A NOTE ON FORWARD-START AMERICAN DERIVATIVE CONTRACTS

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ABSTRACT. The forward start American derivative contract is described and priced in the martingale pricing framework.

1. THE DERIVATIVE CONTRACT

The contract depends on an underlying S, initiates at time 0 and expires at time T_1. At some time τ, the party who is long the contract exercises and receives a payoff of $F(S_\tau)$. It is mandated in the contract that he should not exercise at any time prior to a certain $T_0 < T_1$ and that if he had not exercised by expiry, he has to exercise.

In this note, we are interested in the price V_t of the contract primarily in the cases $F(S) = S - K$ and $F(S) = K - S$ which correspond to what may be called the forward-start American long and short forward contracts respectively.

We will assume that F is bounded below in this note.

2. THE MODEL

Let us assume that the time horizon is $[0, T_1]$, the interest rate r is risk-free and constant and the underlying satisfies the geometric Brownian motion:

$$dS_t/S_t = \mu dt + \sigma dW_t,$$

where W_t is Brownian with respect to a real-world measure $\mathbb{P}$[I]. With the intention to hedge by trading in cash and the underlying, derivative contracts are priced with respect to a measure under which the process $S_t e^{-rt}$ is a martingale. The usual Girsanov argument produces a process $\tilde{W}_t$, Brownian with respect to a certain measure $\tilde{\mathbb{P}}$, under which S_t satisfies

$$dS_t/S_t = rdt + \sigma d\tilde{W}_t.$$

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[I] Basic technical assumption: Underlying filtered probability space is $(\Omega, \mathcal{F})$ and $\mathcal{F}_0$ consists only of sets of measure zero or one.

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3. Theoretical Result

We will now show that the price of the contract is given by

$$V_0 = \sup_{\tau \in \mathcal{T}_{T_0}} \tilde{E}\big[e^{-r\tau} F(S_\tau)\big],$$

where $\mathcal{T}_t$ denotes the collection of stopping times $\tau \in [t, T_1]$. The plausibility of this result is seen by comparison with the price of the ordinary American contract (i.e. without the forward-start clause) which is known to be

$$\sup_{\tau \in \mathcal{T}_0} \tilde{E}\big[e^{-r\tau} F(S_\tau)\big].$$

We hereby state and prove the following result.

Theorem 1.

(1) *The no-arbitrage price at time t of the forward start American derivative contract is given by a process V_t that satisfies*

$$V_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t \vee T_0}} \tilde{E}\big[e^{-r(\tau - t)} F(S_\tau) | \mathcal{F}_t\big] \quad \textit{a.s. } \tilde{\mathbb{P}}.$$

In particular, its price at time 0 is

$$V_0 = \sup_{\tau \in \mathcal{T}_{T_0}} \tilde{E}\big[e^{-r\tau} F(S_\tau)\big].$$

(2) *There exists an optimal stopping time σ which satisfies*

$$\tilde{E}\big[e^{-r(\sigma_t - t)} F(S_{\sigma_t}) | \mathcal{F}_t\big] = V_t \quad \textit{a.s. } \tilde{\mathbb{P}}.$$

(3) *The price $V = V(S,t)$ satisfies the PDE*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

when $t \in [0, T_0)$. When $t \in [T_0, T_1]$, it satisfies the PDE together with the boundary conditions $V(S,t) \geq F(S)$ and $V(S,T) = F(S)$.

Proof. The idea is to regard the derivative as European on $[0, T_0]$ and American on $[T_0, T_1]$.

By a suitable translation, we may assume that F is non-negative. The theory of optimal stopping [MR, App. D] provides us with the Snell envelope X of the process $u \mapsto e^{-ru} F(S_u)$ which allows the price of an American derivative contract at time t to be expressed as

$$e^{rt} X_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \tilde{E}\big[e^{-r(\tau - t)} F(S_\tau) | \mathcal{F}_t\big] \quad \text{a.s. } \tilde{\mathbb{P}}$$

and the stopping time

$$\rho_t := \inf\left\{t \le u \le T_1 : X_u = e^{-ru}F(S_u)\right\}$$

which is optimal in the sense that

$$\tilde{E}\left[e^{-r(\rho_t - t)}F(S_{\rho_t})|\mathcal{F}_t\right] = e^{rt}X_t \quad \text{a.s. } \tilde{\mathbb{P}}$$

for $t \in [0, T_1]$.

Let us define

$$V_t = \begin{cases} \tilde{E}\left[e^{-r(T_0 - t)}e^{rT_0}X_{T_0}|\mathcal{F}_t\right], & \text{if } t \in [0, T_0], \\ e^{rt}X_t, & \text{if } t \in [T_0, T_1]. \end{cases} \quad .$$

On the interval $[0, T_0]$, V is the no-arbitrage price process of an European contingent claim, while on $[T_0, T_1]$, it is an American derivative price process. Standard option pricing theory ([KS, Cor. 3.1.3 and Thm. 5.1.1]) tells us that V is replicable and there is no arbitrage opportunity on $[0, T_1]$. Hence V is the no-arbitrage price of the forward start American derivative contract with payoff function F.

Since

$$\tilde{E}\left[e^{rT_0}X_{T_0}|\mathcal{F}_t\right] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{T_0}} \tilde{E}\left[e^{-r(\tau - T_0)}F(S_\tau)|\mathcal{F}_t\right] \quad \text{a.s. } \tilde{\mathbb{P}}$$

when $t \in [0, T_0]$ ([KS, Prop. D.2]), (1) is proved.

Let us set $\sigma_t = \rho_{t \vee T_0}$. When $t \ge T_0$, (2) follows directly from the definition. When $t < T_0$,

$$\begin{aligned} \tilde{E}\left[e^{-r(\sigma_t - t)}F(S_{\tau_t})|\mathcal{F}_t\right] &= e^{r(t-T_0)}\tilde{E}\left[e^{-r(\rho_{T_0} - T_0)}F(S_{\rho_{T_0}})|\mathcal{F}_t\right] \\ &= e^{r(t-T_0)}\tilde{E}\left[\tilde{E}\left[e^{-r(\rho_{T_0} - T_0)}F(S_{\rho_{T_0}})|\mathcal{F}_{T_0}\right]|\mathcal{F}_t\right] \\ &= \tilde{E}\left[e^{rt}X_{T_0}|\mathcal{F}_t\right] \\ &= V_t. \end{aligned}$$

Lastly, (3) is a consequence of how V is defined above. □

References

[KS] Musiela, M. and Rutkowski, M. (2005). *Martingale Methods in Financial Modelling.* Springer-Verlag Berlin Heidenberg.

[MR] Karatzas, I. and Shreve, S. E. (1998). *Methods of Mathematical Finance.* Springer-Verlag New York, Inc.