-Generation: Liquidity-Seeking AT

- Multi-period optimization of trade scheduling
- Access to all available liquidity, both dark and lit
- Inclusion of order placement and order routing algorithms
- Urgency specification
  - Low: dark, passive
  - Medium: combination of dark and opportunistic participation
  - High urgency: high participation rate
- Optimization of probability of fill (execution certainty), invisibility (stealth), price improvement (passiveness) trading cost (implementation shortfall), and price certainty (risk) subject to client’s specification of urgency.
Lit and Unlit Orders

- In the past, “upstairs” markets and crossing networks allowed institutional block trades to execute.

- Dark pools are the results of technological advancement, intense competition, and new regulatory requirements.

- Most brokers support iceberg orders for lit exchanges, resulting in dark liquidity.

- Is there a smart way to ping for dark liquidity?
ITG’s Classification of Algo Trading Strategies

- Close: Trading at the close
- Dark: Dark pool and unlit orders
- Scheduled: TWAP, VWAP, POV (Participation)
- Implementation Shortfall (IS): Inspired by and generalization of [Almgren and Chriss, 2000]
- Liquidity-Seeking
Preferred BEAT of 20 Buy-Side Desks

[Domowitz and Yegerman, 2011] Sample Period: Q4 2009 to Q4 2010 Q4
## Order Placement Strategies

[Domowitz and Yegerman, 2011] Sample Period: Q4 2009 to Q4 2010

<table>
<thead>
<tr>
<th>Percent of Orders</th>
<th>CLOSE</th>
<th>DARK</th>
<th>IS</th>
<th>LIQ SEEKING</th>
<th>PARTICIPATION</th>
<th>TWAP</th>
<th>VWAP</th>
</tr>
</thead>
<tbody>
<tr>
<td>LT.5%</td>
<td>30%</td>
<td>6%</td>
<td>31%</td>
<td>10%</td>
<td>11%</td>
<td>3%</td>
<td>8%</td>
</tr>
<tr>
<td>.5%-1%</td>
<td>13%</td>
<td>12%</td>
<td>28%</td>
<td>12%</td>
<td>9%</td>
<td>4%</td>
<td>23%</td>
</tr>
<tr>
<td>1%-5%</td>
<td>9%</td>
<td>17%</td>
<td>22%</td>
<td>11%</td>
<td>9%</td>
<td>5%</td>
<td>27%</td>
</tr>
</tbody>
</table>

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Order Placement Strategies (cont’d)

- Demand for liquidity: percentage of median daily volume (MDV) for the parent order
  1. Less than (LT) 0.5% of MDV
  2. 0.5% to 1% of MDV
  3. 1% to 5% (and rarely above) of MDV

- During the “quiet” period (Q4 2009 to Q4 2010 Q4), Almgren-Chriss’ Implementation Shortfall (IS) strategy was most popular.
  - 81% of orders were executed by IS

- VWAP is still dominantly used (58% of orders) by some trading desks.
Trading Strategy Costs By Demand For Liquidity

[Domowitz and Yegerman, 2011] Sample Period: Q4 2009 to Q4 2010 Q4
Summary

- BEAT is a big business and R&D of algorithmic trading systems will continue.

- Generalization to account for stochastic liquidity and volatility (see [Almgren, 2012])

- Permeation of BEAT systems to FX, futures and options, and fixed income securities

- Asia-Pacific markets still have a lot of room to grow the BEAT business.
Unobservable Price versus Transaction Price

- Earlier in Slide 21, we have seen

\[ P_t = S_t + \eta v_t \]

- A model by [Roll, 1984] of bid-ask spread \( C \) from price change \( \Delta P_t := P_t - P_{t-1} \):

\[ P_t = S_t + \frac{C}{2} Q_t + \varepsilon_t, \]

where \( Q_t \) is either +1 (trade @ the ask price), or \(-1\) (trade @ the bid price), or 0 (trade @ the midpoint).

\[ C = 2 \sqrt{-\text{Cov}[\Delta P_t, \Delta P_{t-1}]} . \]
OLS Approach [Ting, 2014]

- Tick-by-tick change of price $\Delta P_{t_i}$ regressed on change in trade sign $\Delta Q_{t_i}$ and on trade duration $\Delta t_i$:

$$\Delta P_{t_i} = c_0 + \frac{C}{2} \Delta Q_{t_i} + b \Delta t_i + \epsilon_{t_i}$$
## Structural Models: A Synthesis

<table>
<thead>
<tr>
<th>Roll</th>
<th>Unobservable Price</th>
<th>Transaction Price</th>
<th>Price Change Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Huang &amp; Stoll</td>
<td>$S_{t-1} + \epsilon_t$</td>
<td>$S_t + \frac{C}{2} x_t$</td>
<td>$\frac{C}{2} \Delta Q_t + \epsilon_t$</td>
</tr>
<tr>
<td>Madhavan et al.</td>
<td>$\frac{\alpha C}{2} Q_{t-1}$</td>
<td>Midquote + $\frac{C}{2} Q_t$</td>
<td>$\frac{C}{2} \Delta Q_t + (\alpha + \beta) \frac{C}{2} Q_{t-1} + \epsilon_t$</td>
</tr>
<tr>
<td>Glosten &amp; Harris</td>
<td>$\theta (Q_t - \rho Q_{t-1})$</td>
<td>$S_t + \phi Q_t$</td>
<td>$(\theta + \phi) \Delta Q_t + \theta (1 - \rho) Q_{t-1} + \epsilon_t$</td>
</tr>
<tr>
<td>Tee &amp; Ting (2015)</td>
<td>$\theta_1 (Q_t - \rho_1 Q_{t-1}) + \theta_2 (V_t - \rho_2 V_{t-1})$</td>
<td>$S_t + \phi_1 Q_t + \phi_2 v_t$</td>
<td>$(\theta_1 + \phi_1) \Delta Q_t + \theta_1 (1 - \rho_1) Q_{t-1} + (\theta_2 + \phi_2) \Delta V_t + \theta_2 (1 - \rho_2) V_{t-1} + \epsilon_t$</td>
</tr>
</tbody>
</table>
Classical Market Microstructure

Signed Volume $V_t$ is Important!

![Graphs showing various contracts and their signed volume evolution over time.](image-url)
Relative Joint Price Impact by Hour

RJPI: $dx \rightarrow dx, x, dv, v$

- TP
- NK
- NO

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Order Flow and Price Change

- According to [Kyle, 1985], the transaction price $P$ depends on the order flow
  - buy order $X^+$
  - sell order $X^-$

- [Evans and Lyons, 2002] show that order flow $X := X^+ - X^-$ is a superior candidate for modelling the variation in foreign exchange rate compared to arguments along the lines of macroeconomics.

- Over a period of time $t$, (e.g. 1 minute, duration of 100 trades, etc), a positive (negative) order flow $X_t$ indicates more (less) buyers than sellers.

- What is the market price of liquidity risk for demanding immediate execution?
Joint Work with Kijima

Stochastic differential equation for the order flow $X_t$ is assumed to be a stationary (mean reverting) process:

$$dX_t = c(m - X_t)dt + \eta \, dw_t,$$

(1)

where

- $w_t$ is the standard Brownian motion.
- $m$ is the long-run average.
- $c$ is the speed of mean-reversion.
- $\eta$ is the volatility of order flow.

The parameters $m, c, \text{ and } \eta$ are assumed to be constants.
Price Impact

- In financial economics, the “true price” $S_t$ is not directly observable.

- Let $\phi(X)$ be the price impact when the order flow is $X_t$ that relates the trade price $P_t$ to the true price $S_t$.

- **Assumption 1: True Price, Trade Price, and Price Impact**

  For a certain function $f(X)$,

  $$\phi(X) = e^{f(X)} \iff f(X) = \log \phi(X)$$

  and

  $$P_t = S_t \phi(X_t) \iff S_t = P_t e^{-f(X_t)}.$$  \hspace{1cm} (2)

- This formalism is similar to [Çetin et al., 2004] and [Huang and Ting, 2008], among others.
Price Impact (cont’d)

- **Assumption 2: Zero Order Flow Condition**
  \[ \phi(0) = 1 \iff f(0) = 0. \]

- **Assumption 3: Trade Price Dynamics**
  \( P_t \) be the time \( t \) trade price and
  \[
  \frac{dP_t}{P_t} = \mu(X_t)dt + \sigma(X_t)dz_t, \tag{3}
  \]
  where \( \mu(x) \) and \( \sigma(x) \) are some smooth functions of order flow \( x \), and where \( w_t \) and \( z_t \) are standard Brownian motions with correlation \( \rho \)
  \[
  dw_t dz_t = \rho \, dt.
  \]

- The trade price \( P_t \) as well as the order flow \( X_t \) are both observable in the market.
Market Prices of Risks

- Let $\lambda_t^z = \lambda_z(X_t)$ and $\lambda_t^w = \lambda_w(X_t)$ denote the market prices of risks associated with $z_t$ and $w_t$, respectively.

- Define
  \[ d\tilde{z}_t = dz_t + \lambda_t^z dt, \quad d\tilde{w}_t = dw_t + \lambda_t^w dt. \]

- Let $\mathcal{Q}$ be a probability measure that makes the processes $\tilde{z}_t$ and $\tilde{w}_t$ standard Brownian motions.

- True price process becomes
  \[
  \frac{dS_t}{S_t} = \left( \mu_S(X_t) - \sigma(X_t)\lambda_z(X_t) + \eta f'(X_t)\lambda_w(X_t) \right) dt \\
  + \sigma(X_t)d\tilde{z}_t - \eta f'(X_t)d\tilde{w}_t
  \]
  under $\mathcal{Q}$.

(4)
The Market Price of Liquidity Risk

- For some positive function $\xi(x) > 0$, the market price of liquidity risk is written as
  \[ \lambda_w(x) = \xi(x) - \eta f'(x). \] (5)

- Let $g(x) = f'(x)$. The ODE then becomes
  \[
  \left\{ c(m - x) + \rho \eta \sigma(x) - \eta \xi(x) \right\} g(x) + \frac{1}{2} \eta^2 \left\{ g'(x) + g^2(x) \right\} = 0. 
  \] (6)
A Special Case

- Let $c = 0$ and $\sigma(x) = \sigma$, a constant.

- For the market price of risk, suppose it is a constant of $x$, i.e., the Sharpe ratio:
  \[ \lambda_z(x) = \frac{\mu - r}{\sigma}. \]

- Moreover, for the market price of liquidity risk,
  \[ \lambda_w(x) = -\eta f'(x). \]
  i.e., $\xi(x) = 0$. 
A Special Case (cont’d)

Let $\mu(x) = \mu - r$. Then

$$r + \rho \sigma \eta f'(x) + \frac{1}{2} \eta^2 \left[ f''(x) + (f'(x))^2 \right] = 0 \quad (7)$$

Consequently, under the same $\mathbb{Q}$, the true price process becomes

$$\frac{dS_t}{S_t} = r \, dt + \sigma \, dz_t + \lambda_w(x) \, dw_t, \quad (8)$$

The ODE (7) admits a closed-form solution:

$$f(x) = \frac{b}{\eta} x + \log \left( \frac{1 - A \exp\left(-\frac{\beta x}{\eta} \right)}{1 - A} \right). \quad (9)$$
Numerical Example of \( f(x) \)

\[ r = 0, \sigma = 0.017, \eta = 10, \text{ and } \rho = 0.16 \]
Market Price of Liquidity Risk

- In this special case

\[ \lambda_w(x) = -\eta f'(x) = - \left( b + \frac{\beta A e^{-\frac{\beta}{\eta} x}}{1 - A e^{-\frac{\beta}{\eta} x}} \right) \]  

(10)

- If \( b \approx 0 \), then the nonlinear term is dominant.

- Market price of liquidity is NEGATIVE. Why?

- (Market) order flow \( x \) demands liquidity and therefore must pay the price for enjoying the liquidity created by liquidity providers (e.g. traders who submit limit orders to trade).
Suppose $x = 0$, i.e., no trade. Then the “Implied bid-ask spread” to be paid by liquidity consumers can be defined as

$$LIX := 2\eta f'(0) = 2 \left( b + \frac{\beta A}{1 - A} \right).$$

LIX is higher if $b$, $A$, and $\beta$ are larger, intuitive indeed.

“Term structure” of LIX: Time interval to compute the order flow

- Wall clock: 1 minute, 2 minutes, 3 minutes, . . . , 10 minutes
- Business clock: 10 trades, 20 trades, 30 trades, . . . , 100 trades
## On-Shore and Off-Shore Nikkei 225 Index Futures

### On-Shore JPX (> 16.5 ms from SGX)
- **Big Nikkei (NK) at Osaka:** tick size 10 points, multiplier ¥1,000
- **Mini Nikkei (NO) at Osaka:** tick size 5 points, multiplier ¥100

### Off-Shore SGX, CME (> 50 ms from SGX)
- **Regular Nikkei (NI) at SGX:** tick size 5 points, multiplier ¥500
- **Regular Nikkei (NX) at CME:** tick size 5 points, multiplier ¥500
- **NI and NX are mutually fungible.**
- **Quanto Nikkei (NH) at CME:** tick size 5 points, multiplier $5

Excellent test bed for studying market fragmentation, order placement, order routing, LIX, etc
EU Referendum Vote Counting  → Brexit

- June 24, 2016

<table>
<thead>
<tr>
<th>Ticker</th>
<th>TV</th>
<th>TOF</th>
<th>$\sigma$ $10^{-3}$</th>
<th>$\eta$</th>
<th>$\rho$</th>
<th>$b_0$ bps</th>
<th>$b$ $10^{-3}$</th>
<th>$A$ $10^{-3}$</th>
<th>$\beta$ $10^{-6}$</th>
<th>SSE</th>
<th>$R^2$ %</th>
<th>LIX bps</th>
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<tbody>
<tr>
<td>NH</td>
<td>54,986</td>
<td>-894</td>
<td>2.25</td>
<td>53.20</td>
<td>0.26</td>
<td>-2.31</td>
<td>-6.70</td>
<td>0.8567</td>
<td>1.18</td>
<td>4.87</td>
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<tr>
<td>NX</td>
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<td>0.38</td>
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<td>0.52</td>
<td>0.0013</td>
<td>1.15</td>
<td>4.30</td>
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<tr>
<td>NI</td>
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<td>158.98</td>
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<td>0.0005</td>
<td>0.16</td>
<td>4.03</td>
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<tr>
<td>NO</td>
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<td>956.44</td>
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<td>NO5</td>
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<td>-10,698</td>
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<td>0.49</td>
<td>0.0302</td>
<td>19.3</td>
<td>3.79</td>
<td>30.45</td>
<td>21.79</td>
</tr>
</tbody>
</table>

- Volatility of log return is $\sigma$.
- Estimation of $b_0$, $b$, $A$, and $\beta$ by constrained nonlinear least squares
- LIX is fairly invariant with different combinations of $b$, $A$, and $\beta$. 
Time Series of LIX (Nov 5, 2014 to Dec 10, 2015)
### Descriptive Statistics for LIX (Nov 5, 2014 to Dec 10, 2015)

<table>
<thead>
<tr>
<th></th>
<th>NH</th>
<th>NX</th>
<th>NI</th>
<th>NK</th>
<th>NO</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>2.11</td>
<td>2.47</td>
<td>2.98</td>
<td>3.51</td>
<td>3.63</td>
</tr>
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<td>28.6%</td>
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Conclusions

- Price impact function is a concave function of order flow.

- Market price of risk is negative for liquidity consumers.

- **LIX** is a useful tool to reflect not only the market price of liquidity risk but also possibly volatility.
  - **Market depth**: larger $\beta$ implies lower depth
  - **Urgency**: larger $A$ implies traders are more anxious when they trade
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