Answer:

Solutions: Lecture 1 Exercises

1. The risk neutral pricing formula says:

$$\begin{aligned}
V_0 &= E^{rn}[\frac{S_1}{1+r}|S_0] \\
&= \tilde{p}\frac{S_1(H)}{1+r} + \tilde{q}\frac{S_1(T)}{1+r} \\
&= \frac{1+r-d}{u-d}\frac{uS_0}{1+r} + \frac{u-1-r}{u-d}\frac{dS_0}{1+r} \\
&= \frac{S_0}{(1+r)(u-d)}(u(1+r-d)+d(u-1-r)) \\
&= S_0
\end{aligned}$$

Q.E.D.

2. Refer to p.16 of Lecture 2—follow the steps shown

$$\begin{aligned}
X_{n+1}(T) &= \Delta_n dS_n + (1+r)(X_n - \Delta_n S_n) \\
&= (1+r)X_n + \Delta_n S_n(d-1-r) \\
&= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}S_n(d-1-r) \\
&= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T)}{uS_n - dS_n}S_n(d-1-r) \\
&= (1+r)V_n - (V_{n+1}(H) - V_{n+1}(T))\frac{1+r-d}{u-d} \\
&= (1+r)V_n - \tilde{p}V_{n+1}(H) + \tilde{p}V_{n+1}(T) \\
&= \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) - \tilde{p}V_{n+1}(H) + \tilde{p}V_{n+1}(T) \\
&= V_{n+1}(T)
\end{aligned}$$

Q.E.D.

3.

(a)

$$\begin{aligned}
V_n(s,y) &= \frac{1}{1+r}(\tilde{p}V_{n+1}(us, y+us) + \tilde{q}V_{n+1}(ds, y+ds)) \\
&= \frac{1/2}{1+\frac{1}{4}}(V_{n+1}(us, y+us) + V_{n+1}(ds, y+ds)) \\
&= \frac{2}{5}(V_{n+1}(us, y+us) + V_{n+1}(ds, y+ds))
\end{aligned}$$

(b)

$$
\begin{aligned}
V_0(4,4) &= \frac{2}{5}(V_1(8,12) + V_1(2,6)) \\
V_1(8,12) &= \frac{2}{5}(V_2(16,28) + V_2(4,16)) \\
V_1(2,6) &= \frac{2}{5}(V_2(4,10) + V_2(1,7)) \\
V_2(16,28) &= \frac{2}{5}(V_3(32,60) + V_3(8,36)) \\
&= \frac{2}{5}((\frac{60}{4} - 4)^+ + (\frac{36}{4} - 4)^+) \\
&= \frac{2}{5}(11+5) = \frac{32}{5} \\
V_2(4,16) &= \frac{2}{5}(V_3(8,24) + V_3(2,18)) \\
&= \frac{2}{5}((\frac{24}{4} - 4)^+ + (\frac{18}{4} - 4)^+) \\
&= \frac{2}{5}(2 + \frac{9}{2} - 4) = 1 \\
V_2(4,10) &= \frac{2}{5}(V_3(8,18) + V_3(2,12)) \\
&= \frac{2}{5}((\frac{18}{4} - 4)^+ + (\frac{12}{4} - 4)^+) \\
&= \frac{2}{5}(\frac{1}{2} + 0) = \frac{1}{5} \\
V_2(1,7) &= \frac{2}{5}(V_3(2,9) + V_3(\frac{1}{2}, 7\frac{1}{2})) \\
&= \frac{2}{5}((\frac{9}{4} - 4)^+ + (\frac{15/2}{4} - 4)^+) \\
&= 0 \\
V_1(2,6) &= \frac{2}{5}(\frac{1}{5} + 0) = \frac{2}{25} \\
V_1(2,6) &= \frac{2}{5}(\frac{32}{5} + 1) = \frac{74}{25} \\
V_0(4,4) &= \frac{2}{5}(\frac{2}{25} + \frac{74}{25}) = \frac{152}{125}
\end{aligned}
$$

(c) Following the proof of Thm 1, $S_n(s,y) = \frac{V_{n+1}(us, y+us) - V_{n+1}(ds, y+ds)}{us-ds}$
(i.e. the reductive argument, modified for the Asian option, will go through with this definition of S_n)

4) Base: USD
Asset: EUR

EUR/USD: S_i $(i = 0, 1)$
Interest rates (risk-free): r_{EUR}, r_{USD}
One period tree:

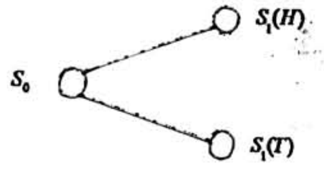

Figure 1: One period tree.

European put option / call option } strike K

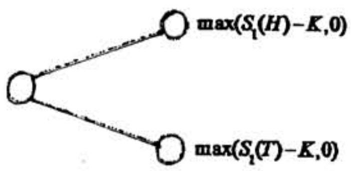

Figure 2: Call option payoff.

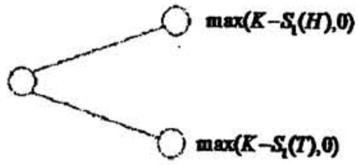

Figure 3: Put option payoff.

Let's replicate the call option
At $t = 0 : a + bS_0$
At $t = 1 : a(1 + r_{USD}) + b(1 + r_{EUR})S_1$
Solve: $\begin{cases} a(1 + r_{USD}) + b(1 + r_{EUR})S_1(H) = \max(S_1(H) - K, 0) \\ a(1 + r_{USD}) + b(1 + r_{EUR})S_1(T) = \max(S_1(T) - K, 0) \end{cases}$

$\Longrightarrow b(1 + r_{EUR})(S_1(H) - S_1(T)) = \max(S_1(H) - K, 0) - \max(S_1(T) - K, 0)$
$\Longrightarrow$

$$
\begin{aligned}
b &= \frac{\max(S_1(H) - K, 0) - \max(S_1(T) - K, 0)}{(1 + r_{EUR})(S_1(H) - S_1(T))} \\
a &= \frac{1}{1 + r_{USD}}(\max(S_1(H) - K, 0) - b(1 + r_{EUR})S_1(H)) \\
&= \frac{1}{1 + r_{USD}}(\max(S_1(H) - K, 0) - \frac{S_1(H)}{S_1(H) - S_1(T)}(\max(S_1(H) - K, 0) - \max(S_1(T) - K, 0))) \\
&= \frac{1}{1 + r_{USD}}(\frac{S_1(H)\max(S_1(T) - K, 0) - S_1(T)\max(S_1(H) - K, 0)}{S_1(H) - S_1(T)})
\end{aligned}
$$

The fair price of the call option at time 0 (=price at which there is no arbitrage opportunity)

$$
\begin{aligned}
&= a + bS_0 \\
&= (S_1(H) - K)^+(\frac{-S_1(T)}{(1 + r_{USD})(S_1(H) - S_1(T))} + \frac{S_0}{(1 + r_{EUR})(S_1(H) - S_1(T))}) \\
&\quad +(S_1(T) - K)^+(\frac{S_1(H)}{(1 + r_{USD})(S_1(H) - S_1(T))} - \frac{S_0}{(1 + r_{EUR})(S_1(H) - S_1(T))}) \\
&= E^{rn}[\frac{(S_1 - K)^+}{1 + r_{USD}}|S_0]
\end{aligned}
$$

Where the risk neutral probabilities are

$$
\begin{aligned}
P(H) &= \frac{-S_1(T)}{S_1(H) - S_1(T)} + \frac{1 + r_{USD}}{1 + r_{EUR}}\frac{S_0}{S_1(H) - S_1(T)} \\
&= \frac{\frac{1+r_{USD}}{1+r_{EUR}}S_0 - S_1(T)}{S_1(H) - S_1(T)} \\
P(T) &= \frac{-\frac{1+r_{USD}}{1+r_{EUR}}S_0 + S_1(H)}{S_1(H) - S_1(T)}
\end{aligned}
$$

Note that for these to be truly probabilities, we require: $S_1(H) > \frac{1+r_{USD}}{1+r_{EUR}}S_0 > S_1(T)$.
Q.E.D.