

**LECTURE 6A: ANSWERS TO SUGGESTED
EXERCISES**

Question 1.2 $\mathbf{a} = (2, 0, 3)$ and $\mathbf{b} = (1, 0, -1)$ find $|\mathbf{a}|$, $|\mathbf{b}|$, $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between the vectors \mathbf{a} and \mathbf{b} ?

1.2)

$$\begin{aligned} |\mathbf{a}| &= \sqrt{2^2 + 0^2 + 3^2} = \sqrt{13} \\ |\mathbf{b}| &= \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2} \\ \mathbf{a} + \mathbf{b} &= (3, 0, 2) \\ \mathbf{a} - \mathbf{b} &= (1, 0, 4) \\ \mathbf{a} \cdot \mathbf{b} &= 2 + 0 - 3 = -1 \\ \angle_{\text{between}} &= \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \cos^{-1} \frac{-1}{\sqrt{26}} \end{aligned}$$

Question 1.3 If $\mathbf{u} = (1, 2, 2)$ and $\mathbf{v} = (-6, 2, 3)$, find the component of \mathbf{u} in the direction of \mathbf{v} and the component of \mathbf{v} in the direction of \mathbf{u} .

1.3)

$$\begin{aligned} \text{pr}_{\mathbf{v}} \mathbf{u} &= \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{-6 + 4 + 6}{\sqrt{36 + 4 + 9}} = \frac{4}{7} \\ \text{pr}_{\mathbf{u}} \mathbf{v} &= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} = \frac{4}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{4}{3} \end{aligned}$$

Question 1.6) *What is the angle between any two diagonals of a cube*

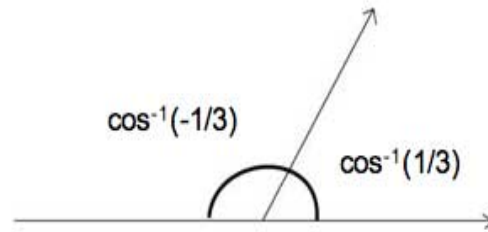
The vertices of a unit cube are $\{(\varepsilon_1, \varepsilon_2, \varepsilon_3) : \varepsilon_i = 0 \text{ or } 1, i = 1, 2, 3\}$
 A diagonal is formed by $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $(1 - \varepsilon_1, 1 - \varepsilon_2, 1 - \varepsilon_3)$ The complementary vector is $(1 - 2\varepsilon_1, 1 - 2\varepsilon_2, 1 - 2\varepsilon_3)$

The angle between $(1 - 2\varepsilon_1, 1 - 2\varepsilon_2, 1 - 2\varepsilon_3) + (1 - 2\varepsilon'_1, 1 - 2\varepsilon'_2, 1 - 2\varepsilon'_3)$ is

$$\cos^{-1}\left(\frac{(1 - 2\varepsilon_1)(1 - 2\varepsilon'_1) + (1 - 2\varepsilon_2)(1 - 2\varepsilon'_2) + (1 - 2\varepsilon_3)(1 - 2\varepsilon'_3)}{\sqrt{3}\sqrt{3}}\right)$$

since each $1 - \varepsilon_i$ or $1 - \varepsilon_j = \pm 1$

Note: that the numerator is either -3, -1 or 1. -3 corresponds to the 2 diagonals being equal. Note: $\cos^{-1}(\frac{-1}{3}) = \pi - \cos^{-1}(\frac{1}{3})$



Hence the acute angle between any 2 distinct diagonals is $\cos^{-1}(\frac{1}{3})$

Question 15.6.12) Find the directional derivative of the function at the given point in the direction of the vector \mathbf{v} .

$$f(x, y) = \ln(x^2 + y^2), \quad (2, 1), \quad \mathbf{v} = \langle -1, 2 \rangle$$

15.6.12)

$$\begin{aligned} D_{\mathbf{v}}f(2, 1) &= Df(2, 1) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right) \Big|_{(x,y)=(2,1)} \cdot \frac{(-1, 2)}{\sqrt{5}} \\ &= \left(\frac{4}{5}, \frac{2}{5} \right) \cdot \frac{(-1, 2)}{\sqrt{5}} \\ &= \frac{1}{5\sqrt{5}}(-4 + 4) = 0 \end{aligned}$$

Question 12.15.6.23) Find the maximum rate of change of f at the given point and the direction in which it occurs.

$$f(x, y) = \sin(xy), \quad (1, 0)$$

15.6.23)

The maximum rate of change occurs in the direction of the gradient

vector :

$$\begin{aligned} \frac{Df(1, 0)}{|Df(1, 0)|} &= \frac{(y\cos(xy), x\cos(xy)) \Big|_{(1,0)}}{|(y\cos(xy), x\cos(xy))|} \\ &= (0, 1) \end{aligned}$$

$$\text{Max. rate of change} = Df(1, 0) \cdot (0, 1) = 1$$

Question 12.15.6.28) Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin(xy)$ at the point $(1, 0)$ has value 1.

15.6.28)

Required: $D_{\mathbf{u}}f(1, 0) = 1$

i.e.

$$\begin{aligned} Df(1, 0) \cdot \mathbf{u} &= (2x + y\cos(xy), x\cos(xy))|_{(x,y)=(1,0)} \cdot \mathbf{u} \\ &= (2, 1) \cdot (a, b) = 1 \end{aligned}$$

This means

$$2a + b = 1, \quad a^2 + b^2 = 1.$$

This means that $a = 4/5, b = -3/5$.

Question 12.15.6.37) Show the operation of taking the gradient of a function has the given property. Assume that \mathbf{u} and \mathbf{v} are differentiable functions of x and y and a, b are constants.

12.15.6.37)

Let $u = u(x, y), v = v(x, y)$

a)

$$\nabla(au + bv) = a \nabla u + b \nabla v$$

$$\begin{aligned} \nabla(au + bv) &= \left(\frac{\delta}{\delta x}(au + bv), \frac{\delta}{\delta y}(au + bv) \right) \\ &= \left(a \frac{\delta u}{\delta x} + b \frac{\delta v}{\delta x}, a \frac{\delta u}{\delta y} + b \frac{\delta v}{\delta y} \right) \\ &= a \left(\frac{\delta u}{\delta x}, \frac{\delta u}{\delta y} \right) + b \left(\frac{\delta v}{\delta x}, \frac{\delta v}{\delta y} \right) \\ &= a \nabla u + b \nabla v \end{aligned}$$

b)

$$\nabla(uv) = u \nabla v + v \nabla u$$

$$\begin{aligned}\nabla(uv) &= \left(\frac{\delta}{\delta x}(uv), \frac{\delta}{\delta y}(uv)\right) \\ &= \left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}, u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) \\ &= u\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) + v\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \\ &= u\nabla v + v\nabla u\end{aligned}$$

c)

$$\nabla\left(\frac{\mathbf{u}}{\mathbf{v}}\right) = \frac{\mathbf{v} \nabla \mathbf{u} - \mathbf{u} \nabla \mathbf{v}}{v^2}$$

$$\begin{aligned} D\left(\frac{u}{v}\right) &= \left(\frac{\delta}{\delta x}\left(\frac{\mathbf{u}}{\mathbf{v}}\right), \frac{\delta}{\delta y}\left(\frac{\mathbf{u}}{\mathbf{v}}\right)\right) \\ &= \left(\frac{\frac{\delta u}{\delta x} \mathbf{v} - \frac{\delta v}{\delta x} \mathbf{u}}{v^2}, \frac{\frac{\delta u}{\delta y} \mathbf{v} - \frac{\delta v}{\delta y} \mathbf{u}}{v^2}\right) \\ &= \frac{1}{v^2} \left(\left(\frac{\delta u}{\delta x}, \frac{\delta u}{\delta y}\right) \mathbf{v} - \left(\frac{\delta v}{\delta x}, \frac{\delta v}{\delta y}\right) \mathbf{u} \right) \\ &= \frac{1}{v^2} (\mathbf{v} D\mathbf{u} - \mathbf{u} D\mathbf{v}) \end{aligned}$$

d)

$$\nabla \mathbf{u}^n = n \mathbf{u}^{n-1} \nabla \mathbf{u}$$

Use Induction

$$n = 1 \quad D\mathbf{u}' = D\mathbf{u}$$

$$1 \mathbf{u}^{1-1} D\mathbf{u} = \mathbf{u}^0 D\mathbf{u} = D\mathbf{u}$$

$$\text{Suppose } D\mathbf{u}^u = n \mathbf{u}^{u-1} D\mathbf{u} \rightarrow (1)$$

$$\text{Then } D\mathbf{u}^{n+1} = D\mathbf{u}^u \cdot \mathbf{u}$$

$$= n \mathbf{u}^{n-1} D\mathbf{u} \cdot \mathbf{u} + D\mathbf{u} \mathbf{u}^n$$

$$= (n + 1) \mathbf{u}^n D\mathbf{u}$$

Hence (1) is true for all $n \geq 1$