SOME TOPICS AROUND THE LOGNORMAL DISTRIBUTION

Learning Objectives:

- The central concept behind the lecture is the lognormal distribution
- The lognormal distribution is related to the normal distribution by the change-of-variables formula
- The moments of the lognormal distribution are computable
- The gadget called the Moment Generating Function encodes in a power series all the moments of a given distribution. The moments of the normal distribution can be easily recalled this way as its MGF has a very nice form.
- In the limit, the distribution of stock price in the binomial model approaches the lognormal distribution.

1. The Change-of-Variables Formula Applied to Densities

Recall the Change-of-Variables Formula from QF203:

Theorem 1 (The Change-of-Coordinates Formula)**.** *Suppose $D \in \mathbb{R}^2$ is a closed and bounded domain and $F : D \to \mathbb{R}$ is a differentiable transformation of the plane that is bijective outside sets of lower dimensions. Then for any continuous function $f : F^{-1}(D) \to \mathbb{R}$, we have*

$$\int\int_D f(X,Y)\, dA_{XY} = \int\int_{F^{-1}(D)} f(F(x,y))||\frac{\partial(X,Y)}{\partial(x,y)}||\, dA_{xy},$$

where

$$\frac{\partial(X,Y)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{pmatrix}$$

is (called) the Jacobian matrix of F.

In higher dimensions, an analogous theorem holds. In one dimension, the analogous theorem is usually called the substitution method of integration.

We will apply this to compare probability densities in this section.

Let V, W be two random variables with densities (over $\mathbb{R}$) f_V, f_W. Suppose there is a strictly increasing transformation $T : \mathbb{R} \to \mathbb{R}$ such that $V = T(W)$. What is the connection between f_V and f_W?

Let us compute:

$$\begin{aligned} \mathbb{P}(W \leq w) &= \mathbb{P}(T(W) \leq T(w)) \\ &= \mathbb{P}(V \leq T(w)) \\ &= \int_{-\infty}^{T(w)} f_V(t)\, dt. \end{aligned}$$

We also know that

$$\mathbb{P}(W \leq w) = \int_{-\infty}^{w} f_W(t)\, dt.$$

Differentiating these with respect to w, we obtain

$$f_V(T(w))T'(w) = f_W(w).$$

Similarly, in two dimensions, if (V_1, V_2) and (W_1, W_2) are two random vectors (over $\mathbb{R}^2$) with densities f_V and f_W, and if $(V_1, V_2) = T(W_1, W_2)$, what is the connection between the two densities?

Let us compute:

$$\begin{aligned}
\mathbb{P}((W_1, W_2) \in D) &= \mathbb{P}(T(W_1, W_2) \in T(D)) \\
&= \mathbb{P}((V_1, V_2) \in T(D)) \\
&= \int\int_{T(D)} f_V(X, Y)\, dA_{XY} \\
&= \int\int_D f_V(T(x, y))||\frac{\partial(X, Y)}{\partial(x, y)}||\, dA_{xy}.
\end{aligned}$$

We also know that

$$\mathbb{P}((W_1, W_2) \in D) = \int\int_D f_W(x, y)\, dA_{xy}.$$

Hence, we obtain

$$f_W(x, y) = f_V(T(x, y))||\frac{\partial(X, Y)}{\partial(x, y)}||.$$

Example 1. *Suppose that $X \sim N(\mu, \sigma^2)$. Normalization means defining $Z = \frac{X-\mu}{\sigma} =: T(X)$. The purpose of normalization is that $Z \sim N(0, 1)$.*

Since the density of a standard normal is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

we have

$$\begin{aligned}
f_X(x) &= f_Z(T(x))T'(x) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}} \cdot \frac{1}{\sigma} \\
&= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.
\end{aligned}$$

The method of argument above is related to a simulation method as follows:

Example 2. *Suppose X has a CDF F. How should we simulate draws of X from the given distribution?*

Suppose we can simulate the uniform distribution $U[0,1]$, and suppose $U \sim U[0,1]$. Then we claim that $X = F^{-1}(U)$ meets the requirement.

To see this, let us compute:

$$\mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U) \leq x)$$

$$= \mathbb{P}(U \leq F(x)) = F(x).$$

This implies that this definition of X indeed follows the distribution implied by F.

Here is another sampling method that makes use of a known distribution. It is called the rejection method and was found by the famous mathematician von Neumann (the man who conceptualized the modern computer).

Example 3. *Suppose that U is uniformly distributed over $[0,1]$, X follows a probability distribution over $\mathbb{R}$ given by the PDF $g(x)$, and X and U are independent. Let $f(x)$ be another PDF over $\mathbb{R}$ and let M be a positive number for which the following inequality holds:*

$$\frac{f(x)}{Mg(x)} \leq 1$$

for all $x \in \mathbb{R}$. Consider the following procedure:

- *Choose a sample U and a sample X.*
- *If $U < \frac{f(X)}{Mg(X)}$, then accept X; otherwise reject X.*

Show that the accepted samples X follow the distribution associated to f by answering the following questions:

(1) *Show that*

$$\mathbb{P}(U < \frac{f(X)}{Mg(X)}) = \frac{1}{M}.$$

(2) *Let I be the random variable defined by*

$$I := \min\{k \geq 1 : U_k < \frac{f(X_k)}{Mg(X_k)}\},$$

where $(U_1, X_1), (U_2, X_2), \ldots$ are independent sampled pairs U and X. Show that I follows the geometric distribution $G(p)$, where $p = \frac{1}{M}$.

(3) *Show that the distribution of X_I is given by the PDF f.*

Solution. (1) Conditional on X, $\mathbb{P}(U < \frac{f(X)}{Mg(X)}) = \frac{f(X)}{Mg(X)}$. Now, take the distribution of X into account, we have

$$\begin{aligned}\mathbb{P}(U < \frac{f(X)}{Mg(X)}) &= \int_{\mathbb{R}} \frac{f(x)}{Mg(x)} \cdot g(x)\, dx \\ &= \frac{1}{M}\int_{\mathbb{R}} f(x)\, dx = \frac{1}{M}.\end{aligned}$$

(2) 'Success' is the event: $U < \frac{f(X)}{Mg(X)}$. And the probability of this has been shown to be $\frac{1}{M}$. Hence the distribution of I is geometric with parameter $p = \frac{1}{M}$.

(3) We have

$$\begin{aligned}\mathbb{P}(X_I \leq y) &= \sum_{k=1}^{\infty} \mathbb{P}(X_k \leq y, I = k) \\ &= \sum_{k=1}^{\infty} \int_{-\infty}^{y} \mathbb{P}(I = k | X_k = x) g(x)\, dx \\ &= \sum_{k=1}^{\infty} \int_{-\infty}^{y} (1-p)^{k-1} \mathbb{P}(U < \frac{f(x)}{Mg(x)}) g(x)\, dx \\ &= \sum_{k=1}^{\infty} (1-p)^{k-1} \int_{-\infty}^{y} \frac{f(x)}{Mg(x)} g(x)\, dx \\ &= \frac{1}{1-(1-p)} \cdot \frac{1}{M} \int_{-\infty}^{y} f(x)\, dx = \int_{-\infty}^{y} f(x)\, dx.\end{aligned}$$

□

Here is another sampling method whose validity is established with the change-of-variables formula in dimension two. It is called the Box-Muller method.

Example 4. *Suppose $U_1, U_2 \sim U[0,1]$ are uniformly distributed and independent, and define*

$$X_1 = \sqrt{-2\log U_1}\cos 2\pi U_2,$$
$$X_2 = \sqrt{-2\log U_1}\sin 2\pi U_2.$$

Show that X_1, X_2 are iid standard normals.

Solution. By the change-of-variables formula,

$$f_U(u_1, u_2) = f_X(T(u_1, u_2))||\frac{\partial(x_1, x_2)}{\partial(u_1, u_2)}||$$
$$= f_X(x_1, x_2)||\frac{\partial(x_1, x_2)}{\partial(u_1, u_2)}||.$$

The Jacobian matrix has components:

$$\frac{\partial x_1}{\partial u_1} = -\frac{1}{2u_1} \cdot \sqrt{\frac{-2}{\log u_1}}\cos 2\pi u_2,$$
$$\frac{\partial x_1}{\partial u_2} = -2\pi\sqrt{-2\log u_1}\sin 2\pi u_2,$$
$$\frac{\partial x_2}{\partial u_1} = -\frac{1}{2u_1} \cdot \sqrt{\frac{-2}{\log u_1}}\sin 2\pi u_2,$$
$$\frac{\partial x_2}{\partial u_2} = 2\pi\sqrt{-2\log u_1}\cos 2\pi u_2.$$

The absolute value of the determinant of the Jacobian is therefore $\frac{2\pi}{u_1}$.

From the defining equations,

$$x_1^2 + x_2^2 = -2\log u_1,$$

which implies that

$$f_X(x_1, x_2) = \frac{u_1}{2\pi}$$
$$= \frac{1}{2\pi}e^{-\frac{1}{2}(x_1^2+x_2^2)} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x_1^2} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x_2^2}.$$

Hence, X_1, X_2 are iid standard normals. □

2. The Lognormal Distribution

We say that a random variable X has a lognormal distribution $\log N(\mu, \sigma^2)$ if the random variable $Y = \log X \sim N(\mu, \sigma^2)$ is normally distributed.

The lognormal distribution is an important distribution in QF because the Black-Scholes Theory assumes geometric Brownian motion for the underlying stock, which leads to lognormal distribution of stock prices at any fixed time of observation.

First, let us write down the density f_X of X:

$$f_X(x) = f_Y(\log x)\frac{1}{x} = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma}(\log x - \mu)^2}.$$

Let us find the basic parameters of the lognormal distribution. These are:

- the mean μ_1
- the standard deviation σ
- the skewness μ_3/σ^3 - what does positive skew look like typically?
- the excess kurtosis $\mu_4/\sigma^4 - 3$ - what does 'excess' refer to?

These parameters affect the shape of the distribution in visible ways, and are all defined from the moments $\mu_n = E[X^n]$. But this is just a rough guide more than anything. Two distributions can have equal first four moments but look rather different in shape.

Let us now compute the moments of the lognormal distribution. We have

$$\begin{aligned} E[X] &= \int_0^\infty x \cdot \frac{1}{\sigma x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right) dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right) dx \end{aligned}$$

Let $y = \frac{\log x - \mu}{\sigma}$ Then $dy = \frac{1}{\sigma x}dx$ or $dx = \sigma x dy = \sigma e^{\sigma y + \mu} dy$.

And we have

$$
\begin{aligned}
E[X] &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}y^2\right) \sigma e^{\sigma y+\mu}\, dy \\
&= \frac{e^{\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2+\sigma y}\, dy \\
&= \frac{e^{\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2-2\sigma y+\sigma^2)+\frac{1}{2}\sigma^2}\, dy \\
&= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-\sigma)^2}\, dy
\end{aligned}
$$

Let $z = y - \sigma$. Then $dz = dy$ and

$$
\begin{aligned}
E[X] &= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz \\
&= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{2\pi}} \times \sqrt{2\pi} = e^{\mu+\frac{\sigma^2}{2}}.
\end{aligned}
$$

The second moment is

$$
\begin{aligned}
E[X^2] &= \int_0^{\infty} x^2 \cdot \frac{1}{\sigma x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right) dx \\
&= \int_0^{\infty} \frac{x}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right) dx \\
&= \int_{-\infty}^{\infty} \frac{e^{\sigma y+\mu}}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) \cdot \sigma e^{\sigma y+\mu}\, dy \\
&= \frac{e^{2\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2+2\sigma y}\, dy \\
&= \frac{e^{2\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2-4\sigma y+4\sigma^2)+2\sigma^2}\, dy \\
&= \frac{e^{2\mu+2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-2\sigma)^2}\, dy.
\end{aligned}
$$

Let $w = y - 2\sigma$. Then $dw = dy$ and

$$E[X^2] = \frac{e^{2\mu+2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2}\, dw$$
$$= \frac{e^{2\mu+2\sigma^2}}{\sqrt{2\pi}} \times \sqrt{2\pi} = e^{2\mu+2\sigma^2}.$$

Hence,

$$Var(X) = E[X^2] - (E[X])^2$$
$$= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$$
$$= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

In fact, the following proposition holds:

Proposition 1. *The n-th moment of the lognormal distribution is given by*

$$E[X^n] = e^{n\mu+n^2\sigma^2/2}.$$

Example 5. *Prove the proposition!*

3. Lognormal Distribution and QF

The lognormal distribution arises in QF from the Geometric Brownian Motion:

$$\frac{dS_t}{S_t} = \mu d_t + \sigma dW_t$$

You will later see that the solution to this at time T is given by

$$\begin{aligned} S_T &= S_0 \; e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} \\ &= (S_0 e^{(\mu - \frac{1}{2}\sigma^2)T}) e^{\sigma W_T} \end{aligned}$$

By the basic properties of the Brownian Motion,

$$W_T \sim N(0, T),$$

hence $e^{\sigma W_T}$ is lognormal (i.e. its log is normal.).

Let's take a look at the shapes of the lognormal density function:

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma}(\log x - \mu)^2} \quad (x > 0)$$

In R, use the command: dlnorm (x,μ,σ)
Do this, for example:

$$\begin{aligned} &x = 1 : 1000/100 \\ &y = dlnorm(x, 0, 1) \\ &plot(x, y) \\ &y = dlnorm(x, 1, 1) \\ &points(x, y) \end{aligned}$$

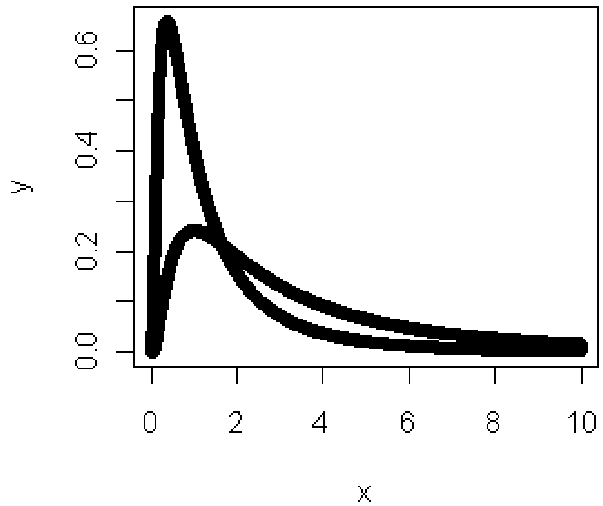

FIGURE 1. (μ,σ)=(0,1)vs(1,1)

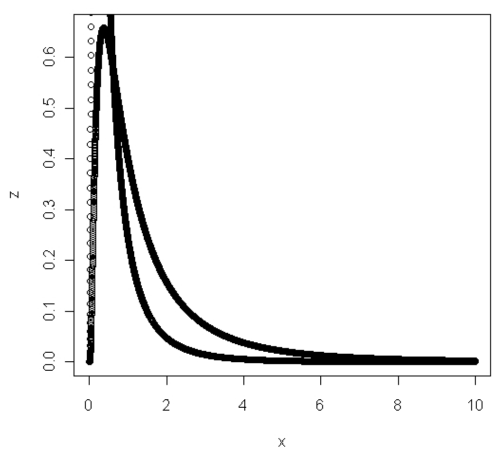

FIGURE 2. (μ,σ)=(0,1)vs(-1,1)

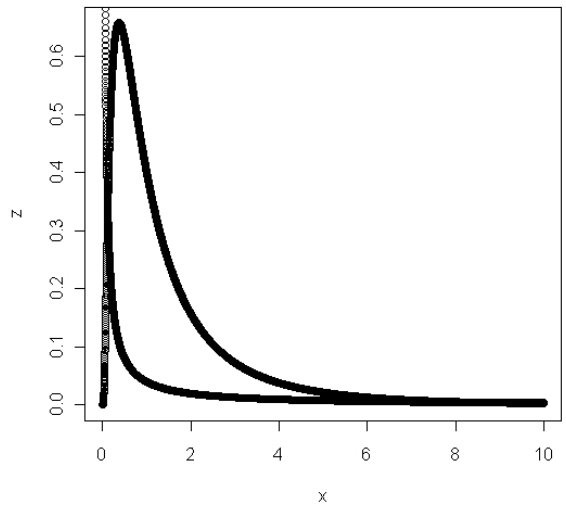

FIGURE 3. (μ,σ)=(0,1)vs(0,10)

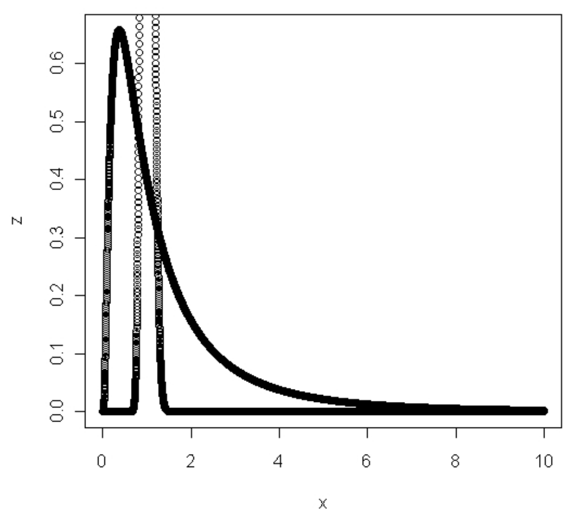

FIGURE 4. (μ,σ)=(0,1)vs(0,0.1)

Let's check how close/far the lognormal assumption is from reality.

Consider daily stock prices:

$$\ldots, S_{(i-1)\delta t}, S_{i\delta t}, S_{(i+1)\delta t}, \ldots$$

and their continuous returns:

$$\ldots, \log \frac{S_{i\delta t}}{S_{(i-1)\delta t}}, \log \frac{S_{(i+1)\delta t}}{S_{i\delta t}}, \ldots$$

If the lognormal assumptions holds,then

$$d \log S_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

-This is a consequence of the Ito's Lemma which will be explained later.

This implies:

$$\log S_{(i+1)\sigma t} - \log S_{i\sigma t} \simeq (\mu - \frac{1}{2}\sigma^2)\delta t + \sigma(W_{(i+1)\delta t} - W_{i\delta t})$$

i.e. approximately

$$\log \frac{S_{(i+1)\delta t}}{S_{i\delta t}} \sim N((\mu - \frac{1}{2}\sigma^2)\delta t, \sigma^2 \delta t)$$

We'll compare the distribution of the daily returns of the S&P500 index over the period 2-Jan-2008 to 6-Mar-2009 with the best-fit normal distribution.

R Code:

```
data1 < −read.table(”C : /users/tanchonghui
/Documents/Work/temp/S&P500.csv”,header=T, sep=”,”)
x = data1$Date
y = data1$Close
length(y)
# length  is  14889
y1 = y[:14888]
y2 = y[2:14889]
y3 = y1/y2
y4 = log(y3)
x[1]
# Start  date  is  6  Mar  2009
x[297]
# Last  date  is  2  Jan  2008
z = y4[1:297]
hist(z)
# observe  there  are  10  columns  between  -0.1  to  0.1
# each  mark  is  .2 / 10 = .02
# total area is thus:
a = .02 * length(z)
xx = (-1000:1000)/10000
yy = dnorm(xx,m,sd)
# re-scale the vertical direction
yy = yy * a
points(xx,yy)
```

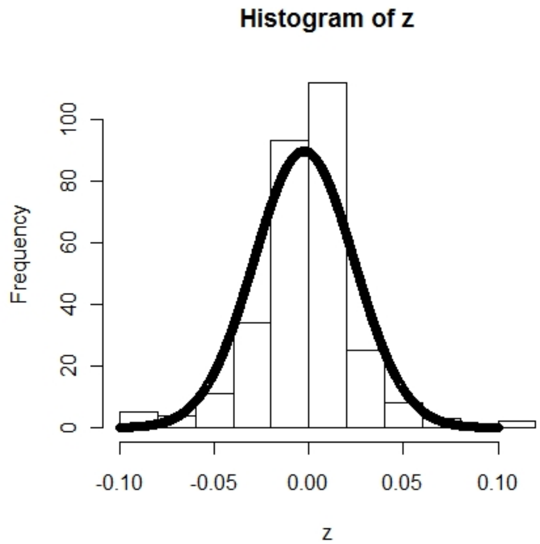

Figure 5. (μ,σ)=(0,1)vs(0,0.1)

4. Other Distributions in QF

To improve the fit between the theoretical distribution and the true distribution of stock price returns, various models have been proposed in QF, the simplest (and in a sense the most important) ones are:

1) Jump diffusion model:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t + dJ_t$$

2) Heston Stochastic Volatility Model:

$$\frac{dS_t}{S_t} = rdt + \sqrt{V_t} dW_t^S$$

$$dV_t = \kappa(\theta - V_t)dt + \xi \sqrt{V_t} dW_t^V$$

The terms W_t, W_t^S,W_t^V are Brownian motions. The processes W_t^S and W_t^V are correlated:

$$dW_t^S dW_t^V = \rho dt$$

(To be interpreted in either expected-value sense or diminishing-standard-deviation sense.)

The term J_t is a pure jump process with Poisson arrivals and normal distributed jump sizes. In the following, let us consider the distribution of $\log(\frac{S_T}{S_0})$ for the two models above.

Jump Diffusion Model

In Lecture 11, we will see that

$$log(\frac{S_T}{S_0}) = (r - \frac{1}{2}\sigma^2)T + \sigma W_T + J_T$$

Let us compute the CDF, namely

$$\mathbb{P}(\log(\frac{S_T}{S_0}) \leq \alpha)$$

This is equal to

$$\mathbb{P}(\sigma W_T + J_T \le \alpha - (r - \frac{1}{2}\sigma^2)T)$$
$$= \sum_{k=0}^{\infty} \mathbb{P}(\sigma W_T + (Z_1 + Z_2 + \ldots + Z_k) \le \alpha - (r - \frac{1}{2}\sigma^2)T) \times \mathbb{P}(N_T = k)$$

If we assume that J_t is independent of W_t,and where N_T is a Poisson process with parameter λT. (λ is the per-unit-time rate).

Since $\sigma W_T + Z_1 + \ldots + Z_k \sim N(0, \sigma^2 T + k)$ the above computation continues.

$$= \sum_{k=0}^{\infty} N(\frac{\alpha - (r - \frac{1}{2}\sigma^2)T - 0}{\sqrt{\sigma^2 T + k}}) e^{-\lambda T} \frac{(\lambda T)^k}{k!}$$

Hence, the PDF is

$$f(\alpha) = \sum_{k=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^k}{k!} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\frac{1}{2}(\alpha - (r - \frac{1}{2}\sigma^2)T)^2}{\sigma^2 T + k}} \cdot \frac{1}{\sqrt{\sigma^2 T + k}}$$

Heston Stochastic Volatility Model

Analytical results concerning the Heston stochastic volatility model are much harder to obtain. For instance, one such result is

$$\mathbb{P}(V_T < \alpha \mid V_t) = F_{\chi'^2}(\frac{\alpha n(t,T)}{e^{-\kappa(T-t)}}; d, V_t \cdot n(t,T)),$$

where

$$d = \frac{4\kappa\theta}{\epsilon^2}$$
$$n(t,T) = \frac{4\kappa e^{-\kappa(T-t)}}{\xi^2(1 - e^{-\kappa(T-t)})}$$

The Heston stochastic volatility model is a popular model because it is a model that improves upon the lognormal distribution (really, by having an additional parameter that admits an intuitive interpretation) and yet remains simple enough for option prices to be found by Fourier transform techniques.

5. Link between the Binomial Model and the Lognormal Distribution

Let the time horizon be $[0, T]$ and let it be discretized in to n equal intervals, each of length $h := T/n$. Let r be the annualized risk-free rate. For time of length h, we have to compound by $1 + r_h := (1+r)^h$.

Let us compute $E[\log(S_T/S_0)]$ and $Var[\log(S_T/S_0)]$:

$$\begin{aligned} E[\log(S_T/S_0)] &= \sum_{i=1}^{n} E[\log(S_{ih}/S_{(i-1)h})] \\ &= n(p\log u + q\log d), \\ Var[\log(S_T/S_0)] &= \sum_{i=1}^{n} Var[\log(S_{ih}/S_{(i-1)h})] \\ &= n\big((p(\log u)^2 + q(\log d)^2) - (p\log u + q\log d)^2\big). \end{aligned}$$

We want to investigate the limiting behaviour of the distribution of the stock price in the binomial model as $n \to \infty$. Thus, it make sense to impose the conditions:

$$\begin{aligned} E[\log(S_T/S_0)] &= \mu T, \\ Var[\log(S_T/S_0)] &= \sigma^2 T. \end{aligned}$$

Combining with the above, there are the following constraints on the parameters u, d, p, q of the binomial model:

$$\begin{aligned} p + q &= 1, \\ n(p\log u + q\log d) &= \mu T, \\ n\big((p(\log u)^2 + q(\log d)^2) - (p\log u + q\log d)^2\big) &= \sigma^2 T. \end{aligned}$$

Set $x = \log u$, $y = \log v$ and $p = \frac{1}{2}$. Then the above equations reduce to

$$\begin{aligned} x + y &= \frac{2\mu T}{n}, \\ x^2 + y^2 &= \frac{2\sigma^2 T}{n} - \frac{2\mu^2 T^2}{n^2}. \end{aligned}$$

Note that when $n >> 0$, then the RHS of the second equation is positive.

Equivalently, we have

$$x + y = \frac{2\mu T}{n},$$

$$xy = 2\Big(\frac{2\mu^2 T^2}{n^2} - \frac{\sigma^2 T}{n}\Big).$$

Exercise 1. *Solve the above.*

What is the limiting distribution of $\log(S_T/S_0)$ as $n \to \infty$? By writing

$$\log(S_T/S_0) = \sum_{i=1}^{n} \log(S_{ih}/S_{(i-1)h}),$$

and noticing that the stock price process is Markovian, we know that the terms $X_i := \log(S_{ih}/S_{(i-1)h})$ are iid and each has the Bernoulli distribution

$$\mathbb{P}(X_i = \log u) = p, \ \ \mathbb{P}(X_i = \log d) = q,$$

with mean $\mu T/n$ and variance $\sigma^2 T/n$ as we have constrained above, we then write it in the form

$$\frac{X_1 + X_2 + \cdots + X_n - \mu T}{\sigma\sqrt{T}}$$

for the application of the Central Limit Theorem.

A remark: the vanilla version of the CLT has the random variables $X_1, X_2, \ldots$ iid regardless of the index n. In the present scenario, the distributions of these Bernoullis change as n changes. It turns out fortunately that the constraints that the first two moments have limits are enough to enable CLT to be suitably extended to hold in this scenario.

By the Central Limit Theorem, we have

$$\frac{X_1 + X_2 + \cdots + X_n - \mu T}{\sigma\sqrt{T}} \to N(0,1)$$

in distributions. In other words,

$$\frac{\log(S_T/S_0) - \mu T}{\sigma\sqrt{T}} \to Z,$$

where $Z \sim N(0,1)$. Put it another way,

$$\log(S_T) \sim N(\log S_0 + \mu T, \sigma^2 T),$$

or S_T is lognormal, in the limit.

Appendix A. Moments and Cumulants

Moment Generating Function

All moments of a distribution are encoded in a single object known as the moment generating function $E_X(z) := E[e^{zX}]$. Notice that

$$\begin{aligned} E(z) &= E\Big[\sum_{m=0}^{\infty} \frac{1}{m!}(zX)^m\Big] \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} z^m E[X^m], \end{aligned}$$

so that

$$E^{(n)}(0) = E[X^n].$$

Hence, if $E(z)$ can be found in a nice closed form, then the moments are found by differentiation. While the normal distribution has a nice MGF, the lognormal distribution does not, even though its moments are quite computable.

Example 6 (Standard Normal Distribution)**.** *We compute:*

$$\begin{aligned} E(z) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{zx-\frac{1}{2}x^2}\, dx \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2-2zx+z^2)} e^{z^2/2}\, dx \\ &= e^{z^2/2} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-z)^2}\, dx \\ &= e^{z^2/2}. \end{aligned}$$

Hence, it is clear that all the odd moments are zero, and

$$E[Z^{2n}] = E^{(2n)}(0) = \frac{(2n)!}{n!2^n}.$$

Example 7. *Find the MGF of the normal distribution $N(\mu, \sigma^2)$.*

Example 8. *A fair coin is tossed until a tail appears. Let X denote the number of heads required. Find:*

(1) *the density function of X*
(2) *the mean and variance of X*
(3) *the MGF of X*

Example 9. *If X, Y are independent random variables, then the MGF E_{X+Y} of $X + Y$ is equal to the product $E_X E_Y$ of the MGFs of X and Y.*

The Moment Problem

If one is given a sequence of numbers $\mu_1,\mu_2,\mu_3,\ldots$, does there exist a distribution F whose moments are $\mu_1,\mu_2,\mu_3,\ldots$.

If two distributions F and G have the same moments $\mu_1,\mu_2,\mu_3,\ldots$,does it follow that F=G?

The common answers to these questions is: it depends. Check out the internet for more information.

Cumulant Generating Function

A notion related to the MGF is the cumulant generating function (CGF): given a random variable X, its cumulant generating function $L(z)$ is defined as

$$L(z) = \log E(z).$$

When written out in full,

$$L(z) = \sum_{n=1}^{\infty} \kappa_n \frac{z^n}{n!},$$

and the coefficients κ_n's are called the cumulants of X.

Let's try to relate cumulants and moments. Since $E(z) = e^{L(z)}$, we have

$$\sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!} = \exp\big(\sum_{n=1}^{\infty} \kappa_n \frac{z^n}{n!}\big).$$

Expanding the RHS, we have

$$\sum_{m=0}^{\infty} \frac{1}{m!}\Big(\sum_{n=1}^{\infty} \kappa_n \frac{z^n}{n!}\Big)^m$$
$$= 1 + \kappa_1 z + \frac{1}{2}(\kappa_2 + \kappa_1^2)z^2 + \cdots .$$

Hence,

$$\mu_1 = \kappa_1$$
$$\mu_2 = \kappa_2 + \kappa_1^2,$$

which implies that the first cumulant is the mean and the second cumulant is

$$\kappa_2 = \mu_2 - \kappa_1^2 = \mu_2 - \mu_1^2,$$

the variance.

In general, the expressions above show that the moments and the cumulants determine one another.

Example 10. *Find the third and fourth cumulants in terms of the first four moments.*

Example 11. *If X, Y are independent random variables, then the CGF L_{X+Y} of $X+Y$ is equal to the sum $L_X + L_Y$ of the CGFs of X and Y.*

Example 12. *Find all the cumulants of the standard normal distribution.*

Solution. The MGF of the standard normal is $E(z) = e^{z^2}$. Hence its CGF is $L(z) = z^2$. This says that apart from $\kappa_2 = 1$, all the other cumulants vanish. □

Example 13. *Find all the cumulants of the normal distribution* $N(\mu, \sigma^2)$.

Example 14. *Find all the cumulants of the lognormal distribution* $\log N(\mu, \sigma^2)$.