EXTENSIONS OF THE BLACK-SCHOLES MODEL

Learning Objectives:

- Three extensions are considered:
  - With transaction costs
  - With time-dependent parameters
  - With jumps in the underlying process
- In addition, we look at the simplest interest rate model - the Vasicek Model

1. Black-Scholes with Transaction Costs

The following argument was put forth by H. E. Leland (Option pricing and replication with transaction costs, J. Finance, 40, 1283-1301, 1985).

Consider a market in which a security is traded with a bid-ask spread $S_{t,ask} - S_{t,bid} = kS_t$. Here S_t represents the midpoint between the bid and ask prices:

$$S_{t,bid} = S_t(1 - \frac{k}{2}),\ S_{t,ask} = S_t(1 - \frac{k}{2})$$

and k is a constant percentage. We assume that the security price satisfies

$$S_t = S_0 e^{\mu + \sigma Z(t)},$$

where $t \in [0, T]$, $Z(t)$ is Brownian motion, and $r = \mu + \frac{1}{2}\sigma^2$.

We want to construct replicating hedging strategies to replicate contingent claims with payoff $f(S_T)$. We form a portfolio $V_t = B_t + \Delta_t S_t$ of cash B_t in risk-free bank and Δ_t shares of stock, and dynamically adjust it in a self-financing manner.

The time interval between successive hedges is assumed to be fixed at δt.

The change in the portfolio value over $[t, t + \delta t]$ is

$$\begin{aligned} V_{t+\delta t} - V_t &= \Delta_t(S_{t+\delta t} - S_t) + (B_t e^{r\delta t} - B_t) - (k/2)S_t|\Delta_{t+\delta t} - \Delta_t| \\ &\approx \Delta_t \delta S_t + rB_t\delta t - (k/2)S_t|\delta\Delta_t|, \end{aligned}$$

where the third term comes from transaction cost and it is this that is postulated by Leland's model.

Assuming that $V_t = V(S_t, t)$, we expand using Ito's Lemma:

$$\delta V = V_S \delta S_t + (V_t + \frac{1}{2}\sigma^2 V_{SS})\delta t,$$

and we equate the above like this:

$$\Delta_t = V_S$$

and

$$rB_t\delta t - \frac{k}{2}S_t|\delta\Delta_t| = (V_t + \frac{1}{2}\sigma^2 V_{SS})\delta t.$$

Now apply Ito's Lemma to $\Delta = V_S$:

$$\delta\Delta_t = V_{SS}\delta S_t + \text{terms of order } \delta t \text{ or smaller.}$$

Hence,

$$\begin{aligned}\frac{k}{2}S_t|\delta\Delta_t| &\approx \frac{k}{2}|V_{SS}||\delta S_t| \\ &\approx \frac{1}{2}k\sigma S_t^2|V_{SS}||\delta Z(t)| \\ &= \frac{1}{2}(\frac{k}{\sigma\sqrt{\delta t}})\sigma^2 S_t^2|V_{SS}||\delta Z(t)|\sqrt{\delta t}.\end{aligned}$$

It is known that $E[|\delta Z(t)|] = \sqrt{\frac{2}{\pi}} \times \sqrt{\delta t}$. Thus, we approximate the above by

$$\frac{\sigma^2 A}{2}S_t^2|V_{SS}|\delta t,$$

where

$$A = \sqrt{\frac{2}{\pi}}(\frac{k}{\sigma\sqrt{\delta t}}).$$

This number A is called the Leland number.

Substituting this and $B_t = V_t - \Delta_t S_t$ back into

$$rB_t\delta t - \frac{k}{2}S_t|\delta\Delta_t| = (V_t + \frac{1}{2}\sigma^2 V_{SS})\delta t,$$

we obtain the PDE

$$V_t + \frac{1}{2}\tilde{\sigma}^2 S^2 V_{SS} + r(SV_S - V) = 0,$$

for $S \in (0,\infty)$, $t < T$, where $\tilde{\sigma}^2 = \sigma^2(1 + A\text{sign}(V_{SS})$, together with the final condition $V(S,T) = f(S)$.

The PDE above is nonlinear unless V_{SS} has the same sign everywhere (i.e. V is everywhere concave or everywhere convex). The Leland number crucially measures the influence of transaction costs in the model. A large Leland number corresponds to either a large bid-ask spread or a small interval between trades.

The analysis of the above non-linear PDE has been undertaken by various authors (Leland, Hoggard-Whalley-Wilmott, Avellaneda-Paras).

2. Time Dependent Parameters

The assumption that the underlying stock price follows the GBM

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

is rather restrictive. Also, the risk-free rate r being a constant may not be a good reflection of reality. In this section, we assume that the volatilty and the risk-free rate are deterministic functions of time, and show that the Black-Scholes formalism leads to analogous formulas for option prices.

Assume that the stock price follows the following SDE:

$$dS_t = \mu(t) S_t dt + \sigma(t) S_t dW_t$$

in the risk-free world with probability measure $\mathbb{P}$, and the risk-free rate is $r = r(t)$.

We seek a measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ and such that the discounted stock price process $e^{-\int_t^T r(s)\,ds} S_t$ is a martingale.

We may rewrite the SDE as:

$$dS_t = r(t) S_t dt + \sigma(t) S_t dW_t + \frac{\mu(t) - r(t)}{\sigma(t)} dt.$$

We would like to define

$$d\tilde{W}_t := dW_t + \frac{\mu(t) - r(t)}{\sigma(t)} dt$$

and have a probability measure with respect to which $\tilde{W}_t$ is Brownian.

This is done by means of the Girsanov Theorem. The process

$$\tilde{W}_t := W_t + \int_0^t \frac{\mu(s) - r(s)}{\sigma(s)}\, ds$$

is Brownian with respect to the measure $\mathbb{Q}$, which is defined by

$$\mathbb{Q}(A) = E^{\mathbb{P}}[\chi_A Z_T(X)],$$

where $Z_t(X)$ is the Doléans-Dade exponential

$$Z_t(X) = \exp\Big(\int_0^t X_s\, dW_s - \frac{1}{2}\int_0^t X_s^2\, ds\Big)$$

and

$$X(t) = -\frac{\mu(t) - r(t)}{\sigma(t)}.$$

Note that we need to assume that $\sigma(t)$ is bounded away from 0 to avoid technical problems from arising.

Under the martingale pricing formalism, the price of the contingent claim X which pays off at time T, at time t, is given by

$$E^{\mathbb{Q}}[e^{-\int_t^T r(s)\,ds} X|\mathcal{F}_t].$$

For the call option, we have

$$C_t = E^{\mathbb{Q}}[e^{-\int_t^T r(s)\,ds} \max(S_T - K, 0)|\mathcal{F}_t].$$

For the put option, we have

$$P_t = E^{\mathbb{Q}}[e^{-\int_t^T r(s)\,ds} \max(K - S_T, 0)|\mathcal{F}_t].$$

Formulas may be found for these prices, in analogy to the case of constant parameters. Thus, for instance, we may write

$$\begin{aligned} C_t &= E^{\mathbb{Q}}[e^{-\int_t^T r(s)\,ds} \max(S_T - K, 0)|\mathcal{F}_t] \\ &= e^{-\int_t^T r(s)\,ds} E^{\mathbb{Q}}[S_T \chi_A|\mathcal{F}_t] - K e^{-\int_t^T r(s)\,ds} E^{\mathbb{Q}}[\chi_A|\mathcal{F}_t], \end{aligned}$$

where A is the set of samples for which $S_T \geq K$.

Let us compute the two terms $E^{\mathbb{Q}}[S_T \chi_A|\mathcal{F}_t]$ and $E^{\mathbb{Q}}[\chi_A|\mathcal{F}_t]$.

First of all, we need to identify the distribution of S_T. From

$$dS(t)/S(t) = r(t)dt + \sigma(t)d\tilde{W}(t),$$

we obtain

$$d\log S(t) = (r(t) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)d\tilde{W}(t),$$

or

$$S(T) = S(0)\exp\Big(\int_0^T (r(t) - \frac{1}{2}\sigma^2(t))\,dt + \int_0^T \sigma(t)\,dW(t)\Big).$$

The random variable $\int_0^T \sigma(t)\,dW(t)$ can be made explicit though a limit computation like this:

$$\sum_{i=1}^n \sigma(t_{i-1})(W(t_i) - W(t_{i-1}))$$

is a sum of independent normals, the i-th term has mean 0 and variance $\sigma(t_{i-1})^2 \frac{T}{n}$. Thus, in the limit, the stochastic integral is a normal with mean 0 and variance

$$\int_0^T \sigma^2(t)\, dt.$$

From the explicit form:

$$S(T) = S(0) \exp\big(\int_0^T r(t)\, dt - \frac{1}{2} \int_0^T \sigma^2(t)\, dt + \sqrt{\int_0^T \sigma^2(t)\, dt Z}\big),$$

we see that the Black-Scholes formula applies with r replaced by $\int_0^T r(t)\, dt$ and σ^2 replaced by $\int_0^T \sigma^2(t)\, dt$.

3. The Merton Jump Diffusion Model

The Merton Model assume that in the risk-neutral world (i.e. with respect to the risk-neutral measure $\mathbb{Q}$), the stock price satisfies

$$dS_t/S_{t-} = rdt + \sigma d\tilde{W}_t + dJ_t,$$

where J_t is a cumulative jump process, in which the occurrences of jumps is a Poisson process with intensity λ and the size of each jump is standard normal and independent from each other. We also assume that all the noise factors are independent of each other. We assume $J_0 = 0$.

To further elaborate and clarify: $t \mapsto J_t$ is a right-continuous and piecewise constant process. Let N_t be the corresponding counting process. Then to say that it is Poisson with intensity λ is to say that $\mathbb{Q}(N_{t+\delta t} - N_t = 1) \approx \lambda \delta t$, when δt is small, and $\mathbb{Q}(N_{t+\delta t} - N_t \geq 2) = o(dt)$. From your basic probability course, you learn that

$$\mathbb{Q}(N_T = n) = \frac{(\lambda T)^n}{n!} e^{-\lambda T}.$$

Let us compute the price of an European call option under this model. The price is given by

$$C_0 = e^{-rT} E^{\mathbb{Q}}[(S_T - K)^+].$$

To do that, we need to make explicit the distribution of S_T: rewrite the SDE in the log form:

$$d \log S_t = (r - \frac{1}{2}\sigma^2)dt + \sigma d\tilde{W}_t + dJ_t,$$

so that

$$S_T = S_0 \exp\big((r - \frac{1}{2}\sigma^2)T + \sigma \tilde{W}_T + J_T\big).$$

As usual $W_T \sim N(0, T)$. The random variable has N_T jumps, of sizes $X_1, X_2, \ldots, X_{N_T}$, each a standard normal, and independent from each other.

We compute the expectation by conditioning upon the number of jumps N_T:

$$C_0 = \sum_{n=0}^{\infty} e^{-rT} E^{\mathbb{Q}}[(S_T - K)^+ | N_T = n] \frac{(\lambda T)^n}{n!} e^{-\lambda T}.$$

Let's compute each term $e^{-rT}E^{\mathbb{Q}}[(S_T - K)^+ | N_T = n]$.

The sum of n independent standard normals is normal with mean 0 and variance n. Hence

$$\sigma W_T + J_T \sim N(0, n + \sigma^2 T),$$

when conditioned upon $N_T = n$.

Hence,

$$e^{-rT}E^{\mathbb{Q}}[(S_T - K)^+ | N_T = n] = S_0 e^{\frac{n}{2}} N(d_1(n)) - K e^{-rT} N(d_2(n)),$$

where

$$d_1(n) = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T + n}{\sqrt{\sigma^2 T + n}},$$

and

$$d_2(n) = \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2 T + n}}.$$

Note that $d_1(n) - d_2(n) = \sqrt{\sigma^2 T + n}$.

Example 1. *Prove the formula above.*

4. The Vasicek Model

The Vasicek Model was proposed by Vasicek (c. 1977) to model short rates. It is probably the first interest rate model proposed. A short rate r_t is a stochastic process such that \$1 invested at t will yield an interest of $r_t \delta t$ at time $t + \delta t$, for small values of δt.

The Vasicek Model incorporates a mean-reversion feature which is missing in the GBM for stock prices:

$$dr_t = (\alpha - \beta r_t)dt + \sigma d\tilde{W}_t$$

for $t \geq 0$. Here, $\alpha, \beta, \sigma > 0$. The mean level is $\frac{\alpha}{\beta}$, while β may be called the speed of mean reversion, and σ is the volatility.

More formally, we assume that there is one noise factor W_t which is Brownian under the real world measure $\mathbb{P}$. We assume the existence of a measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ such that

(1) Under $\mathbb{Q}$, the process $\tilde{W}_t$ is Brownian
(2) (Principle of No-Arbitrage) The primary assets in this model are the ZCBs with various maturities: $B(t,T)$, $(t \in [0,T], T \geq 0)$. These ZCBs with respect to the money market account numeraire $\frac{1}{D(t)}$, where

$$D(t) := \exp\big(-\int_0^t r_u \, du\big),$$

are martingales
(3) (Completeness) All contingent claims are replicable with some self-financing strategies

Remark: We presented the Black-Scholes hedging argument on bonds in Lecture 6. There, we saw that the argument works provided the notion of the market price of risk is well-defined. It turns out that the market price of risk is an equivalent formulation for the martingale measure. Existence allows us to price the primary assets via the martingale framework and uniqueness allows us to price contingent claims. The assumptions here will force the ZCBs to have a specific form for their defining SDEs.

Let us use this model to compute two things:

- ZCB which matures at time s and has face 1. This may be expressed as

$$B(0,s) = E^{\mathbb{Q}}[\exp\big(-\int_0^s r_u, du\big)],$$

since the payoff of the bond at s is 1 and by martingale measure assumption.

- European call option which expires at time T, with underlying a ZCB which matures at time $s > T$ and has face 1. This may be expressed as

$$C(0) = E^{\mathbb{Q}}[e^{-\int_0^T r_u\,du}(B(T,s) - K)^+],$$

where

$$B(T,s) = E^{\mathbb{Q}}[\exp\big(-\int_T^s r_u, du\big)|r_T].$$

This is valid by virtue of the completeness assumption.

4.1. **Computation of ZCB Price.** We have seen earlier that the solution to the SDE defining the Vasicek Model is given by

$$r_t = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,d\tilde{W}_s \ (t \geq 0).$$

For r_T given and time horizon being $[T,s]$, we use the following equation:

$$r_t = e^{-\beta(t-T)} r_T + \frac{\alpha}{\beta}(1 - e^{-\beta(t-T)}) + \sigma e^{-\beta(t-T)}\int_T^t e^{\beta(u-T)}\,d\tilde{W}_u$$

$$= e^{-\beta(t-T)} r_T + \frac{\alpha}{\beta}(1 - e^{-\beta(t-T)}) + \sigma e^{-\beta t}\int_T^t e^{\beta u}\,d\tilde{W}_u \ (t \in [T,s]).$$

Let us compute the integral $\int_T^s r_u\,du$:

$$\int_T^s r_u\,du = \Big[-\frac{e^{-\beta(t-T)}}{\beta} r_T + \frac{\alpha}{\beta}(t + \frac{e^{-\beta(t-T)}}{\beta})\Big]_T^s + \sigma\int_T^s e^{-\beta t}\int_T^t e^{\beta u}\,d\tilde{W}_u dt$$

$$= (\frac{\alpha}{\beta} - r_T)(\frac{e^{-\beta(s-T)}}{\beta} - \frac{1}{\beta}) + \frac{\alpha}{\beta}(s - T) + \sigma\int_T^s\int_u^s e^{-\beta t} e^{\beta u}\,dt d\tilde{W}_u.$$

The term

$$\int_T^s \int_u^s e^{-\beta t} e^{\beta u}\, dt d\tilde{W}_u = \int_T^s \frac{1}{\beta}(1 - e^{-\beta(s-u)})\, d\tilde{W}_u$$

is normal with mean 0 and variance

$$\int_T^s \Big(\frac{1}{\beta}(1-e^{-\beta(s-u)})\Big)^2 du = \frac{1}{\beta^2}\big(s-T-\frac{2}{\beta}(1-e^{-\beta(s-T)})+\frac{1}{2\beta}(1-e^{-2\beta(s-T)})\big).$$

Hence,

$$\int_T^s r_u\, du = D(T,s) + \sigma E(T,s) Z,$$

where

$$\begin{aligned} D(T,s) &= (\frac{\alpha}{\beta} - r_T)(\frac{e^{-\beta(s-T)}}{\beta} - \frac{1}{\beta}) + \frac{\alpha}{\beta}(s-T) \\ &= F(T,s) r_T + G(T,s), \\ E(T,s) &= \frac{1}{\beta^2}\big(s - T - \frac{2}{\beta}(1-e^{-\beta(s-T)}) + \frac{1}{2\beta}(1-e^{-2\beta(s-T)})\big), \\ F(T,s) &= -\frac{e^{-\beta(s-T)}}{\beta} + \frac{1}{\beta}, \\ G(T,s) &= \frac{\alpha}{\beta}(\frac{e^{-\beta(s-T)}}{\beta} - \frac{1}{\beta}) + \frac{\alpha}{\beta}(s-T), \end{aligned}$$

and $Z \sim N(0,1)$.

The expectation has a closed form formula:

$$\begin{aligned} B(T,s) &= E^{\mathbb{Q}}[\exp\big(-D(T,s) - \sigma E(T,s) Z\big)|r_t] \\ &= e^{-D(T,s)} E^{\mathbb{Q}}[\exp\big(-\sigma E(T,s) Z\big)|r_t] \\ &= e^{-D(T,s)} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\sigma E(T,s) z - \frac{1}{2}z^2}\, dz \\ &= e^{-D(T,s) + \frac{1}{2}\sigma^2 E(T,s)^2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(z + \sigma E(T,s))^2}\, dz \\ &= e^{-D(T,s) + \frac{1}{2}\sigma^2 E(T,s)^2}. \end{aligned}$$

Replacing T by 0, we have thus found the ZCB price $B(0,T)$.

4.2. **Computation of Call Option Price using the Forward Measure.** Let us now compute the call option price. We will indicate how it can be done while leaving the tedious computation to you if you're keen to get your hands dirty. We shall use the change-of numeraire technique again, and in the process introduce a notion very important in interest rate models - the forward measure.

Fix T and allow $t \in [0, T]$ to vary. Set the ZCB $B(t) = B(t, T)$ as numeraire.

We've assumed that the ZCBs are martingales - that's why we're able to compute $B(0, T)$ from the martingale pricing formula. This means that it must satisfy a driftless SDE

$$d(D(t)B(t)) = \sigma^*(t, T)D(t)B(t)d\tilde{W}(t),$$

where $\sigma^*(t, T)$ is some volatility process.

Example 2. *Find* $\sigma^*(t, T)$.

Rearranging, we obtain

$$D(t)B(t) = D(0)B(0) \exp\big(\int_0^t \sigma^*(u, T)\, d\tilde{W}_u - \frac{1}{2} \int_0^t \sigma^*(u, T)^2\, du \big).$$

By the Girsanov Theorem, under the measure $\mathbb{Q}^T$ defined by

$$\mathbb{Q}^T(A) = \int_A Z_T(\omega)\, d\mathbb{Q}(\omega) \text{ (for all events } A),$$

where

$$Z_t = \exp\big(\int_0^t \sigma^*(u, T)\, d\tilde{W}_u - \frac{1}{2} \int_0^t \sigma^*(u, T)^2\, du \big) = \frac{D(t)B(t)}{B(0)},$$

the process

$$\tilde{W}_t^T := - \int_0^t \sigma^*(u, T)\, du + \tilde{W}_t$$

is Brownian. The measure $\mathbb{Q}^T$ is called the T*-forward measure*. The Radon-Nikodym derivative is

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = Z_T = \frac{D(T)B(T)}{D(0)B(0)} = \frac{\frac{B(T)}{B(0)}}{\frac{D(T)^{-1}}{D(0)^{-1}}}$$

and the following holds

$$X_{t'} = D_{t'}^{-1} E^{\mathbb{Q}}[D(t)X(t)|\mathcal{F}_{t'}] = B(t')E^{\mathbb{Q}^T}[B(t)^{-1}X(t)|\mathcal{F}_{t'}]$$

for $t' \leq t \leq T$ and any asset X_t replicable with the primary ones.

With respect to the T-forward measure $\mathbb{Q}^T$, the option price may be expressed as

$$\begin{aligned} C(0) &= E^{\mathbb{Q}}[D(T)(B(T,s)-K)^+] \\ &= D(0)^{-1}E^{\mathbb{Q}}[D(T)(B(T,s)-K)^+] \\ &= B(0)E^{\mathbb{Q}^T}[B(T)^{-1}(B(T,s)-K)^+] \\ &= B(0,T)E^{\mathbb{Q}^T}[(B(T,s)-K)^+]. \end{aligned}$$

We can thereby witness the usefulness of the forward measure in working with stochastic interest rates - the discount factor can be 'taken out' from within the expectation operator. This allows us to focus on the integration of the payoff function as was the case with the Black-Scholes Model.

In order to evaluate $E^{\mathbb{Q}^T}[(B(T,s)-K)^+]$, we need to know the distribution of $B(T,s)$ under $\mathbb{Q}^T$. To do that, we compute the differential of $B(T,s)$, expressing it in terms of $\tilde{W}^{\mathbb{Q}^T}$ (remember that s is fixed):

$$\begin{aligned} d\log B(t,s) &= d\big(-F(t,s)r_t - G(t,s) + \frac{1}{2}\sigma^2 E(t,s)^2\big) \\ &= -F_t r_s dt - F dr_t - G_t dt + \sigma^2 E E_t dt. \end{aligned}$$

The volatility process $\sigma^*(t,T)$ and the functions E, F, G can all be explicitly written down. We can already see that $B(T,s)$ is lognormally distributed. Lots of patience will allow us to find its mean M and variance V^2 and hence the option price.

Example 3. *Find M and V.*

Then we will be able to write

$$\begin{aligned} E^{\mathbb{Q}^T}[(B(T,s)-K)^+] &= \frac{1}{\sqrt{2\pi}}\int_{\frac{\log K - M}{V}}^{\infty} (e^{M+Vz} - K)e^{-\frac{1}{2}z^2}\,dz \\ &= e^{M+\frac{1}{2}V^2}N(V - \frac{\log K - M}{V}) - KN(-\frac{\log K - M}{V}), \end{aligned}$$

as usual.