AMERICAN OPTIONS

Learning Objectives:

- In the perspective of probability, the notion of stopping times is needed to define American options.

- In general, the prices of American options do not admit closed form formulas.

- The American call option on non-dividend paying stock is the same as its European counterpart.

- The price of an American put option on non-dividend paying stock does not (seem to) admit closed form formula. Researchers have considered simpler cases or approximations. Here we consider two: approximating the American put option with Bermudan and European put options; and the perpetual put option.

- American option prices can also be expressed as a free boundary problem. The free boundary that characterizes the American put option is called is exercise boundary. The exercise boundary is hard to pin down analytically when the maturity $T < \infty$, and this is related to the absence of a known closed form formula for the option price. When $T = \infty$, the exercise boundary becomes flat, and accordingly the price is easily found.

1. Stopping Times

Some exotic options involve a time during the lifespan of the option at which something happens - such as the option gets exercised, comes alive, or the level of the stock price is noted at that time. The time is not determined at the start of the contract - it is random.

Two classes of options involve this notion:

(1) Barrier Options - these are derivatives that expire (knock-out options) or come alive (kick-in) contingent upon the spot price reaching a certain level during their lives - the random time is the time at which the stock price attains this level
(2) American Options - these are derivatives which give the right to the holder to exercise at any time during the lifespan of the option

We will consider the pricing of American options in the following. But let us first define mathematically the central notion of a random time - called 'stopping time' in the literature - here.

Definition 1 (Stopping Time). *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space. A random variable taking values in $[0, T]$ is a stopping time if for every $t \geq 0$,*

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

In short, we may say that the stopping time is a random variable that is adapted to the given filtration.

The intuitive meaning of being adapted to the given filtration is this. Randomness is due to uncertainty of future outcomes. Probability theory models randomness via the notion of a filtered probability space. To say that a stopping time is adapted to the given filtration is to say that once we learn about the information that underlies the filtration, we will also know about the stopping time. For instance, it is common to have the filtration generated by the Brownian motions that function as basic noise drivers in a mathematical model. These Brownian motions drive the stock prices and interest rate in the mathematical economy. The stopping time is a time that is contingent upon the outcome of the stock prices and rate moves, so that when the moves are revealed, we will also know what the stopping time is.

Related to stopping time is the following notion:

Definition 2 (Stopped Process). *For any stochastic process S and any stopping time τ, we denote by S^τ the process stopped at τ:*

$$S_t^\tau(\omega) := S_{t\wedge\tau(\omega)}(\omega),$$

for all $\omega \in \Omega$ and for all $t \in [0, T]$.

A fundamental fact regarding stopping times is Doob's Optional Sampling Theorem.

Theorem 1 (Optional Sampling). *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a probability space and S be an adapted process such that $S_t \in \mathcal{L}^1$ for each t. Then M is a martingale if and only if M^τ is a martingale for any stopping time τ.*

The theorem basically says: a martingale that is stopped at a stopping time remains a martingale. It is plausible because a constant is trivially a martingale and the only way the stopping time could destroy the martingale property is it could decide to stop the process on one side in order that the process has a tendency for the other side. Thus, for instance, the stopping time would have to stop the process at times when it would be going down while not stop at times when it would be going up to destroy the martingale property and make it into a strict submartingale. This would require the stopping time to be able to look ahead. But since a stopping time is adapted, this is impossible.

2. A Brief on American Options

We begin with an informal introduction. We work within the risk neutral world and let $\mathbb{Q}$ be the martingale measure.

The holder of an American option has the right to exercise the option at any time during the lifespan $[0, T]$ of the option. Suppose she exercises at the time τ. The choice of this time can only be made based on what has unfold up till then. Thus τ is a stopping time. We may think of τ as a particular strategy that the option holder assumes.

Let $t \in [0, T]$. How should we price an American option at time t? For a particular stopping time τ in $[t, T]$, the value at time t of an American put option is

$$E^{\mathbb{Q}}[e^{-r(T-\tau)}(K - S_\tau)^+ | S_t = x],$$

where $\mathbb{Q}$ is the risk neutral measure of the Black-Scholes world. The option holder is assumed to be rational and will therefore optimize the value of the option. Hence, from the perspective of the option seller who needs to manage the risk in being short the option, the value of the option is

$$V^{put}(t, x) = \max_{\tau \in \mathcal{T}_{t,T}} E^{\mathbb{Q}}[e^{-r(T-\tau)}(K - S_\tau)^+ | S_t = x],$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times that lie in $[t, T]$.

Similarly, the American call option is valued by

$$V^{call}(t, x) = \max_{\tau \in \mathcal{T}_{t,T}} E^{\mathbb{Q}}[e^{-r(T-\tau)}(K - S_\tau)^+ | S_t = x].$$

The pricing of American options is a difficult problem in general as there is not formula and the numerical solution is computationally intensive and requires approximations for feasibility.

We consider the most basic issues regarding the American options with an underlying which follows a Geometric Brownian Motion (and without dividend) in the following.

3. American Call

Theorem 2. *The price of an American call option on a stock that does not give dividends is equal to the price of an European call option with the same strike and maturity. In other words, the American call option should not be exercised early.*

Proof. By the fact that $x \mapsto (x-K)^+$ is convex, we have

$$\begin{aligned} E^{\mathbb{Q}}[e^{-r(T-t)}(S_T-K)^+|S_t] &\geq E^{\mathbb{Q}}[(e^{-r(T-t)}S_T-K)^+|S_t] \\ &\geq (E^{\mathbb{Q}}[e^{-r(T-t)}S_T|S_t]-K)^+ \\ &= (e^{rt}E^{\mathbb{Q}}[e^{-rT}S_T|S_t]-K)^+ \\ &= (e^{rt}e^{-rt}S_t-K)^+ \\ &= (S_t-K)^+, \end{aligned}$$

which shows that the value of the European call option always exceeds the intrinsic value of its American counterpart. □

4. The Geske-Johnson Approximation

Geske and Johnson finds a sequence of values $P_1, P_2, \ldots$ such that

$$\lim_{n\to\infty} P_n =: P_\infty = P = P(K,T),$$

the value of the American put with strike K and maturity T.

The value of P_1 is just the value of the European put option $p(K,T)$ with the same strike K as the given American put option.

The value of P_2 is the Bermudan put option with exercise times $T/2, T$. At $T/2$, the critical price $\overline{S}_{T/2}$ at $T/2$ (i.e. exercise if $S_{T/2} \leq \overline{S}_{T/2}$) solves

$$K - S = p(S, K, T/2),$$

i.e. it is exercised if the payoff is greater than holding the option till maturity - this is equivalent to holding an European put option with time-to-maturity $T/2$.

The value P_3 is the Bermudan put option with exercise times $T/3, 2T/3, T$. At $T/2$, the critical price $\overline{S}_{T/3}$ at $T/3$ solves

$$K - S = P_2(S, K, 2T/3).$$

AT $2T/3$, the critical price $\overline{S}_{2T/3}$ at $2T/3$ solves

$$K - S = p(S, K, T/3).$$

The values $P_4, P_5, \ldots$ are defined likewise. It is clear that the limit above holds. The indices denote step size $T, \frac{1}{2}T, \frac{1}{3}T, \ldots$ respectively. The Richardson extrapolation takes linear combinations of such approximations to yield a higher order of convergence. Namely, if

$$P_{1/h} = P_\infty + \sum_{i=1}^{\infty} a_i h^i,$$

i.e. the Taylor expansion, then

$$P_{1/(\alpha h)} = P_\infty + \sum_{i=1}^{\infty} a_i (\alpha h)^i,$$

so that for $h = 1/3$ and $\alpha = 1, \frac{3}{2}, 3$, a suitable linear combination will result in an approximation of P_∞, by knocking out the h and h^2 terms, to an order of $O(h^3)$.

With the three terms P_1, P_2, P_3, Richardson extrapolation reads

$$P_\infty \approx P_3 + \frac{7}{2}(P_3 - P_2) - \frac{1}{2}(P_2 - P_1).$$

This then is the Geske-Johnson approximation of the American put option value (as it is commonly known; otherwise the Geske-Johnson approximation is this scheme of approximating the American put option by suitable Bermudan options with exercise dates designated at regular intervals, with the interval lengths diminishing to 0).

Example 1. *Verify that the Richardson approximation does what it is supposed to do, i.e. approximate up to $O(h^3)$.*

Now, how are the critical values $\overline{S}_{kT/n}$ and the values P_n determined?

The critical values may be found using standard root-finding numerical schemes (such as the Excel goal-seek function).

Let us illustrate how to find P_2. The values of $P_3, P_4, \ldots$ may be found similarly.

To be certain, the Bermudan option P_2 is

$$P_2 = E^{\mathbb{Q}}[e^{-rT/2}(K - S_{T/2})^+ 1_{S_{T/2} \le \overline{S}_{T/2}} + e^{-rT}(K - S_T)^+ 1_{S_{T/2} > \overline{S}_{T/2}}]$$

$$= e^{-rT/2} E^{\mathbb{Q}}[(K - S_{T/2})^+ 1_{S_{T/2} \le \overline{S}_{T/2}}] + e^{-rT} E^{\mathbb{Q}}[(K - S_T)^+ 1_{S_{T/2} > \overline{S}_{T/2}}]$$

$$= e^{-rT/2} E^{\mathbb{Q}}[(K - S_{T/2}) 1_{S_{T/2} \le \overline{S}_{T/2}}] + e^{-rT} E^{\mathbb{Q}}[(K - S_T) 1_{S_{T/2} > \overline{S}_{T/2}, S_T \le K}]$$

$$= e^{-rT/2} K \mathbb{Q}(S_{T/2} \le \overline{S}_{T/2}) - e^{-rT/2} E^{\mathbb{Q}}[S_{T/2} 1_{S_{T/2} \le \overline{S}_{T/2}}]$$

$$+ K e^{-rT} \mathbb{Q}(S_{T/2} > \overline{S}_{T/2}, S_T \le K) - e^{-rT} E^{\mathbb{Q}}[S_T 1_{S_{T/2} > \overline{S}_{T/2}, S_T \le K}]$$

$$= e^{-rT/2} K \mathbb{Q}(S_{T/2} \le \overline{S}_{T/2}) - S_0 \mathbb{T}(S_{T/2} \le \overline{S}_{T/2})$$

$$+ K e^{-rT} \mathbb{Q}(S_{T/2} > \overline{S}_{T/2}, S_T \le K) - S_0 \mathbb{T}(S_{T/2} > \overline{S}_{T/2}, S_T \le K),$$

where in the last lines, I've applied the change-of-numeraire method, with the numeraire S_t. Recall from Lecture 8 that $x_t = \frac{Ke^{-r(T-t)}}{S_t}$ is a martingale with respect to the measure $\mathbb{T}$, and in the notation from Lecture 8,

$$dx_t = -\sigma x_t d\overline{W}_t,$$

or

$$d\log(x_t) = -\frac{1}{2}\sigma^2 dt - \sigma d\overline{W}_t.$$

Let's continue: the above is equal to

$$e^{-rT/2}K\mathbb{Q}(\log S_{T/2} \le \log \overline{S}_{T/2}) - S_0\mathbb{T}(\log x_{T/2} \ge \log \frac{Ke^{-rT/2}}{\overline{S}_{T/2}})$$
$$+Ke^{-rT}\mathbb{Q}(\log S_{T/2} > \log \overline{S}_{T/2}, \log S_T \le \log K)$$
$$-S_0\mathbb{T}(\log x_{T/2} < \log \frac{Ke^{-rT/2}}{\overline{S}_{T/2}}, \log x_T \ge 0)$$
$$= e^{-rT/2}KN_{\log S_0+(r-\frac{1}{2}\sigma^2)T/2;\sigma\sqrt{T/2}}\big(\log \overline{S}_{T/2}\big)$$
$$-S_0\mathbb{T}\Big(\log x_{T/2} - \log x_0 + \frac{1}{2}\sigma^2 T/2 \ge -\log x_0 + \frac{1}{2}\sigma^2 T/2 + \log \frac{Ke^{-rT/2}}{\overline{S}_{T/2}}\Big)$$
$$+Ke^{-rT}\mathbb{Q}(Z^S_{T/2} > \frac{\log \overline{S}_{T/2} - \log S_0 - (r-\frac{1}{2}\sigma^2)T/2}{\sigma\sqrt{T/2}},$$
$$Z^S_T \le \frac{\log K - \log S_0 - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})$$
$$-S_0\mathbb{T}(Z^x_{T/2} < \frac{\log \frac{Ke^{-rT/2}}{\overline{S}_{T/2}} - \log x_0 + \frac{1}{2}\sigma^2 T/2}{\sigma\sqrt{T/2}}, Z^x_T \ge \frac{-\log x_0 + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}),$$

where $Z^S_{T/2}, Z^S_T, Z^x_{T/2}, Z^x_T$ are standard normals and $Cor(Z^S_{T/2}, Z^S_T) = Cor(Z^x_{T/2}, Z^x_T) = -\frac{1}{\sqrt{2}}$.

(*Why?*)

Continuing, the above is equal to

$$e^{-rT/2}KN\big(\frac{\log(\overline{S}_{T/2}/S_0) - (r-\frac{1}{2}\sigma^2)T/2}{\sigma\sqrt{T/2}}\big)$$
$$-S_0N\Big(\frac{\log(\overline{S}_{T/2}/S_0) - (r+\frac{1}{2}\sigma^2)T/2}{\sigma\sqrt{T/2}}\Big)$$
$$+Ke^{-rT}N\Big(\frac{\log(S_0/\overline{S}_{T/2}) + (r-\frac{1}{2}\sigma^2)T/2}{\sigma\sqrt{T/2}}, -\frac{\log(S_0/K) + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}; -\frac{1}{\sqrt{2}}\Big)$$
$$-S_0N\Big(\frac{\log(S_0/\overline{S}_{T/2}) + (r+\frac{1}{2}\sigma^2)T/2}{\sigma\sqrt{T/2}}, -\frac{\log(S_0/K) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}; -\frac{1}{\sqrt{2}}\Big).$$

Exercise 1. *Find P_3.*

5. Perpetual American Put

Just as the European option satisfies a PDE with a suitable terminal condition, the American put option is the solution to a *free boundary value problem.*

The free boundary alluded to is called the exercise boundary. If S_t is sufficiently small, one exercises the put option to deposit the quantity close to the strike price at the risk-free rate. If it is sufficiently large, then one does not exercise as it makes more sense to wait for the stock price to fall further. Thus, it is conceivable that there is an exercise boundary $b(t)$, such that if $S_t > b(t)$, one does not exercise the option, and once S_t rises to $b(t)$, one exercises. We may also describe the stopping time τ^* to be

$$\tau^* = \inf\{t \in [0, T] : S_t \geq b(t)\}.$$

The free boundary problem is stated as follows:

Let $\mathcal{G}$ be an open domain in $\mathbb{R}_+ \times [0, T)$ (T possible ∞) with a $b(t)$, i.e.

$$\mathcal{G} = \{(s, t) : s > b(t)\}.$$

The function $v : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ satisfies the following PDE on $\mathcal{G}$:

$$v_t + \frac{1}{2}\sigma^2 s^2 v_{ss} + rsv_s - rv = 0,$$

and

$$\begin{aligned}
v(s,t) &> (K-s)^+, && \forall (s,t) \in \mathcal{G},\\
v(s,t) &= (K-s)^+, && \forall (s,t) \in \mathcal{G}^c,\\
v(s,T) &= (K-s)^+, && \forall s \in \mathbb{R}_+,\\
\lim_{s \downarrow b(t)} v_s(s,t) &= -1, && \forall t \in [0,T).
\end{aligned}$$

Then $v(s, t)$ is the value at time t of the American put option with strike price K and maturity T, and $b(t)$ is the exercise boundary of the option. Conversely, the value function of the American put option is a solution of the free boundary problem above.

The last boundary condition above is called the *smooth fit principle* since the slope of $(K - s)^+$ is -1 when $s \leq b(t) \leq K$.

Justification of the statements above may be found in the paper 'On Optimal Stopping and Free Boundary Problem' by van Moerbeke.

Let us apply the facts above to price the perpetual put option ($T = \infty$). Since the option does not mature, the dependence of t disappears! Let us reap the benefit of this observation.

The exercise boundary is is flat: $b(t) = b \leq K$.

The PDE becomes an ODE:

$$\frac{1}{2}\sigma^2 s^2 v_{ss} + rsv_s - rv = 0,$$

where now $v = v(s)$ and $s \in (b, \infty)$. The boundary conditions become:

$$v(b) = K - b$$

and

$$\lim_{s \downarrow b} v_s(s) = -1.$$

We look for a solution of the form

$$v(s) = cs^d.$$

Substituting this into the ODE above, we have

$$\frac{1}{2}\sigma^2 d(d-1) + rd - r = 0,$$

$$cb^d = K - b,$$

and

$$cdb^{d-1} = -1.$$

This implies

$$d = 1, -\frac{2r}{\sigma^2}.$$

If $d = 1$, then $c = -1$, which will make v negative - inadmissible. Next, from

$$K - b = -\frac{b}{d},$$

we get

$$b = \frac{K}{1 - \frac{1}{d}} = \frac{2rK}{2r + \sigma^2}.$$

Finally, we have

$$v(s) = cs^d = cb^d(\frac{s}{b})^d = (K - b)(\frac{b}{s})^{\frac{2r}{\sigma^2}}.$$

Remark. The free boundary problem formulation of the price function of the American put option allows the price to be found by numerical methods developed for PDEs. An alternative approach - the *variational inequalities approach* - developed mathematically by Bensoussan and Lions and applied to American options by Jaillet et. al. does not require making the exercise boundary explicit and state the mathematical problem as a system of inequalities.