OPTION PRICING IN THE BINOMIAL MODEL

Abstract. Many factors affect the movements of asset prices (such as prices of stocks and commodities). No model can truly capture the complexity of the movements. Nevertheless formulating a mathematical model is an initial step towards understanding the intrinsic complexity. Mathematical models of asset prices also allow one to price financial derivatives and manage the associated risk. The binomial model is one of the simplest mathematical model of asset prices.

Learning Objectives:

- Learn about the binomial model of asset prices
- Learn about the principles of no-arbitrage and replication and how they are used in the pricing of derivatives
- Be familiar with how to find the risk-neutral price of derivatives using the binomial model

1. One-Period Binomial Model

1.1. The Basic Setup. To fix ideas, let us imagine ourselves in a simplified world in which there are only two times: time 0 and time 1.

We consider a stock whose price is known to be S_0 at time 0. Its price at time 1 depends on the throw of a biased coin which lands H with probability p and T with probability $q = 1 - p$.

We introduce two numbers: $d < u$. Let us stipulate that the value of the stock at time 1 is $S_1(H) = uS_0$ if the coin lands H, and it is $S_1(T) = dS_0$ if the coin lands T.

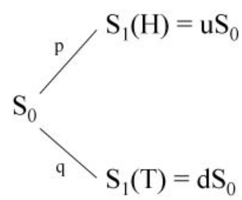

Figure 1. The One-Period Binomial Model

Suppose there is a market interest rate r. An amount of \$$X$ deposited at time 0 in the bank grows into the amount $X(1+r)$ at time 1.

A **trading strategy** is a portfolio containing cash and a certain number of the stock that is put together at time 0 and settled for P&L at time 1. An **arbitrage opportunity** is a trading strategy that has zero chance of making any loss and a positive probability of making profit. **In any reasonable model of the financial market, arbitrage opportunities should not exist.**

Proposition 1 (No-Arbitrage Condition). *The following inequalities hold:*

$$0 < d < 1 + r < u.$$

Proof. First, note that $d > 0$ since stock prices can never reach zero.

For the other two inequalities, we argue by contradiction.

Suppose $d \geq 1 + r$ instead. Then by the rule of buy-low-sell-high, we may borrow X from the bank at an interest rate of r and buy X of the stock at time 0. At time 1, we sell the stock and return the money with interest to the bank. The profit of this transaction is at least

$$X(d - 1 - r) \geq 0.$$

There is a positive chance that the stock price is uS_0 at time 1, in which case the profit will be

$$X(u - 1 - r) > 0.$$

This is an arbitrage opportunity which cannot occur. Hence $d < 1 + r$.

Suppose $u \leq 1 + r$ instead. Then by the rule of buy-low-sell-high, we may short sell X of the stock and lend out the same amount at an interest rate of r. At time 1, we collect our loan plus interest and buy back the stock in order to return the stock the we borrow for short-selling. The profit of this transaction is at least

$$X(1 + r - u) \geq 0.$$

There is a positive chance that the stock price is dS_0 at time 1, in which case the profit will be

$$X(1+r-d) > 0.$$

This is an arbitrage opportunity which cannot occur. Hence $u > 1+r$.

□

1.2. **The Replication Strategy.** Let us try to price a **European call option** with **strike price** K. This is a contract that gives the payoff $\max(0, S_1 - K)$ at time 1. If the outcome of the coin toss is H, then the payoff is $\max(0, S_1(H) - K)$; otherwise, the payoff is $\max(0, S_1(T) - K)$. Pricing the call option means finding its price at time 0.

The price of the option depends on the principle employed to price it. In **arbitrage pricing theory**, the principle is to **replicate** the option by trading in the stock and money market.

Let's see what replication entails in the following example.

Example 1. *Consider the One-Period Binomial Model with the following data:*

$$S_0 = 4,\ u = 2,\ d = \frac{1}{2},\ r = \frac{1}{4},\ K = 5.$$

This implies that

$$S_1(H) = 8,\ S_1(T) = 2.$$

Suppose we have an initial wealth of $X_0 = 1.20$ and we buy $\Delta_0 = \frac{1}{2}$ shares of stock at time 0. Our cash position is thus $X_0 - \Delta_0 S_0 = -0.80$ (i.e. a loan of 0.80 from the money market).

Let's compare the value of this portfolio with the value of the option at time 1.

If the coin lands H, then the value of the portfolio is

$$\Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) = \frac{1}{2}(8) + (1 + \frac{1}{4})(-0.80) = 3,$$

while the value of the option is

$$\max(0, S_1(H) - K) = 3.$$

If the coin lands T*, then the value of the portfolio is*

$$\Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) = \frac{1}{2}(2) + (1+\frac{1}{4})(-0.80) = 0,$$

while the value of the option is

$$\max(0, S_1(T) - K) = 0.$$

This shows that the portfolio replicates the option in having the same payoff at time 1*.*

This means that the price of the option at time 0 ought to be 1.20, the initial wealth in the above example. For if not, then the no-arbitrage principle will be violated:

- If the price of the option at time 0 is less than 1.20, say 1: buying the option and selling the replicating portfolio earns the profit of $(1+r)(1.20-1)$ at time 1
- If the price of the option at time 0 is more than 1.20, say 1.40: selling the option and buying the replicating portfolio earns the profit of $(1+r)(1.40-1.20)$ at time 1

1.3. **Pricing a Derivative Security.** Let us now consider the pricing of a **derivative security** in the One-Period Binomial Model generally. A derivative security is a security that pays off $V_1(H)$ if the coin-toss is H, and $V_1(T)$ if the coin-toss is T. We are interested in the following question:

What is V_0*, the value of the security at time* 0*?*

We shall apply the replication strategy as above to answer this question.

Let us use the following notation, the index $i = 0, 1$ indicating time:

- Wealth (or value of portfolio) : X_i
- Stock price : S_i
- Number of shares in portfolio: : Δ_i
- Value of option : V_i

At time 0:
We have wealth X_0. We buy Δ_0 shares of stock and invest the remaining amount of money $X_0 - \Delta_0 S_0$ in the money market (it's a loan if the sign is negative).

At time 1:
Stock value becomes $\Delta_0 S_1$. Cash becomes $(1+r)(X_0 - \Delta_0 S_0)$. Therefore the wealth at time 1 is

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0).$$

Note that this is an equation of random variables, since there are 2 states: H and T. So it actually amounts to the following 2 equations:

$$X_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0),$$

$$X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0).$$

In order to replicate the security with the portfolio, we require the condition: $X_1 = V_1$. This actually amounts to:

$$X_1(H) = V_1(H),\ X_1(T) = V_1(T).$$

We have two free variables - X_0 and Δ_0 - to help us do that.

In other words, we need to solve the following equations for X_0 and Δ_0:

$$\Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) = V_1(H),$$

$$\Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) = V_1(T).$$

They are solved as follows:

Dividing both equations by $1+r$, multiply the first equation by $\tilde{p}$ and the second by $\tilde{q} = 1 - \tilde{p}$, then sum them together, give

$$X_0 + \Delta_0\Big(\frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] - S_0\Big) = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)].$$

Choose $\tilde{p}$ so that

$$S_0 = \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] = \frac{1}{1+r}[\tilde{p}uS_0 + \tilde{q}dS_0].$$

In fact,

$$\tilde{p} = \frac{1+r-d}{u-d}, \ \tilde{q} = \frac{u-1-r}{u-d}.$$

Then the equation above reduces to

$$X_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)].$$

Taking the difference of the original two equations gives:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}.$$

Summary:

We are able to replicate the derivative security with a portfolio of of stock and cash. The amount of shares in the portfolio is

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}.$$

This is called the **delta hedging formula**. The word 'hedge' refers to the elimination of the risk in a short position in the derivative security by holding that amount of shares. In other words, the random variable

$$\Delta_0 S_1 - V_1$$

is in fact deterministic (i.e. does not depend on the toss of the coin).

The **no-arbitrage price** of the security at time 0 is

$$V_0 = X_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)].$$

This formula is also known as the **risk neutral pricing formula**.

1.4. **Some Reflections on the Model.**

(1) Apart from the more obvious features of the model such as the stock price being able to move only two ways at any one time, we have also assumed that stocks are arbitrarily divisible and not considered spreads in buying/selling of stock and borrowing/lending of money.

(2) By the no-arbitrage condition, $\tilde{p}, \tilde{q} > 0$ and $\tilde{p}+\tilde{q} = 1$. Therefore they may be interpreted as probabilities. These probabilities are not real world probabilities that are p and q. They are deduced from the no-arbitrage pricing principle eluded in the argument above and are called **risk neutral probabilities**. The formula

$$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]$$

says that V_0 may be interpreted as the expected discounted value of V_1, computed with respect to the risk neutral probabilities. It is as though an agent who prices the security by the no-arbitrage pricing principle lives in another world with a different set of probabilities - the risk neutral world.

In contrast, if the same formula were considered with real world probabilities instead of risk neutral probabilities:

$$V_0 = \frac{1}{1+r}[pV_1(H) + qV_1(T)],$$

the implied principle underlying the pricing is essentially that of taking bets on the outcome of the stock price at maturity. In fact, the model does not even require the knowledge of the real world probabilities p and q.

(3) To be realistic, we can impose on p and q the condition

$$S_0 < \frac{1}{1+r}[pS_1(H) + qS_1(T)].$$

For the RHS represents the expected discounted value of the stock at time 1, and it should be greater than the current value of the stock otherwise it would be wiser to deposit the cash in the bank safely without risk.

2. Multiperiod Binomial Model

2.1. **The Basic Setup.** Suppose now there are N periods and $N+1$ times: $T_0, T_1, T_2, \ldots, T_N$.

The diagram below shows the two-period model:

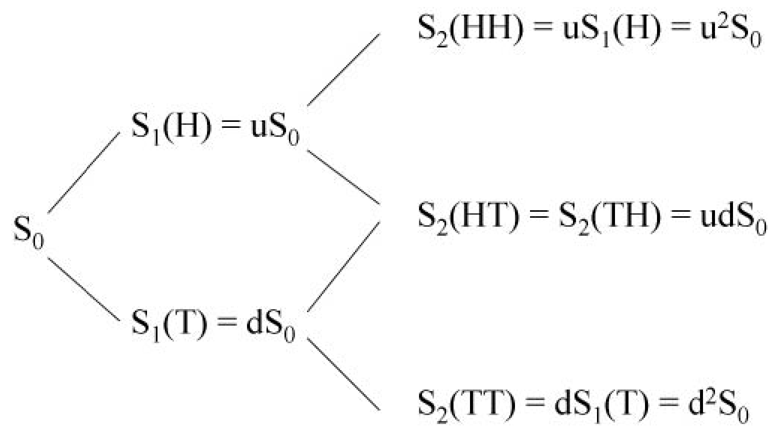

Figure 2. The Two-Period Binomial Model

We will begin with wealth X_0 and maintain a portfolio of stock and cash. The following notation will be used, $i = 0, 1, \ldots, N$ indicating time:

- Wealth (or value of portfolio) : X_i
- Stock price : S_i
- Number of shares in portfolio: : Δ_i
- Value of security : V_i

The i-th toss of the coin is denoted by ω_i = H,T.

The wealth at time $n+1$ comes from holding the portfolio at time n, and is equal to the **wealth equation**:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n),$$

which comes from owning Δ_n shares at time n, and the growth by interest of the portfolio cash.

Our goal is to replicate the security by dynamically adjusting the portfolio with the appropriate delta hedging ratios Δ_i ($i = 0, 1, \ldots, N-$

1), the value of the security at every time, i.e.

$$X_i = V_i$$

for each $i = 0, 1, \ldots, N$.

2.2. **Replication in the Multiperiod Binomial Model.** The following theorem tells us that it is possible to replicate the security with cash and stock in the Multiperiod Binomial Model and shows us how to do so by means of an explicit formula. The no-arbitrage price of the derivative security is also derived as a consequence.

Theorem 1. *Consider the N-period Binomial Model, with*

$$0 < d < 1 + r < u$$

and

$$\tilde{p} = \frac{1 + r - d}{u - d}, \; \tilde{q} = \frac{u - 1 - r}{u - d}.$$

Let V_N be a random variable that depends on the first N tosses of the coin (the security payoff). Define the following quantities:

(1) *Starting from V_N, the sequence of random variables $V_{N-1}, V_{N-2}, \ldots, V_1, V_0$ is defined by backward recursion, for every set of first n coin tosses $\omega_1\omega_2 \ldots \omega_n$, by*

$$V_n(\omega_1\omega_2 \ldots \omega_n) = \frac{1}{1 + r}[\tilde{p}V_{n+1}(\omega_1\omega_2 \ldots \omega_n H) + \tilde{q}V_{n+1}(\omega_1\omega_2 \ldots \omega_n T)].$$

(2) *Define the sequence of random variables $\Delta_0, \Delta_1, \ldots, \Delta_{N-2}, \Delta_{N-1}$ by*

$$\Delta_n(\omega_1\omega_2 \ldots \omega_n) = \frac{V_{n+1}(\omega_1 \ldots \omega_n H) - V_{n+1}(\omega_1 \ldots \omega_n T)}{S_{n+1}(\omega_1 \ldots \omega_n H) - S_{n+1}(\omega_1 \ldots \omega_n T)}.$$

If $X_0 := V_0$ and $X_1, X_2, \ldots, X_N$ is defined by the wealth equation, then

$$X_n = V_n$$

for $n = 1, 2, \ldots, N$.

Proof. The proof proceeds by induction.

Let $S(n)$ be the statement:

$$X_n(\omega_1\omega_2 \ldots \omega_n) = V_n(\omega_1\omega_2 \ldots \omega_n)$$

for all tosses $\omega_1\omega_2 \ldots \omega_n$.

Let's assume that $S(n)$ is true. We need to show that $S(n+1)$ is true. Let us try to show that

$$X_{n+1}(\omega_1\omega_2\ldots\omega_n H) = V_{n+1}(\omega_1\omega_2\ldots\omega_n H).$$

The other case in which the last toss is T is handled similarly. Let us also suppress writing down the first n tosses for simplicity.

Then we have

$$\begin{aligned}
X_{n+1}(H) &= \Delta_n u S_n + (1+r)(X_n - \Delta_n S_n) && \text{Wealth Equation}\\
&= (1+r)X_n + \Delta_n S_n(u-1-r) && \text{Rearrange}\\
&= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} S_n(u-1-r) && \text{Formula for } \Delta_n\\
&= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T)}{uS_n - dS_n} S_n(u-1-r) && \text{Definition of up and down moves}\\
&= (1+r)V_n + (V_{n+1}(H) - V_{n+1}(T))\frac{u-1-r}{u-d} && \text{Simplify}\\
&= (1+r)V_n + \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) && \text{Definition of } \tilde{q}\\
&= [\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)] + \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) && \text{Formula for } V_n\\
&= V_{n+1}(H).
\end{aligned}$$

□

3. Computational Considerations

In practice, binomial model of 100 or more periods are implemented. This means that there are at least $2^{100} \approx 10^{30}$ sequences of coin tosses (i.e. outcomes).

If the payoff of the derivative security at maturity depends on all possible sequences of coin tosses (i.e. the payoff of the derivative depends on how the price of the stock evolved before maturity), such as the Asian option, then the computation of the risk neutral price of the derivative will be rather infeasible.

On the other hand, the payoff of many options depends only on the price of the stock at maturity (e.g. European call option). In these cases, the computation of the risk neutral price can be made efficient as there are only 101 different stock prices at maturity and one can easily tabulate the option payoff on a computer.

Example 2. *Given the following data:* $S_0 = 1$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$. *Find the price at time* 0, *on a two-period binary tree, of the European call option with strike equals to* 1.

Solution:

The risk neutral probabilities are

$$\tilde{p} = \tilde{q} = \frac{1}{2}.$$

The problem can be solved on the binary tree in two passes - one forward and one backward.

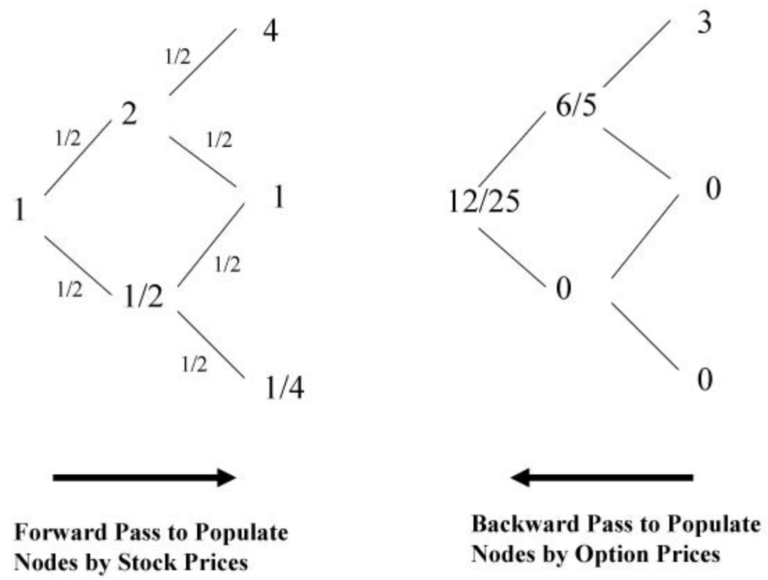

FIGURE 3. European Option Pricing

The numbers 6/5 and 24/25 are computed based on the formula for derivative valuation in Theorem 1:

$$\frac{6}{5} = \frac{1}{1+\frac{1}{4}}(\frac{1}{2}(0) + \frac{1}{2}(3)),$$

$$\frac{24}{25} = \frac{1}{1+\frac{1}{4}}(\frac{1}{2}(0) + \frac{1}{2}(\frac{6}{5})).$$

Notice that if all coin tosses were tabulated, we would have to compute the option payoff at maturity on 4 instead of 3 nodes above.

Example 3. *Given the following data:* $S_0 = 1$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$. *Find the price at time* 0, *on a 100-period binary tree, of the European call option with strike equals to* 1.

Solution:

We cannot possibly do this by hand. The following is an R script that implements the binary tree algorithm.

```
binarytree<-function(S0, u, r, N, K)
{

# Define the down factor
    d = 1/u

# Initialize array
    nodes = rep(0,N+1)
    nodes[1] = S0

# Compute risk neutral probabilities
    p = (1+r-d)/(u-d)

# Populate leaves of binomial tree with stock prices at maturity
        nodes[1:(N+1)] = u^(0:N)*d^(N:0)

# Option payoff
    nodes[1:(N+1)] = pmax(0,nodes[1:(N+1)]-K)

# Backward pass to populate array with option prices
    for (i in N:1)
        nodes[1:i] = (nodes[1:i]*(1-p) + nodes[2:(i+1)]*p)/(1+r)

# Output
    nodes[1]

}
```

In the following example of a lookback option, the history of how the stock price evolves matters - in other words, the option is path-dependent. Notice then that it is not possible to perform a backward induction on the tree to obtain the option price at time 0. If we were to have a graphical structure like a tree whose nodes contain the prices of the lookback option at all times between 0 and maturity and for all tosses of the coin, this structure will need to have $1 + 2 + \cdots + 2^N = 2^{N+1} - 1$ nodes in total (N is the total number of tosses at maturity). The tree is said to be *non-recombining*.

Example 4. *Given the following data:* $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$. *Find the price at time* 0, *on a three-period binary tree, of the* **lookback option**, *whose payoff at maturity (i.e. time* 3*), is given by*

$$V_3 = \max_{0 \le n \le 3} S_n - S_3.$$

Solution:

First note that the risk neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$.

Step 1: Forward evolve the stock price

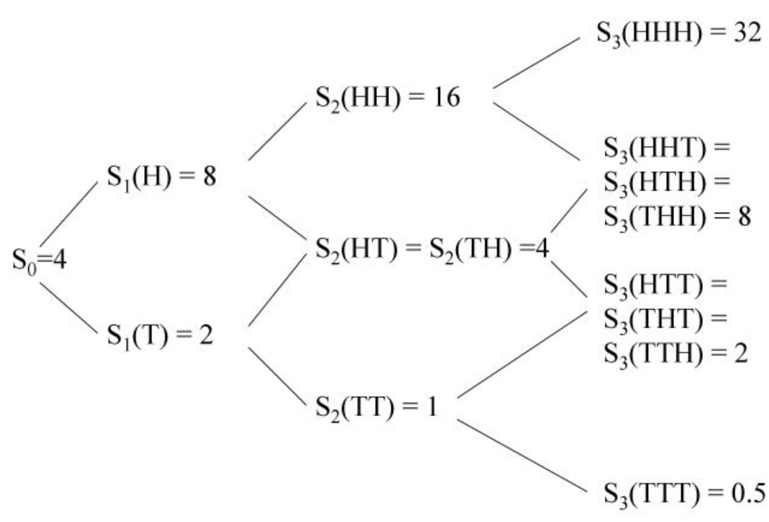

Figure 4. The Three-Period Binomial Model

Step 2: Backward recursion to obtain the current option price

Time 3:

$$\begin{aligned}
V_3(HHH) &= S_3(HHH) - S_3(HHH) && = 32 - 32 = 0,\\
V_3(HHT) &= S_2(HH) - S_3(HHT) && = 16 - 8 = 8,\\
V_3(HTH) &= S_1(H) - S_3(HTH) && = 8 - 8 = 0,\\
V_3(HTT) &= S_1(H) - S_3(HTT) && = 8 - 2 = 6,\\
V_3(THH) &= S_3(THH) - S_3(THH) && = 8 - 8 = 0,\\
V_3(THT) &= S_2(TH) - S_3(THT) && = 4 - 2 = 2,\\
V_3(TTH) &= S_0 - S_3(TTH) && = 4 - 2 = 2,\\
V_3(TTT) &= S_0 - S_3(TTT) && = 4 - 0.5 = 3.5.
\end{aligned}$$

Time 2:

$$V_2(HH) = \frac{4}{5}\Big(\frac{1}{2}V_3(HHH) + \frac{1}{2}V_3(HHT)\Big) = 3.2,$$

$$V_2(HT) = \frac{4}{5}\Big(\frac{1}{2}V_3(HTH) + \frac{1}{2}V_3(HTT)\Big) = 2.4,$$

$$V_2(TH) = \frac{4}{5}\Big[\frac{1}{2}V_3(THH) + \frac{1}{2}V_3(THT)\Big] = 0.8,$$

$$V_2(TT) = \frac{4}{5}\Big[\frac{1}{2}V_3(TTH) + \frac{1}{2}V_3(TTT)\Big] = 2.2.$$

Time 1:

$$V_1(H) = \frac{4}{5}\Big[\frac{1}{2}V_2(HH) + \frac{1}{2}V_2(HT)\Big] = 2.24,$$

$$V_1(T) = \frac{4}{5}\Big[\frac{1}{2}V_2(TH) + \frac{1}{2}V_2(TT)\Big] = 1.20.$$

Time 0:

$$V_0 = \frac{4}{5}\Big[\frac{1}{2}V_1(H) + \frac{1}{2}V_1(T)\Big] = 1.376.$$

Delta hedging ratio equals

$$\Delta_0 - \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{2.24 - 1.20}{8 - 2} = 0.1733.$$

Therefore the shares of the stock cost $0.1733 \times 4 = 0.6933$. The cash that is invested is worth $1.376 - 0.6933 = 0.6827$. At time 1, the replicating portfolio is $\Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$. If the stock goes up in price, the value is $0.1733 \times 8 + (1+\frac{1}{4}) \times (1.376 - 0.1733 \times 4) = 2.24$. If the stock goes down in price, the value is $0.1733 \times 2 + (1 + \frac{1}{4}) \times (1.376 - 0.1733 \times 4) = 1.2$.