EXOTIC OPTIONS

Learning Objectives:

- As an application of the risk-neutral pricing formula, the following exotic options are priced:
  - Package options
  - Forward start options
  - Chooser options
  - Compound options
  - Digital options

- The change-of-numeraire method is introduced and the following exotic options are priced with the method:
  - Exchange options
  - Best-of and Worst-of options

1. PACKAGE OPTIONS

A package option is a contract whose payoff at maturity is a piecewise linear function of the terminal price of the underlying asset. It can be decomposed into a combination of standard options, cash and the underlying asset.

1.1. Collars.

A collar is a contract whose payoff at maturity is

$$CL_T = \min(\max(S_T, K_1), K_2),$$

where $K_2 > K_1 > 0$.

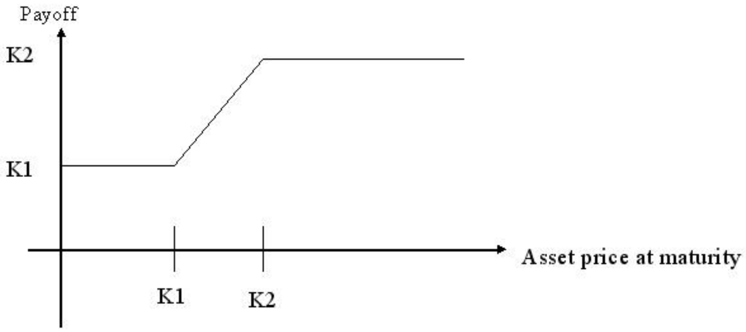


FIGURE 1

The payoff can be written as

$$CL_T = K_1 + (S_T - K_1)^+ - (S_T - K_2)^+,$$

from which it can be seen that a collar is a package of cash, long one option and short another option. The value of the collar at time t is thus given by the BS formula:

$$CL_t = K_1 e^{-r(T-t)} + C(S_t, T-t, K_1) - C(S_t, T-t, K_2).$$

That is not the only way to decompose the payoff. An alternative expression is this:

$$CL_T = S_T + (K_1 - S_T)^+ - (S_T - K_2)^+.$$

This leads to the time-t price of

$$CL_t = S_t + P(S_t, T-t, K_1) - C(S_t, T-t, K_2).$$

1.2. **Break Forwards.** A break forward is a modification of the forward contract. Its payoff is

$$BF_T = \max(S_T, F) - K,$$

where $F = S_0 e^{rT}$.

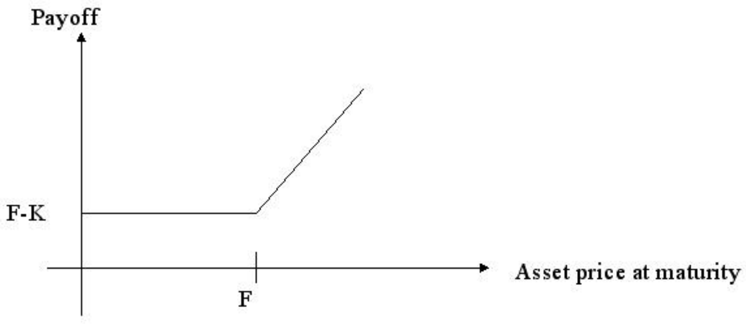


FIGURE 2

The payoff can be written as

$$BF_T = (S_T - F)^+ + F - K,$$

from which it can be seen that a break forward is a package of cash and option. The value of a break forward at time t is thus given by the BS formula:

$$BF_t = C(S_t, T-t, F) + (F-K)e^{-r(T-t)}.$$

2. FORWARD START OPTIONS

Let $t \leq T_0 < T$ be three times, with t representing the present. A forward start option is a contract that is struck at time t, in which the holder receives at time T_0 an option which matures at T and has strike equal to KS_{T_0}.

The payoff at maturity is

$$FS_T = (S_T - KS_{T_0})^+.$$

The price of the contract at time T_0 is equal to the price of the vanilla option:

$$FS_{T_0} = C(S_{T_0}, T - T_0, KS_{T_0})$$

$$= S_{T_0} C(1, T - T_0, K),$$

the last equality following from the BS formula.

Since the term $C(1, T - T_0, K)$ is non-random, we have

$$FS_t = S_t C(1, T - T_0, K),$$

since we may hold $C(1, T - T_0, K)$ of the stock to precisely hedge the risk of the contract at time T_0.

3. CHOOSER OPTIONS

Let $t \leq T_0 < T$ be three times, with t representing the present. A chooser option gives the right to choose at T_0 an option, which may be a call or a put, and which expires at time T and has strike K.

The time-T_0 value of a chooser option is

$$CH_{T_0} = \max(C(S_{T_0}, T - T_0, K), P(S_{T_0}, T - T_0, K)),$$

which we may rewrite, using the put-call parity, as

$$\max(C(S_{T_0}, T - T_0, K), C(S_{T_0}, T - T_0, K) - S_{T_0} + Ke^{-r(T-T_0)})$$

$$= C(S_{T_0}, T - T_0, K) + \max(0, -S_{T_0} + Ke^{-r(T-T_0)}).$$

This is a portfolio of a call option and a put option with maturities at T and T_0 respectively.

Hence, the time t price of the chooser option is

$$CH_t = C(S_t, T - t, K) + P(S_t, T_0 - t, Ke^{-r(T-T_0)}).$$

4. COMPOUND OPTIONS

A compound option is an option with another option as the underlying asset. We may distinguish four types: call on call, call on put, put on call and put on put.

Let $t \le T_0 < T$ be three times, with t representing the present. The payoff at time T_0 of a call on call is

$$CO_{T_0} = (C(S_{T_0}, T - T_0, K) - K_0)^+.$$

In the BS framework, the time-t price of this option is the risk neutral expectation, properly discounted:

$$CO_t = E^{rn}[e^{-r(T_0-t)} \max(0, C(S_{T_0}, T - T_0, K) - K_0)|\mathcal{F}_t].$$

That is expectation is taken with respect to the risk neutral probability measure means that

$$S_{T_0}(x) = S_t \exp\big((r - \frac{1}{2}\sigma^2)(T_0 - t) + \sigma\sqrt{T_0 - t}x\big),$$

where $x \sim N(0, 1)$.

Thus,

$$CO_t = e^{-r(T_0-t)} \int_{x_0}^{\infty} (C(S_{T_0}(x), T - T_0, K) - K_0)n(x)\, dx$$

$$= e^{-r(T-T_0)} \int_{x_0}^{\infty} (S_{T_0}(x)N(\hat{d}_1) - Ke^{-r(T-T_0)}N(\hat{d}_2) - K_0)n(x)\, dx,$$

where $n(x)$ is the standard normal PDF,

$$\hat{d}_i = d_i(S_{T_0}(x), T - T_0, K),$$

and

$$x_0 = \inf\{x \in \mathbb{R} : C(S_{T_0}(x), T - T_0, K) \ge K_0\}.$$

Remark: The integral does not seem to admit a closed form formula (like the Black-Scholes formula). However, the fact that the price of the compound option is expressible as an explicit integral means that it is amenable to numerical methods. Indeed, straightforward quadrature methods will give very good approximations of the price in an instant on the computer.

5. Digital/Binary Options

The simplest examples of digital options are the cash-or-nothing options and the asset-or-nothing options. The payoff of a cash-or-nothing call option is

$$BCC_T = X\chi_{\{S_T>K\}};$$

the payoff of a cash-or-nothing put option is

$$BCP_T = X\chi_{\{S_T<K\}};$$

the payoff of an asset-or-nothing call option is

$$BAC_T = S_T\chi_{\{S_T>K\}};$$

the payoff of an asset-or-nothing put option is

$$BAP_T = S_T\chi_{\{S_T<K\}}.$$

The pricing formula for such an option is obtained by applying the martingale pricing formula. For example,

$$BAC_t = e^{-r(T-t)}E^{rn}[S_T\chi_{\{S_T>K\}}|\mathcal{F}_t] = S_tN(d_1).$$

Other digital options are the gap options. Gap call options have payoffs

$$GC_T = (S_T - X)\chi_{\{S_T>K\}} = BAC_T - BCC_T.$$

Hence it may be regarded as a package of two vanilla binary options. Similarly, gap put options have payoffs

$$GC_T = (X - S_T)\chi_{\{S_T<K\}} = BCP_T - BAP_T.$$

Example 1. *The payoff of a supershare is*

$$SS_T = \frac{S_T}{K_1}\chi_{K_1<S_T<K_2},$$

for some positive constants $K_1 < K_2$. Price it.

6. MAGRABE'S TRICK AND THE CHANGE-OF-NUMÉRAIRE METHOD

6.1. **Probabilistic Interpretation of $N(d_1)$.** Let's first take a look at the Black-Scholes formula

$$C(S_t, t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2).$$

We showed that the terms are separately:

$$S_t N(d_1) = e^{-r(T-t)} E^{rn}[S_T \chi_A | \mathcal{F}_t],$$
$$N(d_2) = E^{rn}[\chi_A | \mathcal{F}_t].$$

Thus the formula has the intuitive interpretation that the call option price is equal to the difference between the expected stock price at maturity and the average taking by the option writer in the risk neutral world. In other words, the price of the call option is equal to the difference between the average of what the option writer pays and the average of what she receives at maturity in the risk neutral world.

Analogously, the price of a put option is given by

$$P(S_t, t) = -S_t N(-d_1) + Ke^{-r(T-t)} N(-d_2),$$

with the probabilistic interpretations

$$S_t N(-d_1) = e^{-r(T-t)} E^{rn}[S_T \chi_B | \mathcal{F}_t],$$
$$N(-d_2) = E^{rn}[\chi_B | \mathcal{F}_t],$$

where B is the event that $S_T \leq K$.

Is there a probabilistic interpretation of $N(d_1)$?

The following discussion shows that there is. The discussion works in another world that is different from the Black-Scholes. In other words, the underlying assets are different. However, the formalism of martingale pricing continues to apply.

Recall the Black-Scholes setting:

- Risk-free rate is r
- Risky asset S_t satisfies

  $$dS_t = \mu S_t dt + \sigma S_t dW_t$$

  with respect to the real world measure $\mathbb{P}$

- Change measure to $\mathbb{Q}$, so that S_t satisfies
$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$$
- Pricing formula for call option V_t:
$$\begin{aligned} V_t &= E^{\mathbb{Q}}[e^{-r(T-t)}V_T|\mathcal{F}_t] \\ &= S_t N(d_1) - Ke^{-r(T-t)}N(d_2), \end{aligned}$$
where for call options,
$$\begin{aligned} d_1 &= \frac{\log(S_T/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log(S_T/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

Instead of taking e^{rt} as the unit of measure (numeraire), we may change the numeraire to something else, say 1. In other words, we imagine a world that needs no discounting.

- Let numéraire be 1, and the risk asset be $\frac{S_t}{e^{rt}}$, so that relative to the numéraire, stock price is $\frac{S_t}{e^{rt}}$ itself
- The SDE becomes
$$d\frac{S_t}{e^{rt}} = \sigma \frac{S_t}{e^{rt}} d\tilde{W}_t,$$
which shows that $S_t e^{-rt}$ is a martingale with respect to $\mathbb{Q}$
- Risky asset is thus $\frac{S_t}{e^{rt}}$, and risk-free rate is 0 (look at the SDE and compare with the one above!)
- Martingale pricing formula for contingent claim V_T:
$$e^{-rt}V_t = E^{\mathbb{Q}}[e^{-rT}V_T|\mathcal{F}_t]$$

Since $e^{-rT}V_T = (S_T/e^{rT} - K/e^{rT})^+$, the expectation on the RHS may be interpreted as the time t value of a call option with strike K/e^{rT} and underlying asset S_t/e^{rt} in a world with risk-free rate 0. By the Black-Scholes formula, therefore,
$$E^{\mathbb{Q}}[e^{-rT}V_T|\mathcal{F}_t] = \frac{S_t}{e^{rt}}N(d_1') - \frac{K}{e^{rT}}N(d_2'),$$
where
$$d_1' = \frac{\log(\frac{S_t}{e^{rt}}/\frac{K}{e^{rT}}) + (0 + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_2' = \frac{\log(\frac{S_t}{e^{rt}}/\frac{K}{e^{rT}}) + (0 - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Note that this agrees with the original formulation.

The following discussion (and the rest of this lecture) illustrates the change-of-numéraire technique, which depends on the following theorem, which spells out the link between numeraires and martingale measures.

Theorem 1. *Assume there exists a numeraire N and a probability measure $\mathbb{Q}$, equivalent to the initial $\mathbb{P}$, such that the price of any traded asset X (without intermediate payments) relative to N is a martingale under $\mathbb{Q}$, i.e.*

$$X_t/N_t = E^{\mathbb{Q}}[X_T/N_T|\mathcal{F}_t] \ (0 \le t \le T).$$

Let U be an arbitrary numeraire. Then there exists a probability measure $\mathbb{T}$, equivalent to the initial $\mathbb{P}$, such that the price of any attainable claim Y normalized by U is a martingale under $\mathbb{T}$, i.e.

$$Y_t/U_t = E^{\mathbb{T}}[Y_T/U_T|\mathcal{F}_t] \ (0 \le t \le T).$$

Moreover, the Radon-Nikodym derivative defining the measure $\mathbb{T}$ is given by

$$\frac{d\mathbb{T}}{d\mathbb{Q}} = \frac{\frac{U_T}{U_0}}{\frac{N_T}{N_0}}.$$

(Reference: Interest Rate Models - Theory and Practice, by D. Brigo and F. Mercurio, Proposition 2.2.1)

In other words,

$$Y_t = U_t E^{\mathbb{T}}[\frac{Y_T}{U_T}|\mathcal{F}_t] = N_t E^{\mathbb{Q}}[\frac{Y_T}{N_T}|\mathcal{F}_t],$$

i.e. for any attainable claim Y_T, we may compute its no-arbitrage price at time t, namely Y_t, using different martingale measures which correspond to choices of numeraires. We will apply this fact in the following.

Write the call option payoff as

$$\max(S_T - K, 0) = S_T \max(1 - \frac{K}{S_T}, 0).$$

This allows the European call option to be interpreted as an European put option. Since S_T appears in the denominator, this motivates declaring it as numéraire, hence leading us to the following 'world':

- Risky asset S_t satisfies
$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$$
with respect to the measure $\mathbb{Q}$
- Let numéraire be S_t. The contingent claims S_T and K are attainable: by S_t and $Ke^{-r(T-t)}$ respectively. Measured relative to the numeraire S_t, they become 1 and $x_t := \frac{Ke^{-r(T-t)}}{S_t}$ respectively
- With respect to the measure $\mathbb{Q}$, x_t satisfies
$$\begin{aligned} dx_t &= rx_t dt - \frac{1}{S_t} x_t dS_t + \frac{1}{S_t^2} x_t dS_t^2 \\ &= rx_t dt - \frac{1}{S_t} x_t (rS_t dt + \sigma S_t d\tilde{W}_t) + \frac{1}{S_t^2} x_t S_t^2 \sigma^2 dt \\ &= \sigma^2 x_t dt - \sigma x_t d\tilde{W}_t \end{aligned}$$
- Change measure from $\mathbb{Q}$ to $\mathbb{T}$ so that the stochastic process $\overline{W}_t$ defined by
$$d\overline{W}_t = -\sigma dt + d\tilde{W}_t$$
is Brownian (Girsanov Theorem)
- The SDE under $\mathbb{T}$ becomes
$$dx_t = -\sigma x_t d\overline{W}_t,$$
which shows that x_t is a martingale with respect to $\mathbb{T}$ and which verifies that $\mathbb{T}$ is the correct measure as mentioned in the Theorem
- In this world, the risk-free rate is 0 - just look at this last SDE above!
- Martingale pricing formula for contingent claim V_T:
$$V_t/S_t = E^{\mathbb{T}}[V_T/S_T|\mathcal{F}_t].$$

For another perspective of the situation, consider the put-call parity

$$C_t - P_t = S_t - Ke^{-r(T-t)}.$$

Dividing by S_t gives

$$\frac{C_t}{S_t} - \frac{P_t}{S_t} = 1 - x_t,$$

where

$$x_t = \frac{Ke^{-r(T-t)}}{S_t}.$$

This means that if we want to price the call option value in units of the stock, i.e. $\frac{C_t}{S_t}$, we would imagine a zero-interest-rate world that has one risk asset x_t. In this world, the derivative $\frac{C_t}{S_t}$ has value at maturity equal to

$$\frac{C_T}{S_T} = \max(0, 1 - x_T),$$

i.e. a put option with strike 1.

The theorem above says that

$$V_t = e^{rt}E^{\mathbb{Q}}[V_T/e^{rT}|\mathcal{F}_t] = S_tE^{\mathbb{T}}[V_T/S_T|\mathcal{F}_t],$$

i.e. the risk-neutral price can as well be computed under the measure $\mathbb{T}$.

Since x_t satisfies

$$dx_t = -\sigma x_t d\overline{W}_t,$$

the price of the put option at time t is

$$\begin{aligned}\frac{C_t}{S_t} &= E^{\mathbb{T}}[C_T/S_T|\mathcal{F}_t] \\ &= E^{\mathbb{T}}[(1-x_T)^+|\mathcal{F}_t] \\ &= -x_t \times N(-\delta_1) + 1 \times N(-\delta_2),\end{aligned}$$

where

$$\delta_1 = \frac{\log(x_t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

$$\delta_2 = \frac{\log(x_t) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

This implies:

$$C_t = -Ke^{-r(T-t)}N(-\delta_1) + S_tN(-\delta_2).$$

We may compare this with the usual BS formula for call option price:

$$C_t = S_tN(d_1) - Ke^{-r(T-t)}N(d_2).$$

From these equations, we obtain

$$N(d_1) = N(-\delta_2).$$

This is the probabilistic interpretation of $N(d_1)$: it is the probability that $x_T \leq 1$ in the world defined by the measure $\mathbb{T}$.

6.2. **Exchange Option.** Let's consider the pricing of an exchange option whose payoff at maturity is

$$\max(S_T^1 - S_T^2, 0).$$

It is called an exchange option because the vanilla case can be interpreted as a right to exchange stock for cash; here it is S^1 for S^2.

We shall assume that S^1 and S^2 follow the GBM with correlated Brownian motions:

$$dS_t^i = \mu_i S_t^i dt + \sigma_i S_t^i dW_t^i, \ (i = 1, 2),$$
$$dW_t^1 dW_t^2 = \rho dt.$$

By the two-dimensional version of the Girsanov Theorem, the above SDEs are transformed into

$$dS_t^i = r S_t^i dt + \sigma_i S_t^i d\tilde{W}_t^i, \ (i = 1, 2),$$
$$d\tilde{W}_t^1 d\tilde{W}_t^2 = \rho dt,$$

where $\tilde{W}^1, \tilde{W}^2$ are Brownian with respect to a measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$.

Magrabe's Trick or the change-of-numeraire method reduces the number of variables from two to one. The gist of what we want is this: we want to price the exchange option at time t, and this is given by

$$E^{\mathbb{Q}}[e^{-r(T-t)}(S_T^1 - S_T^2)^+ | \mathcal{F}_t].$$

This may be written as

$$E^{\mathbb{Q}}[e^{-r(T-t)} S_T^2 (x_T - 1)^+ | \mathcal{F}_t],$$

where $x_t := S_t^1 / S_t^2$ for $t \in [0, T]$. The method gives

$$\begin{aligned} E^{\mathbb{Q}}[e^{-r(T-t)} S_T^2 (x_T - 1)^+ | \mathcal{F}_t] &= e^{rt} E^{\mathbb{Q}}[e^{-rT} S_T^2 (x_T - 1)^+ | \mathcal{F}_t] \\ &= S_t^2 E^{\mathbb{T}}[S_T^2 (x_T - 1)^+ / S_T^2 | \mathcal{F}_t] \end{aligned}$$

$$= S_t^2 E^{\mathbb{T}}[(x_T - 1)^+ | \mathcal{F}_t].$$

In order to compute this last expectation, we need to work out the SDE for x_t in the 'world' defined by the numeraire S_t^2 and the probability measure $\mathbb{T}$. We know that x_t satisfies an SDE found by applying Itô's Lemma to $f(S_t^1, S_t^2) = \frac{S_t^1}{S_t^2}$:

$$\begin{aligned}
dx_t &= f_{S^1} dS_t^1 + f_{S^2} dS_t^2 + \frac{1}{2} f_{S^1 S^1} (S_t^1)^2 (dS_t^1)^2 \\
&\quad + f_{S^1 S^2} S_t^1 S_t^2 dS_t^1 dS_t^2 + \frac{1}{2} f_{S^2 S^2} (S_t^2)^2 (dS_t^2)^2 \\
&= \frac{1}{S_t^2}(rS_t^1 dt + \sigma_1 S_t^1 d\tilde{W}_t^1) - \frac{S_t^1}{(S_t^2)^2}(rS_t^2 dt + \sigma_2 S_t^2 d\tilde{W}_t^1) \\
&\quad - \frac{1}{(S_t^2)^2} S_t^1 S_t^2 \sigma_1 \sigma_2 \rho dt + \frac{S_t^1}{(S_t^2)^3}(S_t^2)^2 \sigma_2^2 dt \\
&= x_t \Big((\sigma_2^2 - \rho\sigma_1\sigma_2) dt + \sigma_1 d\tilde{W}_t^1 - \sigma_2 d\tilde{W}_t^2 \Big) \\
&= x_t \Big((\sigma_2^2 - \rho\sigma_1\sigma_2) dt + \Sigma \big(\frac{\sigma_1}{\Sigma} d\tilde{W}_t^1 - \frac{\sigma_2}{\Sigma} d\tilde{W}_t^2 \big) \Big),
\end{aligned}$$

where $\Sigma := \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$. The stochastic process $\frac{\sigma_1}{\Sigma}\tilde{W}_t^1 - \frac{\sigma_2}{\Sigma}\tilde{W}_t^2$ is Brownian. hence the SDE may be written as

$$dx_t = \Sigma x_t d\overline{W}_t,$$

where $\overline{W}_t := \frac{\sigma_1}{\Sigma}\tilde{W}_t^1 - \frac{\sigma_2}{\Sigma}\tilde{W}_t^2 + \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\Sigma} t$ is Brownian with respect to the measure $\mathbb{T}$, as provisioned by the Girsanov Theorem.

Question: Why is the stochastic process $\frac{\sigma_1}{\Sigma} W_t^1 - \frac{\sigma_2}{\Sigma} W_t^2$ Brownian ?

The risky asset in this world of $\mathbb{T}$ is x_t and the risk-free rate is 0. Hence the option price at time t is

$$\begin{aligned}
S_t^2 E^{\mathbb{T}}[(x_T - 1)^+ | \mathcal{F}_t] &= S_t^2 \big(x_t \times N(d_1') - 1 \times N(d_2') \big) \\
&= S_t^1 N(d_1') - S_t^2 N(d_2'),
\end{aligned}$$

where

$$\begin{aligned}
d_1' &= \frac{\log(x_t/1) + (0 + \frac{1}{2}\Sigma^2)(T - t)}{\Sigma\sqrt{T - t}}, \\
d_2' &= \frac{\log(x_t/1) + (0 - \frac{1}{2}\Sigma^2)(T - t)}{\Sigma\sqrt{T - t}}.
\end{aligned}$$

6.3. **Best/Worst-of Options.** Now let's consider the best-of-two call option whose payoff at maturity is

$$\max(\max(S_T^1, S_T^2) - K, 0).$$

We will retain the assumption about the GBM dynamics for S^1, S^2.

Changing from the real world measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$, we have the SDEs:

$$dS_t^i = rS_t^i dt + \sigma_i S_t^i d\tilde{W}_t^i, \ (i = 1, 2),$$

$$d\tilde{W}_t^1 d\tilde{W}_t^2 = \rho dt.$$

The time-t price of the option is:

$$E^{\mathbb{Q}}[e^{-r(T-t)} \max(\max(S_T^1, S_T^2) - K, 0)|\mathcal{F}_t]$$

$$\begin{aligned}
&= E^{\mathbb{Q}}[e^{-r(T-t)}(\max(S_T^1, S_T^2) - K)1_{\max(S_T^1,S_T^2)>K}|\mathcal{F}_t] \\
&= E^{\mathbb{Q}}[e^{-r(T-t)} \max(S_T^1, S_T^2)1_{\max(S_T^1,S_T^2)>K}|\mathcal{F}_t] - E^{\mathbb{Q}}[e^{-r(T-t)}K1_{\max(S_T^1,S_T^2)>K}|\mathcal{F}_t] \\
&= E^{\mathbb{Q}}[e^{-r(T-t)}S_T^1 1_{S_T^1>S_T^2,S_T^1>K}|\mathcal{F}_t] + E^{\mathbb{Q}}[e^{-r(T-t)}S_T^2 1_{S_T^2>S_T^1,S_T^2>K}|\mathcal{F}_t] \\
&\quad - Ke^{-r(T-t)}E^{\mathbb{Q}}[1 - 1_{\max(S_T^1,S_T^2)<K}|\mathcal{F}_t] \\
&= E^{\mathbb{Q}}[e^{-r(T-t)}S_T^1 1_{S_T^1>S_T^2,S_T^1>K}|\mathcal{F}_t] + E^{\mathbb{Q}}[e^{-r(T-t)}S_T^2 1_{S_T^2>S_T^1,S_T^2>K}|\mathcal{F}_t] \\
&\quad - Ke^{-r(T-t)} + Ke^{-r(T-t)}\mathbb{Q}(S_T^1, S_T^2 < K|\mathcal{F}_t),
\end{aligned}$$

which shows that we need to compute the terms $E^{\mathbb{Q}}[e^{-r(T-t)}S_T^1 1_{S_T^1>S_T^2,S_T^1>K}|\mathcal{F}_t]$, $E^{\mathbb{Q}}[e^{-r(T-t)}S_T^2 1_{S_T^2>S_T^1,S_T^2>K}|\mathcal{F}_t]$ and $\mathbb{Q}(S_T^1, S_T^2 < K|\mathcal{F}_t)$.

Let us compute $E^{\mathbb{Q}}[e^{-r(T-t)}S_T^2 1_{S_T^2>S_T^1,S_T^2>K}]$ by the change of numéraire method. The computation for $E^{\mathbb{Q}}[e^{-r(T-t)}S_T^1 1_{S_T^1>S_T^2,S_T^1>K}]$ will be similar.

Let us write

$$E^{\mathbb{Q}}[e^{-r(T-t)}S_T^2 1_{S_T^2>S_T^1,S_T^2>K}|\mathcal{F}_t] = E^{\mathbb{Q}}[e^{-r(T-t)}S_T^2 1_{1>\frac{S_T^1}{S_T^2},1>\frac{K}{S_T^2}}|\mathcal{F}_t].$$

This suggests taking S_t^2 to be the numeraire and looking at the dynamics of the two risky assets $\frac{S_t^1}{S_t^2}$ and $\frac{Ke^{-r(T-t)}}{S_t^2}$.

By the change-of-numeraire method, corresponding to the numeraire S_t^2 is a measure $\mathbb{T}$ such that

$$\begin{aligned} E^{\mathbb{Q}}[e^{-r(T-t)}S_T^2 1_{1>\frac{S_T^1}{S_T^2},1>\frac{K}{S_T^2}}|\mathcal{F}_t] &= e^{rt}E^{\mathbb{Q}}[e^{-rT}S_T^2 1_{1>\frac{S_T^1}{S_T^2},1>\frac{K}{S_T^2}}|\mathcal{F}_t] \\ &= S_t^2 E^{\mathbb{T}}[S_T^2 1_{1>\frac{S_T^1}{S_T^2},1>\frac{K}{S_T^2}}/S_T^2|\mathcal{F}_t] \\ &= S_t^2\mathbb{T}(1_{1>\frac{S_T^1}{S_T^2},1>\frac{K}{S_T^2}}|\mathcal{F}_t). \end{aligned}$$

Now we need to compute $\mathbb{Q}(S_T^1, S_T^2 < K|\mathcal{F}_t)$ and $\mathbb{T}(1_{1>\frac{S_T^1}{S_T^2},1>\frac{K}{S_T^2}}|\mathcal{F}_t)$, which we rewrite respectively as

$$\mathbb{Q}(\log S_T^1, \log S_T^2 < \log K|\mathcal{F}_t)$$

and

$$\mathbb{T}(1_{0>\log\frac{S_T^1}{S_T^2},0>\log\frac{K}{S_T^2}}|\mathcal{F}_t).$$

From the SDEs:

$$dS_t^i = rS_t^i dt + \sigma_i S_t^i d\tilde{W}_t^i,\ (i=1,2),$$
$$d\tilde{W}_t^1 d\tilde{W}_t^2 = \rho dt,$$

we obtain

$$d\log S_t^i = (r - \frac{1}{2}\sigma_i^2)dt + \sigma_i d\tilde{W}_t^i,\ (i=1,2),$$
$$d\tilde{W}_t^1 d\tilde{W}_t^2 = \rho dt.$$

Hence

$$\begin{aligned} &\mathbb{Q}(\log S_T^1, \log S_T^2 < \log K|\mathcal{F}_t) \\ &= \mathbb{N}_{\log S_t^1+(r-\frac{1}{2}\sigma_1^2)(T-t),\log S_t^2+(r-\frac{1}{2}\sigma_2^2)(T-t);\sigma_1\sqrt{T-t},\sigma_2\sqrt{T-t};\rho}(\log K, \log K), \end{aligned}$$

where $\mathbb{N}_{\mu_1,\mu_2;\sigma_1,\sigma_2;\rho}$ is the CDF of the two-dimensional normal distribution with means μ_1 and μ_2, standard deviations σ_1 and σ_2, and correlation ρ.

To compute $\mathbb{T}(1_{0>\log\frac{S_T^1}{S_T^2},0>\log\frac{K}{S_T^2}}|\mathcal{F}_t)$, we need the dynamics of $\log\frac{S_T^1}{S_T^2}$ and $\log\frac{Ke^{-r(T-t)}}{S_T^2}$. We've earlier shown that

$$d(\frac{S_t^1}{S_t^2}) = \Sigma(\frac{S_t^1}{S_t^2})d\overline{W}_t,$$

where $\Sigma := \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$ and

$$\overline{W}_t = \frac{\sigma_1}{\Sigma}\tilde{W}_t^1 - \frac{\sigma_2}{\Sigma}\tilde{W}_t^2 + \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\Sigma}t$$

is Brownian with respect to $\mathbb{T}$. We've also shown that

$$d\big(\frac{Ke^{-r(T-t)}}{S_t^2}\big) = -\sigma_2\big(\frac{Ke^{-r(T-t)}}{S_t^2}\big)d\hat{W}_t,$$

where $\hat{W}_t = \tilde{W}_t^2 - \sigma_2 t$ is Brownian with respect to $\mathbb{T}$. Consequently, we have

$$d\hat{W}_t d\overline{W}_t = (-\frac{\sigma_2}{\Sigma} + \frac{\sigma_1\rho}{\Sigma})dt.$$

Therefore,

$$\mathbb{T}(1_{0>\log\frac{S_T^1}{S_T^2}, 0>\log\frac{K}{S_T^2}}|\mathcal{F}_t)$$

$$= \mathbb{N}_{\log\frac{S_t^1}{S_t^2} - \frac{1}{2}\Sigma^2(T-t), \log\frac{Ke^{-r(T-t)}}{S_t^2} - \frac{1}{2}\sigma_2^2(T-t); \Sigma\sqrt{T-t}, \sigma_2\sqrt{T-t}; \frac{\sigma_2}{\Sigma} - \frac{\sigma_1\rho}{\Sigma}}(0,0).$$

Example 2. *So, what's the time-t price of the best-of call option?*

Example 3. *Compute the time t-price of the best-of put option, i.e. the option whose payoff at maturity T is*

$$\max(K - \max(S_T^1, S_T^2), 0).$$

6.4. **Parity Relations.** Is there any relationship between the best-of put and best-of call options akin to the put-call parity?

Let us recall how the put-call parity is derived: starting from the relationship

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K,$$

apply $E^{\mathbb{Q}}[e^{-r(T-t)} \bullet |\mathcal{F}_t]$ on both sides to obtain

$$C_t - P_t = S_t - e^{-r(T-t)}K.$$

Let us do the same for the best-of options:

$$(\max(S_T^1, S_T^2) - K)^+ - (K - \max(S_T^1, S_T^2))^+ = \max(S_T^1, S_T^2) - K.$$

We may remove the maximum by using the relationship

$$\max(S_T^1, S_T^2) + \min(S_T^1, S_T^2) = S_T^1 + S_T^2.$$

Since the minimum is now involved, we might as well throw in the relationship

$$(\min(S_T^1, S_T^2) - K)^+ - (K - \min(S_T^1, S_T^2))^+ = \min(S_T^1, S_T^2) - K.$$

Adding, we obtain

$$\begin{aligned}&(\max(S_T^1, S_T^2) - K)^+ - (K - \max(S_T^1, S_T^2))^+\\ &+(\min(S_T^1, S_T^2) - K)^+ - (K - \min(S_T^1, S_T^2))^+\\ &= S_T^1 + S_T^2 - 2K.\end{aligned}$$

Finally, we apply $E^{\mathbb{Q}}[e^{-r(T-t)} \bullet |\mathcal{F}_t]$ to both sides to obtain

$$BC_t - BP_t + WC_t - WP_t = S_t^1 + S_t^2 - 2Ke^{-r(T-t)},$$

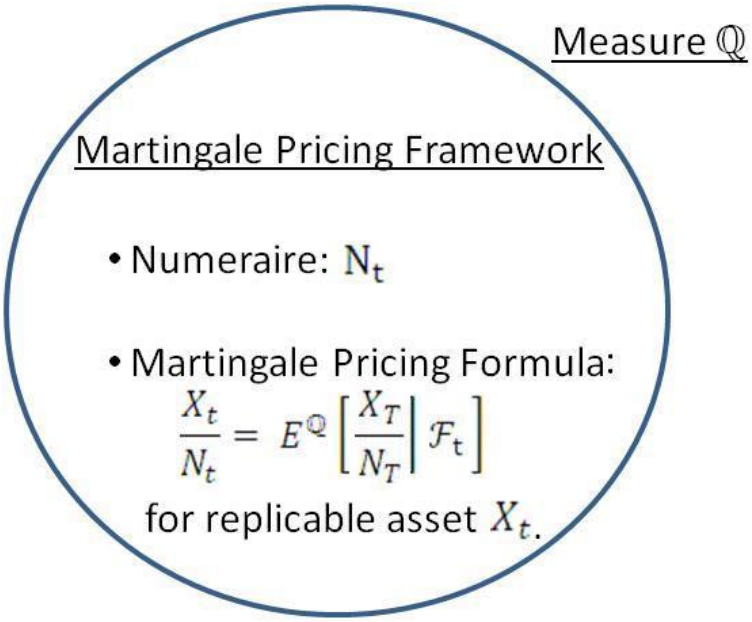
where BC stands for best-of call, BP stands for best-of put, WC stands for worst-of call and WP stands for worst-of put.

6.5. **Summary of the Change-of-Numeraire Technique.** The background setting comprises stochastic processes S_t^1,S_t^2,... representing asset prices. We're interested in the pricing derivatives of these processes.

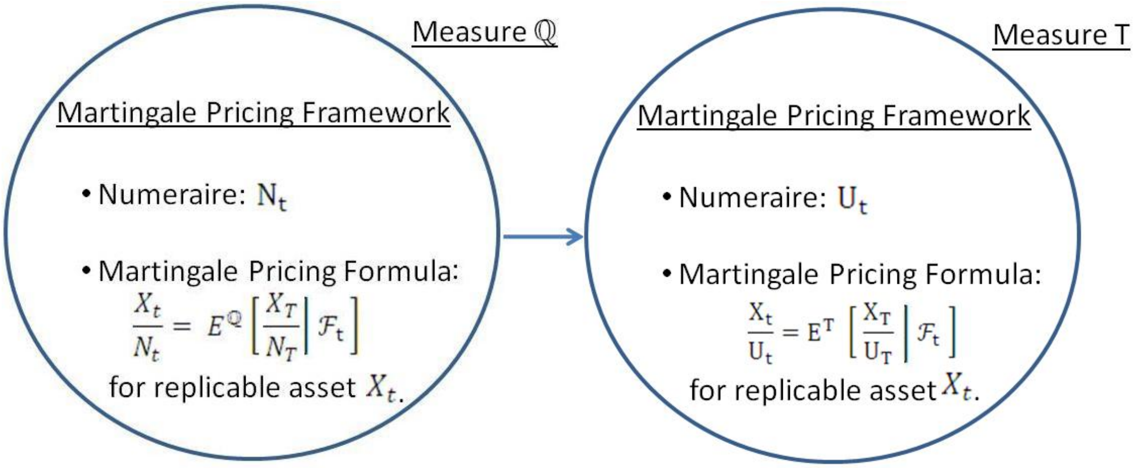
We may think this way: pricing may be carried out

- within the martingale pricing framework
- with respect to a certain probability measure (i.e. a "world")

This is a "world":



"Worlds" are connected by Girsanov's Theorem and Theorem 1:



The pair $(\mathbb{Q}, N_t(t \in [0,T]))$ and the pair $(\mathbb{T}, U_t(t \in [0,T]))$ are connected by:

$$\frac{d\mathbb{T}}{d\mathbb{Q}} = \frac{\frac{U_T}{U_0}}{\frac{N_T}{N_0}}$$

Pricing of replicable assets X_t is consistent across "worlds":

$$X_t = U_t E^{\mathbb{T}}[\frac{X_T}{U_T} \mid \mathcal{F}_t] = N_t E^{\mathbb{Q}}[\frac{X_T}{N_T} \mid \mathcal{F}_t]$$

Computationally, to obtain a formula for X_t. we need to compute either the expectation $E^{\mathbb{T}}[\frac{X_T}{U_T} \mid \mathcal{F}_t]$ or $E^{\mathbb{Q}}[\frac{X_T}{N_T} \mid \mathcal{F}_t]$, which requires us to know about the distribution of $\frac{X_T}{U_T}$ under $\mathbb{T}$, or the distribution of $\frac{X_T}{N_T}$ under $\mathbb{Q}$, respectively.

This in turn is found by analyzing the SDEs satisfied by $\frac{X_t}{U_t}$ and $\frac{X_t}{N_t}$. The noise factors driving the SDEs are respectively $W^{\mathbb{T}}$ and $W^{\mathbb{Q}}$, Brownian with respect to $\mathbb{T}$ and $\mathbb{Q}$ respectively, and connected by Girsanov's Theorem.

The crux of the change-of-numeraire technique is that, by changing numeraire (and hence "worlds") from $\mathbb{Q}$ to $\mathbb{T}$, the term in the expectation operator is simplified, and computation becomes easier. In a sense, the computational complexity gets absorbed by the probability density function as effected by the change in SDEs.