Market Model and CAPM as Applications of Simple Linear Regression

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Learning Objectives

• Describe the market model and connect it with simple (univariate) OLS regression model

• Recall the familiar concept of CAPM and appreciate how it is further developed in the context of QF

• Gain a deep understanding of conditional mean and conditional variance in CAPM

• Develop a working knowledge of
  • alpha and beta of CAPM
  • systematic risk versus unsystematic (idiosyncratic) risk

• Gain a deeper understanding of how alpha, beta, systematic and unsystematic risks are estimated.

• Describe and discuss security market line, capital market line, market risk premium, Sharpe’s ratio, and other risk-adjusted performance measures
Market Model

OLS Approach to Market Model

Market portfolio’s log return is denoted by $r_{mt}$, and portfolio $i$’s log return by $r_{it}$:

Market model: A linear regression model of $r_{it}$ on $r_{mt}$ is

$$r_{it} = a + b r_{mt} + e_{it}.$$ (1)

Properties of residuals $e_{it}$:

- Uncorrelated with $r_{mt}$:
  $$E(e_{it} \mid r_{mt}) = 0$$

- By the law of iterated expectations, $E(e_{it}) = 0$. 
Market Model’s $a$ and $b$

- The covariance of $r_{it}$ and $r_{mt}$ is denoted by $\sigma_{im}$.

$$
\sigma_{im} := \mathbb{C}(r_{it}, r_{mt}) = \mathbb{C}(a + br_{mt} + e_{it}, r_{mt})
$$

$$
= \mathbb{C}(a, r_{mt}) + \mathbb{C}(br_{mt}, r_{mt}) + \mathbb{C}(e_{it}, r_{mt})
$$

$$
= 0 + b \mathbb{C}(r_{mt}, r_{mt}) + 0
$$

$$
= b \sigma^2_m
$$

- Hence,

$$
b = \frac{\sigma_{im}}{\sigma^2_m}.
$$

- Taking expectation operation on both sides of (1):

$$
\mathbb{E}(r_{it}) = a + b \mathbb{E}(r_{mt}) + 0
$$

- Hence,

$$
a = \mathbb{E}(r_{it}) - \frac{\sigma_{im}}{\sigma^2_m} \mathbb{E}(r_{mt}).
$$
Given the market log return $r_{mt}$, the conditional mean of portfolio $i$’s log return $r_{it}$

$$
\mathbb{E}(r_{it} | r_{mt}) = \mathbb{E}(r_{it}) + \frac{\sigma_{im}}{\sigma_m^2} (r_{mt} - \mathbb{E}(r_{mt}))
$$

$$
= \left( \mathbb{E}(r_{it}) - \frac{\sigma_{im}}{\sigma_m^2} \mathbb{E}(r_{mt}) \right) + \frac{\sigma_{im}}{\sigma_m^2} r_{mt}.
$$
From (1), by applying the variance operator on both sides, we find that the unconditional variance of $r_{it}$ has two parts

$$\sigma_i^2 = b^2 \sigma_m^2 + \sigma_e^2$$

- **Systematic volatility:** $\sigma_m$
- **Unsystematic volatility, idiosyncratic, or diversifiable volatility:** $\sigma_e$

Now, the variance of $r_{it}$ given $r_{mt}$ is simply

$$\mathbb{V}(r_{it}|r_{mt}) = \mathbb{V}(a + b r_{mt} + e_{it}|r_{mt}) = \sigma_e^2$$

because $r_{mt}$ is known and thus zero variance.

Hence, from (2), the conditional variance of $r_{it}$ given $r_{mt}$ is

$$\mathbb{V}(r_{it}|r_{mt}) = \sigma_i^2 - \frac{\sigma_{im}^2}{\sigma_m^2}.$$
When economic equilibrium is added to the market model, then $a$ is restricted to

$$a = r_f(1 - b)$$

where $r_f$ is the riskfree rate.

Imposing this theoretical CAPM restriction, the OLS regression of market model becomes

$$r_{it} = r_{ft} + b(r_{mt} - r_{ft}) + e_{it}$$

Capital Asset Pricing Theory

$$\mathbb{E}(r_{it}) = r_{ft} + b_i \mathbb{E}(r_{mt} - r_{ft})$$

OLS econometric model of portfolio $i$’s excess return:

$$r_{it} - r_{ft} = a_i + b_i(r_{mt} - r_{ft}) + e_{it}$$

(3)
Estimation of Alpha and Beta

- Time series of returns \( \{r_{it}, r_{mt}\}_{t=1,2,...,T} \)

- OLS estimation of beta

\[
\hat{b}_i = \frac{\sum_{t=1}^{T} (r_{mt} - \bar{r}_m)(r_{it} - \bar{r}_i)}{\sum_{t=1}^{T} (r_{mt} - \bar{r}_m)^2}
\]

- Theoretically alpha \( a_i \) is zero in equilibrium. But
  - if \( \hat{a}_i > 0 \), then positive abnormal return
  - if \( \hat{a}_i < 0 \), then negative abnormal return

- Alpha is also called the Jensen measure in the context of portfolio theory.

- Practical issues
  - What is the ideal sampling frequency?
  - What is the ideal sampling size?
Linear regression model: \[ r_{it} - r_{ft} = a_i + b_i (r_{mt} - r_{ft}) + e_{it} \]

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.003119</td>
<td>0.002653</td>
<td>1.175459</td>
<td>0.2426</td>
</tr>
<tr>
<td>MKT EXC RET</td>
<td>0.624225</td>
<td>0.080491</td>
<td>7.755199</td>
<td>0.0000</td>
</tr>
<tr>
<td>( R )-squared</td>
<td>0.375559</td>
<td>Mean dependent var.</td>
<td>0.000904</td>
<td></td>
</tr>
<tr>
<td>Adjusted ( R )-squared</td>
<td>0.369314</td>
<td>S.D. dependent var.</td>
<td>0.033544</td>
<td></td>
</tr>
<tr>
<td>S.E. of regression</td>
<td>0.026639</td>
<td>Akaike info criterion</td>
<td>-4.393447</td>
<td></td>
</tr>
<tr>
<td>Sum squared resid</td>
<td>0.070965</td>
<td>Schwarz criterion</td>
<td>-4.341977</td>
<td></td>
</tr>
<tr>
<td>Log likelihood</td>
<td>226.0658</td>
<td>( F )-statistic</td>
<td>60.14311</td>
<td></td>
</tr>
<tr>
<td>Durbin-Watson stat</td>
<td>2.513910</td>
<td>Prob(( F )-statistic)</td>
<td>0.000000</td>
<td></td>
</tr>
</tbody>
</table>
Example (Cont’d)

- The standard error of \( \hat{a}_i \) is “Std Error of C”:
  \[
  \hat{\sigma}_e \sqrt{\frac{1}{T} + \frac{\bar{X}^2}{\sum_{t=1}^{T} (X_t - \bar{X})^2}}
  \]

- The standard error of \( \hat{b}_i \) is “Std Error of Coefficient of MKT EXC RET”:
  \[
  \hat{\sigma}_e \sqrt{\frac{1}{\sum_{t=1}^{T} (X_t - \bar{X})^2}}
  \]

- The SSR, “Sum squared resid” is SSR = \( \sum_{t=1}^{T} \hat{e}_t^2 \)

- The standard error of \( e_t \), “S.E. of regression” is \( \sigma_e = \sqrt{\frac{1}{T-2} \text{SSR}} \).
A line in a graph of expected return versus beta:

$$E(r_j) = r_f + \beta_j (E(r_m) - r_f)$$

or

$$E(r_j) - r_f = \frac{C(r_j, r_m)}{V(r_m)} (E(r_m) - r_f)$$

The market portfolio has a beta of 1.

Estimation of beta with OLS without the intercept

$$r_{jt} - r_{ft} = \beta_j (r_{mt} - r_{ft}) + e_{jt}$$

$$\hat{\beta}_j = \frac{\sum_{t=1}^{T} (r_{mt} - \bar{r}_m)(r_{jt} - \bar{r}_j)}{\sum_{t=1}^{T} (r_{mt} - \bar{r}_m)^2}$$
Illustration of SML

\[ \mathbb{E}(r_m) \]

\[ r_f \]

\[ \mu \]

\[ \beta \]

SML
Capital Market Line

CML is the line containing all possible portfolios of investors. Each portfolio is a linear combination of 2 assets — the market portfolio and the riskfree asset. Any portfolio with return $r_p$ and volatility $\sigma_p$ has the same Sharpe ratio as the market portfolio has:

$$\frac{\mathbb{E}(r_p - r_f)}{\sigma_p} = \frac{\mathbb{E}(r_m - r_f)}{\sigma_m} =: \lambda$$

So

$$\mathbb{E}(r_m - r_f) = \lambda \sigma_m$$

Estimation of market risk premium $\lambda$ with realized returns

$$r_{mt} - r_{ft} = \lambda \sigma_{mt} + u_t$$

Merton’s proposal

$$\mathbb{E}(r_{mt} - r_{ft}) = \lambda \sigma_m^2$$

where $\lambda$ is interpreted as the relative risk aversion.
Illustration of CML

\[ \mathbb{E}(r_p) - r_f \]

\[ \mathbb{E}(r_m) - r_f \]
Performance Measures: Treynor and Jensen

- Treynor measure is the expected excess portfolio return \( r_{pt} \) per unit of portfolio beta \( b_p \):

\[
E\left( r_{pt} - r_{ft} \right) / b_p.
\]

- If Jensen measure indicates superior performance, so does Treynor measure:

\[
a = E\left( r_{pt} - r_{ft} \right) - b_p E\left( r_{mt} - r_{ft} \right) > 0 \quad \iff \quad \frac{E\left( r_{pt} - r_{ft} \right)}{b_p} > E\left( r_{mt} - r_{ft} \right)
\]
Sharpe and Information Ratios

**Sharpe ratio**

\[
\frac{\mathbb{E}(r_{pt} - r_{ft})}{\sigma_p}
\]

shows how well the portfolio is performing relative to CML with slope \(\frac{\mathbb{E}(r_{mt} - r_{ft})}{\sigma_m}\)

\[
\frac{\mathbb{E}(r_{pt} - r_{ft})}{\sigma_p} > \frac{\mathbb{E}(r_{mt} - r_{ft})}{\sigma_m} \iff \alpha > 0
\]

**Information ratio**

\[
\frac{\text{Average Tracking Error}}{\text{Volatility of Tracking Error}} = \frac{\mathbb{E}(r_{pt} - r_{bt})}{\sigma(r_{pt} - r_{bt})}
\]

where \(r_{bt}\) is the return on a benchmark (such as S&P 500 index).
**Model with No Intercept**

△ Merton’s model is a constrained linear regression model:

\[ Y_i = b X_i + e_i , \quad i = 1, 2, \cdots, n \]

△ Minimization of least squares

\[ \min_{\hat{b}} \sum_{i=1}^{n} \hat{e}_i^2 \equiv \min_{\hat{b}} \sum_{i=1}^{n} (Y_i - \hat{b} X_i)^2 \]

△ FOC

\[ \frac{\partial}{\partial \hat{b}} \sum_{i=1}^{n} \hat{e}_i^2 = -2 \sum_{i=1}^{n} X_i (Y_i - \hat{b} X_i) = 0 \]

△ Solution

\[ \hat{b} = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2} \]
Finite Sample Properties

\( \hat{b} = \frac{\sum_{i=1}^{n} X_i (b X_i + e_i)}{\sum_{i=1}^{n} X_i^2} = b + \frac{\sum_{i=1}^{n} X_i e_i}{\sum_{i=1}^{n} X_i^2} \)

\[ \implies \mathbb{E}(\hat{b}) = b \]

\[ \mathbb{V}(\hat{b}) = \mathbb{E}(\hat{b} - b)^2 = \mathbb{E}\left( \frac{\sum_{i=1}^{n} X_i e_i}{\sum_{i=1}^{n} X_i^2} \right)^2 = \frac{\sigma_e^2}{\sum_{i=1}^{n} X_i^2} \]

\( \hat{b} \sim N\left(b, \sigma_e^2 \frac{1}{\sum_{i=1}^{n} X_i^2}\right) \)
Application: Stock Picking

+ Ideal stock to hold:
  - positive alpha
  - large beta during bull market
  - small beta during bear market
  - large Sharpe ratio

+ Ideal stock to short is the reverse

+ Long the “good” stocks, short the “bad” stocks. Will this quant strategy work?

+ So exactly how could one search for those good and bad stocks?
Which is Better?

★ Price-weighted portfolio
  - Same number of shares for each stock

★ Equally-weighted portfolio
  - Same dollar amount of the fund to buy shares of each stock

★ Market-cap-weighted portfolio
  - The number of shares for each stock is based on the ratio of its market cap to the total market cap
Suppose you have picked three stocks.

<table>
<thead>
<tr>
<th>Stock</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price ($)</td>
<td>10</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>Shares (millions)</td>
<td>40</td>
<td>60</td>
<td>60</td>
</tr>
</tbody>
</table>

You have 90 million dollars.

- To construct a price-weighted portfolio, let \( x \) be the number of shares to buy

\[
10x + 15x + 20x = 90
\]

Therefore, you buy 2 million shares of each stock.

- To construct a portfolio of equal weight, you allocate equal amount of 30 million dollars to buy each stock. Therefore, you buy 3 million shares of A, 2 million shares of B, and 1.5 million share of C.
To construct a portfolio weighted by market cap, first compute the total market cap

$$10 \times 40 + 15 \times 60 + 20 \times 60 = 2500$$

(A) Stock A’s fraction is 4/25. The amount allocated to Stock A is
$$4 \times 90/25 = 14.4$$ million dollars. Therefore, the number of shares bought is 14.4/10 = 1.44 million shares.

(B) Stock B’s fraction is 9/25. The amount allocated to Stock B is
$$9 \times 90/25 = 32.4$$ million dollars. Therefore, the number of shares bought is 32.4/15 = 2.16 million shares.

(C) Stock C’s fraction is 12/25. The amount allocated to Stock C is
$$12 \times 90/25 = 43.2$$ million dollars. Therefore, the number of shares bought is 43.2/20 = 2.16 million shares as well.
Summary

• Market portfolio is the cornerstone of both the market model and CAPM.

• Systematic versus idiosyncratic volatilities

• Beta is the sensitivity of a portfolio’s return to the market factor.

• Security market line is a pictorial presentation of CAPM with beta being the x-axis.

• Capital market line is closely related to the concept of risk-adjusted return, with the volatility being the x-axis.

• Performance measure: Jensen, Treynor, Sharpe, and Information Ratio

• Price-weighted, equally weighted, market-cap weighted