On the Ross recovery under the single-factor spot rate model

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Risk management of interest-rate sensitive products has become more important than ever due to the drastic change of the monetary policies, in particular after the credit crunch.

For example, many countries have adopted the so-called zero interest-rate policy (ZIRP) and now started introducing negative interest rates as in Japan.

As a result, it becomes harder and harder to predict the future dynamics of the interest rates because of the untraditional policies.

Scenarios based on historical data often become useless to predict such drastic changes (e.g., Swiss Franc shock).

Practitioners want to have forward-looking scenarios for the risk management rather than the traditional backward-looking ones.
Motivation, Continued

- For the risk evaluation purpose, it is important to distinguish the risk-neutral measure $Q$ from the physical (observed) measure $P$.
- While $Q$ is needed for the pricing of financial assets, $P$ is used to generate future interest-rate scenarios.
- In practice, the common way to construct two models under $Q$ and $P$ simultaneously is first to assume a financial model under either $P$ or $Q$ and then identify the other model under $Q$ or $P$, respectively, by estimating the market price of risk from market data.
- E.g., Kijima et al. (2014) assume that the spot-rate process under $P$ follows a QG model whose parameters are estimated by GMM, and assuming the market price of risk is a piecewise linear function of time, they calibrate the parameters of the model under $Q$. 
On the other hand, as in Dai and Singleton (2000), it is possible to start from a (affine) model under $\mathbb{Q}$ and, by assuming that the market price of risk follows a (affine) model, a (affine) model under $\mathbb{P}$ can be estimated from the term structure observed in the market.

Note that, in either models, estimation of the parameters typically requires heavy computation, because the optimization problem involved is usually non-linear and difficult to converge because of the existence of many local-optima.

Also, in these approaches, the estimation relies on the historical data and so, the resulting risk evaluation is a backward-looking.

 Practitioners require a simple and fast method to evaluate, e.g., P/L distributions of derivatives that are consistent to the current derivative prices based on forward-looking risk scenarios.

For this purpose, they typically assume $\mathbb{P} = \mathbb{Q}$ or a zero drift model.
Ross (2015) proposes the idea to recover the model under $\mathbb{P}$ from the model under $\mathbb{Q}$; i.e., it can avoid the estimation of the market price of risk, which is presumably the most difficult and contentious.

Because this method uses the market data of derivatives only and derivative prices are determined by future dynamics of the underlying assets, the resulting risk evaluation is considered to be a forward-looking, in contrast to the existing methods.

Qin and Linetsky (2016) state that such an identification would be of great interest to finance researchers and market participants, as it would open avenues for extracting market’s assessment of physical probabilities that could be incorporated in investment decisions and supply scenarios for risk management.
Following Ross (2015), there appear a vast literature that discuss the Ross recovery theorem.

The first set of papers such as Audrino et al. (2015) and Kiriu and Hibiki (2015) use the market data of option prices to estimate the transition law under $\mathbb{Q}$ by using the MP method and apply the theorem to translate them to that under $\mathbb{P}$.

However, the resulting MP problem is rather huge and ill-posed, which makes the problem very difficult to solve.

The second set of papers such as Audrino et al. (2015) and Borovička, et al. (2014) provide empirical analyses of the theorem.

They test whether or not the recovery yields predictive information beyond what can be gleaned from risk-neutral densities.

The results often appear very opposite; some are affirmative and some are very negative.
The third set of papers such as Park (2015) and Qin and Linetsky (2016) intend to extend the recovery theorem to a continuous-time setting.

The continuous-time model fails to recover a physical measure from a risk-neutral measure in general.

If $X_t$ is recurrent under $\mathbb{P}$, then recovery is possible.

Park (2015) investigates what information is sufficient to recover when $X_t$ is transient.

Qin and Linetsky (2016) extend the theorem to recurrent Borel right processes (time-homogeneous).

For the one-dimensional diffusion including the CIR and Vasicek models (with positive exponential jumps), they provide a complete answer to the problem.
In This Paper

- We study the Ross recovery when the spot-rate process follows a one-dimensional diffusion (possibly inhomogeneous).
- In particular, we study the (inhomogeneous) Hull–White model, which is the most common term-structure model for practitioners.
- We construct a discrete-time skip-free random walk under $\mathbb{Q}$ from the market data of option prices.
- The transition laws of the random walk under $\mathbb{P}$ can be calculated by the recovery theorem by applying a novel bisection search.
- A numerical example is given to support the usefulness of our method for the risk evaluation purpose.
- Our method can be extended to the case that the spot-rate process follows a one-dimensional diffusion with one-side jumps.
Consider a discrete-time economy with time index \( \{0, 1, 2, \ldots, T\} \).

Let \( r_n \) be the riskfree spot rate at time \( n \) and denote
\[
B_n = e^{\sum_{u=0}^{n-1} r_u \Delta t},
\]
where \( \Delta t > 0 \) is the time interval.

For any asset price \( S_t \), we have
\[
S_n = \mathbb{E}^P_n \left[ \frac{\eta_{n+1}}{\eta_n} S_{n+1} \right] = \mathbb{E}^Q_n \left[ \frac{B_n}{B_{n+1}} S_{n+1} \right]
\]
where \( \eta_n \) denotes the state price density at time \( n \).

Let \( X_n \) be a finite Markov chain (possibly inhomogeneous) with transition probability matrix \( Q_n = (q_{ij}(n)) \) under \( Q \).

Its transition matrix under \( P \) is denoted by \( P_n = (p_{ij}(n)) \).

Uncertainty is driven by \( X_n \) only and we shall denote
\[
S_n = s_n(X_n), \quad r_n = r_n(X_n), \quad \eta_n = \eta_n(X_n)
\]

Note that these functions are time-dependent.
Suppose $X_n = i$, so that

$$s_n(i) = \sum_j \frac{\eta_{n+1}(j)}{\eta_n(i)} p_{ij}(n) s_{n+1}(j) = \frac{1}{R_i(n)} \sum_j q_{ij}(n) s_{n+1}(j)$$

where $R_i(n) = e^{r_n(i) \Delta t}$, which implies

$$\frac{\eta_{n+1}(j)}{\eta_n(i)} p_{ij}(n) = \frac{1}{R_i(n)} q_{ij}(n) = \tilde{q}_{ij}(n), \quad \forall i, j$$

Ross (2015) basically assumes there are $\delta(n)$ and $\xi_i(n)$ such that

$$\frac{\eta_{n+1}(j)}{\eta_n(i)} = \delta(n) \frac{\xi_j(n)}{\xi_i(n)} \iff \delta(n) \frac{\xi_j(n)}{\xi_i(n)} p_{ij}(n) = \tilde{q}_{ij}(n), \quad \forall i, j$$

In other words, there is $\delta(n)$ such that $\eta_{n+1}(j) = \delta(n) \eta_n(j)$ for all $j$; i.e., $\delta(n)$ is independent of $j$, called transition independent.
The Ross Recovery Theorem

- Let $D = \text{diag}(R_i)$ and $E = \text{diag}(\xi_i)$ be the diagonal matrices with diagonals $R_i$ and $\xi_i$, respectively (suppress $(n)$ for simplicity).
- The above equations can be formally written in matrix form as
  $$\delta E^{-1}PE = D^{-1}Q \iff P = \delta^{-1}E(D^{-1}Q)E^{-1}$$
- If $D^{-1}Q$ is primitive, then there exist $\lambda$ and $z$ such that
  $$(D^{-1}Q)z = \lambda z, \quad \lambda > 0, \quad z > 0$$
due to the Perron-Frobenius (PF) theorem.
- Further suppose that $P$ is stochastic; we then have
  $$Pe = \delta^{-1}E(D^{-1}Q)y = e, \quad y = E^{-1}e,$$
  where $e$ denotes the column vector with all entities being unity.
- By the uniqueness, we obtain $\delta = \lambda$ and $y = z$.
- It follows that $\xi_i = 1/z_i$ and we thus have
  $$p_{ij} = \frac{z_j}{\lambda R_i z_i} q_{ij}$$
1 Suppose that $R_i = R$, i.e., the spot rate does not depend on the underlying Markov chain $X_n$. Then, since the transition matrix $Q$ is stochastic (if so is $P$), we have $z = e$, $\lambda = 1/R$, which implies that $p_{ij} = q_{ij}$.

2 It may be common to assume that $\eta_t = \eta(X_t)$ for some time-independent function $\eta(x)$. But, this case yields $\delta = 1$ so that we have $p_{ij} = q_{ij}$ again.

3 If we assume $S_n = s(X_n)$ for some time-independent function $s(x)$, we have arbitrage opportunities. Hence, the asset price $S_n$ should be dependent on time such as derivatives with finite maturities.
The Setup

- Consider the one-dimensional (possibly inhomogeneous) recurrent diffusion $X_t$ under $\mathbb{Q}$ with appropriate boundary conditions:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dz_t$$

- Such a diffusion can be approximated by a finite random walk.
- Let $\Delta t > 0$ and define $\Delta x = \xi \sqrt{\Delta t}$ for some $\xi > 0$.
- We discretize the model by $t_n = n \Delta t$ and $x_i = i \Delta x$.
- The transition probabilities $q_{ij}(n)$ are determined by

$$\mu(t_n, x_i) = \frac{1}{\Delta t} [q_{i,i+1}(n) \Delta x - q_{i,i-1}(n) \Delta x]$$

and

$$\sigma^2(t_n, x_i) = \frac{1}{\Delta t} [q_{i,i+1}(n)(\Delta x)^2 + q_{i,i-1}(n)(\Delta x)^2] - \mu^2(t_n, x_i) \Delta t$$
Consider the **tri-diagonal matrix** $Q_n = (q_{ij}(n))$ under $Q$.

The transition probability at time $T$ from $i$ to $j$ of the random walk under $Q$ can be obtained by $(Q_0 Q_1 \cdots Q_{T-1})_{ij}$.

Let $R_i(n) = e^{r_n(x_i)\Delta t}$ and $D_n = \text{diag}(R_i(n))$.

From the recovery theorem, we then have, for each $n$,

$$p_{ij}(n) = \frac{z_j(n)}{\lambda(n) R_i(n) z_i(n)} q_{ij}(n), \quad \forall i, j$$

where $\lambda(n)$ and $z(n) = (z_i(n))$ are the PF eigenvalue and the associated eigenvector of $D_n^{-1} Q_n$.

The transition probability under $\mathbb{P}$ is obtained by $(P_0 P_1 \cdots P_{T-1})_{ij}$. 
In order to solve the PF equation, consider in general
\[
\lambda \begin{pmatrix}
z_1 \\
z_2 \\
\vdots \\
z_{N-1} \\
z_N
\end{pmatrix} = \begin{pmatrix}
m_1 & u_1 & 0 & \cdots & 0 \\
d_2 & m_2 & u_2 & \cdots & 0 \\
0 & \cdots & d_{N-1} & m_{N-1} & u_{N-1} \\
0 & \cdots & 0 & d_N & m_N
\end{pmatrix} \begin{pmatrix}
z_1 \\
z_2 \\
\vdots \\
z_{N-1} \\
z_N
\end{pmatrix}
\]

Namely, we solve the following system of linear equations:
\[
\begin{align*}
\lambda z_1 &= m_1 z_1 + u_1 z_2 \\
\lambda z_2 &= d_2 z_1 + m_2 z_2 + u_2 z_3 \\
\vdots \\
\lambda z_{N-1} &= d_{N-1} z_{N-2} + m_{N-1} z_{N-1} + u_{N-1} z_N \\
\lambda z_N &= d_N z_{N-1} + m_N z_N
\end{align*}
\]
Defining $\chi_i = z_{i+1}/z_i$, the above equations become

\[\lambda = m_1 + u_1 \chi_1\]
\[\lambda = d_2 \chi_1^{-1} + m_2 + u_2 \chi_2\]
\[\vdots\]
\[\lambda = d_{N-1} \chi_{N-2}^{-1} + m_{N-1} + u_{N-1} \chi_{N-1}\]
\[\lambda = d_N \chi_{N-1}^{-1} + m_N\]

Choose $\lambda > 0$ arbitrary, and solve the equations from above as

\[\chi_1(\lambda) = [\lambda - m_1]/u_1\]
\[\chi_2(\lambda) = [\lambda - d_2 \chi_1^{-1}(\lambda) - m_2]/u_2\]
\[\vdots\]
\[\chi_{N-1}(\lambda) = [\lambda - d_{N-1} \chi_{N-2}^{-1}(\lambda) - m_{N-1}]/u_{N-1}\]
\[\chi_N(\lambda) = \lambda - d_N \chi_{N-1}^{-1}(\lambda) + m_N\]
If $\chi_i(\lambda) > 0$ and $\chi_N(\lambda) = 0$, then this $\lambda$ is the PF eigenvalue.

By induction, it is shown that the functions $\chi_i(\lambda)$ are continuous and strictly increasing in $\lambda$ for all $i$.

Moreover, if $\chi_i(\lambda) > 0$, we have the following relationship:

$$
\begin{align*}
\chi_N(x) < 0 & \iff x < \lambda, & \chi_N(x) = 0 & \iff x = \lambda, \\
\chi_N(x) > 0 & \iff x > \lambda
\end{align*}
$$

Therefore, by using the standard search (e.g., bisection method), we can easily find the PF eigenvalue $\lambda$ that satisfies $\chi_N(\lambda) = 0$, and at the same time $\chi_i = \chi_i(\lambda)$ numerically.
The Hull–White Model

- Suppose that the spot rate $r_t$ follows

\[ dr_t = (\phi(t) - ar_t)dt + \sigma(t)dz_t \]

where $\phi(t)$ and $\sigma(t)$ are deterministic functions of time $t$, $a$ is a positive constant, and $z_t$ denotes a standard BM under $\mathbb{Q}$.

- $\phi(t)$ is chosen to be consistent with the current term structure.

- It is well known that

\[ r_t = \theta(t) + x_t, \quad dx_t = -ax_t dt + \sigma(t)dz_t \]

- The shift function $\theta(t)$ is given by

\[ \theta(t) = e^{-at} \left( \int_0^t e^{as} \phi(s) ds + r_0 \right) \]

- Note that $r_0 = \theta(0)$ implies $x_0 = 0$. 

Discretization

- For the time interval $\Delta t$, according to Hull and White (1994), we discretize the state as $\Delta x = \sigma(t_n)\sqrt{3\Delta t}$, i.e., $\xi = \sigma(t_n)\sqrt{3}$
- Construct the nodes $(n, i)$ when time is $n$ and state is $i$ so that the realization of $x_t$ is given by $x(n, i) = x_i \equiv i\Delta x$.
- The state space is $\mathcal{S} = \{-I, -I + 1, \ldots, I - 1, I\}$
- The transition probabilities under $\mathbb{Q}$ are defined as ($i \neq \pm I$)

\[
\begin{align*}
q_{i,i+1}(n) &= \frac{1}{6} + \frac{a^2i^2(\Delta t)^2 - ai\Delta t}{2} \\
q_{i,i}(n) &= \frac{2}{3} - a^2i^2(\Delta t)^2 \\
q_{i,i-1}(n) &= \frac{1}{6} + \frac{a^2i^2(\Delta t)^2 + ai\Delta t}{2}
\end{align*}
\]

- In order for these probabilities to be positive, we must have

\[-\frac{\sqrt{6}}{3a\Delta t} < -I \leq i \leq I < \frac{\sqrt{6}}{3a\Delta t}\]

- $q_{ij}(n) = q_{ij}$ on $\mathcal{S}$ are determined as far as $a$ is estimated.
Suppose that the risk-neutral transition probabilities \( Q = (q_{ij}) \) are estimated by some means (equivalently, \( a \) is given).

It follows that

\[
R_i(n) = e^{r_n(i)\Delta t} = e^{(\theta_n+x_i)\Delta t} = e^{\theta_n\Delta t} R^x_i(n), \quad R^x_i(n) = e^{x_i\Delta t}
\]

Note that the constant term in spot rates does not affect the estimate of the transition probabilities \( p_{ij}(n) \) under \( \mathbb{P} \).

Denoting \( D^x_n = \text{diag}(R^x_i(n)) \), we have the PF equation

\[
((D^x_n)^{-1}Q)z_n = \lambda_n z_n, \quad \lambda_n > 0, \quad z_n = (z_i(n)) > 0
\]

Defining \( \chi_i(n) = z_{i+1}(n)/z_i(n) \) for the solution \( z_n \), we have

\[
p_{i,i+1}(n) = \frac{\chi_i(n)}{\lambda_n R^x_i(n)} q_{i,i+1}, \quad p_{i,i-1}(n) = \frac{1}{\lambda_n R^x_i(n) \chi_{i-1}(n)} q_{i,i-1}
\]
The transition probabilities $q_{ij}$ depend on $a$, but not on the volatility $\sigma(t)$. That is, once the speed of mean reversion $a$ is estimated by some means, the parameters $\Delta t$ and $a$ alone determine the transition probabilities under $Q$.

The volatility $\sigma(t)$ affects the discretization of the state space.

Since $R^x_i(n)$ depends on $\Delta x$ which depends on $\sigma(t_n)$, the solution of the PF equation also depends on the volatility $\sigma(t)$.

Hence, the transition probabilities $p_{ij}(n)$ depend on the volatility $\sigma(t)$, and the recovery is inhomogeneous in time.

Also, note that $R^x_i < 1$ for $i < 0$. 
Comparison with Qin and Linetsky (2016)

- When $\theta(t) = \theta$ and $\sigma(t) = \sigma$, the Hull–White is reduced to the Vasicek: $dr_t = \alpha(\theta - r_t)dt + \sigma dz_t^Q$

- They show that, under $\mathbb{P}$, the spot rate process follows

$$dr_t = \alpha \left( \theta - \frac{\sigma^2}{2\alpha^2} - r_t \right) dt + \sigma dz_t^\mathbb{P}$$

for a choice of eigenvalue $\lambda = \theta - \sigma^2/(2\alpha^2)$

- In the Hull–White model, their results are not directly applicable, because it is not time-homogeneous.

- Note. Bloomberg recommends to fix $\alpha = 3\%$ and calibrate $\sigma(t)$ as a piecewise constant. The volatility function is usually calibrated by swaption prices in the market.
Date: Sep. 30, 2015 (Source: Reuters)
Estimated Result: $a = 3\%$, $\sigma = 1.02\% \Rightarrow (\sigma/a)^2 = 11.56\%$
Suppose that the spot rate $r_t$ follows the SDE under $\mathbb{Q}$

$$r_t = (x_t + \alpha(t))^2 + \theta(t), \quad dx_t = -a x_t dt + \sigma dz_t$$

Here, $\theta(t)$ is the shift function (used to fit the initial term structure) and, when $\theta(t) < 0$, the spot rates can be negative.

When $\theta(t) = 0$, the model is the QG model of Pelsser (1997).

Kijima et al. (2014) derive the closed form solution for discount bond as well as cap/floor prices when $\alpha(t)$ is a piecewise linear function.

Suppose that $Q_n = (q_{ij}(n))$ is estimated by some means. Then,

$$R_i(n) = e^{r_n(x_i)\Delta t} = e^{\theta_n\Delta t} R_i^x(n), \quad R_i^x(n) = e^{(x_i + \alpha_n)^2\Delta t}$$

Hence, the Ross Recovery can be applied by the same way as in the Hull–White model.
Consider the one-dimensional recurrent diffusion with jumps under $\mathbb{Q}$ with appropriate boundary conditions:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dz_t + dJ_t$$

where $J_t$ is a compound Poisson with negative (or positive) jumps.

- $J_t$ and $z_t$ are mutually independent.
- Such a jump-diffusion process can be approximated by a skip-free random walk (positive or negative, respectively).
- The associated transition probabilities can be calculated easily because of the independence.
The skip-free positive Markov chain has the transition matrix of the form

\[
Q = \begin{pmatrix}
  m_1 & u_{12} & 0 & \cdots & 0 \\
  d_2 & m_2 & u_2 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  d_{N-1,1} & \cdots & d_{N-1} & m_{N-1} & u_{N-1} \\
  d_{N,1} & \cdots & d_{N,N-2} & d_{N,N-1} & m_N
\end{pmatrix}
\]

Kijima (1993) develops a bisection algorithm to solve the PF equation, when the Markov chain is skip-free to the one direction.

Using the solution of the PF equation, we can apply the recovery theorem to obtain the transition matrix under \( \mathbb{P} \).
Consider a two-dimensional recurrent Markov chain \((X_t, Y_t)\) with transition matrix \(P = (p_{(i,j),(k,\ell)})\) and \(Q = (q_{(i,j),(k,\ell)})\) under \(\mathbb{P}\) and \(\mathbb{Q}\), respectively (assume time-homogeneity for simplicity).

Assume that

\[
S_t = s_t(X_t, Y_t), \quad r_t = r(X_t), \quad \eta_t = \eta_t(X_t, Y_t)
\]

so, \(X_t\) drives the interest-rate dynamics.

For \(X_t = i, Y_t = j\), we have \(R_i = e^{r(i)\Delta t}\) and

\[
s_t(i, j) = \sum_{k, \ell} \frac{\eta_T(k, \ell)}{\eta_t(i, j)} p_{(i,j),(k,\ell)} s_T(k, \ell) = \frac{1}{R_i} \sum_{k, \ell} q_{(i,j),(k,\ell)} s_T(k, \ell)
\]

It follows that, under the Ross' assumption,

\[
\delta \frac{\eta_{k,\ell}}{\eta_{i,j}} p_{(i,j),(k,\ell)} = \frac{1}{R_i} q_{(i,j),(k,\ell)}, \quad \forall i, j, k, \ell
\]
The Setup for the Two-Dim. Model

- We label the states \((i, j)\) in a lexicographical order.
- Suppose we have \(N\) states for \(X_t\) and \(M\) states for \(Y_t\), and define the subspace

\[
S_i = \{(i, 1), (i, 2), \ldots, (i, M - 1), (i, M)\}
\]

- The state space is \(S = \sum_{i=1}^{N} S_i\), and the total number is \(N \times M\).
- The transition matrix under \(\tilde{Q}\) has the following form:

\[
\tilde{Q} = \begin{pmatrix}
Q_{11} & Q_{12} & \cdots & Q_{1N} \\
Q_{21} & Q_{22} & \cdots & Q_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{N1} & Q_{N2} & \cdots & Q_{NN}
\end{pmatrix}, \quad Q_{ik} = \left(q_{(i,j),(k,\ell)}\right)_{j,\ell=1}^{M}
\]

- \(Q_{ik}\) is the transition matrix from \(S_i\) to \(S_k\).
Define the diagonal matrix $\tilde{D} = \text{diag}(R_i e)$ and consider the PF equation

$$(\tilde{D}^{-1} \tilde{Q}) z = \lambda z, \quad \lambda > 0, \quad z > 0$$

The recovery is obtained by

$$p_{(i,j),(k,\ell)} = \frac{z_{k,\ell}}{\lambda R_i z_{i,j}} q_{(i,j),(k,\ell)}$$

Here, $\lambda$ is the PF eigenvalue and $z = (z_i)$ is the associated eigenvector with $z_i = (z_{i,j})$ being the sub-vector on the sub-space $S_i$. 
Consider the model
\[
\begin{align*}
\frac{dS_t}{S_t} &= r_t dt + \sigma_S dw_t, \\
\quad dr_t &= a(\theta - r_t) dt + \sigma_r dz_t,
\end{align*}
\]
where \( dw_t dz_t = \rho dt \).

The option pricing formula is available for the model, so assume that the parameters are all estimated under \( Q \).

Define
\[
y_t = \frac{1}{\sigma_S} \log S_t - \frac{\rho}{\sigma_r} r_t
\]
so that
\[
dy_t = \left( \left( \frac{1}{\sigma_S} \frac{a \rho}{\sigma_r} \right) r_t - \frac{a \rho}{\sigma_r} \theta - \frac{\sigma_S}{2} \right) dt + dw_t - \rho dz_t
\]
Also, define

\[ \Sigma dW_t = dw_t - \rho dz_t, \quad \Sigma = \sqrt{1 - \rho^2} \]

so that \( dW_t dz_t = 0 \)

We then have mutually independent processes

\[
\begin{align*}
    dx_t &= -ax_t dt + \sigma_r dz_t \\
    dy_t &= \left( \left( \frac{1}{\sigma_S} + \frac{a \rho}{\sigma_r} \right) (\theta + x_t) - \frac{a \rho \theta}{\sigma_r} - \frac{\sigma_S}{2} \right) dt + \Sigma dW_t
\end{align*}
\]

Using these processes, we have

\[
\begin{align*}
    r_t &= \theta + x_t \\
    S_t &= \exp \left\{ \sigma_S y_t + \frac{\rho \sigma_S}{\sigma_r} (\theta + x_t) \right\}
\end{align*}
\]
We construct a two-dim. tri-nominal model for \((x_t, y_t)\) as follows.

Take \(\Delta t > 0\) and define

\[
\Delta x = \sigma_r \sqrt{3\Delta t}, \quad \Delta y = \Sigma \sqrt{3\Delta t}
\]

Transition probabilities of \(X_n\) are given as before, because \(x_t\) is the OU (\(r_t\) is the Vasicek).

Transition probabilities of \(Y_n\) are obtained by

\[
\mu_y(y_j) = \frac{1}{\Delta t} \left[ q_{j,j+1}^y \Delta y - q_{j,j-1}^y \Delta y \right]
\]

\[
\sigma_y^2(y_j) = 3\Sigma^2 \left[ q_{j,j+1}^y + q_{j,j-1}^y \right] - \mu_y^2(y_j) \Delta t
\]
Solving these equations, we get
\[
\begin{align*}
q_{y,j,j+1} &= \frac{1}{6} + \frac{m_i^2(\Delta t)^2 + m_i \Delta t}{2} \\
q_{y,j,j-1} &= \frac{1}{6} + \frac{m_i^2(\Delta t)^2 - m_i \Delta t}{2}
\end{align*}
\]

Because \(X_n\) and \(Y_n\) are mutually independent, we finally have
\[
q_{(i,j),(k,\ell)} = q_{x}^{x} q_{y}^{y}
\]

The resulting transition matrix under \(Q\) becomes a block tri-diagonal of the form
\[
D^{-1}Q = 
\begin{pmatrix}
M_{-I} & U_{-I} & O & \cdots & O \\
D_{-I+1} & M_{-I+1} & U_{-I+1} & \cdots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O & \cdots & D_{I-1} & M_{I-1} & U_{I-1} \\
O & \cdots & O & D_{I} & M_{I}
\end{pmatrix}
\]
In this paper, we develop a numerical method for the Ross recovery when the spot rate follows a one-dimensional diffusion (possibly inhomogeneous and with one-side jumps).

We approximate such a process by a discrete-time, finite random walk with skip-free nature.

The PF equation can be solved easily by applying a novel bisection search.

Our method can be applied to equity options under stochastic interest-rate economy.

There remain many problems; in particular, lots of numerical examples to test the usefulness of the Ross recovery in risk management (and also investment decisions).
Thank You for Your Attention