LECTURE 5A: ANSWERS TO SUGGESTED EXERCISES

10.6.1.14 Given that the function $h(x) = x^3 + 2x + 1$ for $x \in \mathbb{R}$ has an inverse h^{-1} on $\mathbb{R}$, find the value of $(h^{-1})'(y)$ and the points corresponding to $x = 1, 1, -1$.

$$\begin{aligned} y &= h(x) \\ x &= h^{-1}(y) \\ (h^{-1})'(y) &= \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = \frac{1}{3x^2+2} \end{aligned}$$

When $x = 0$, $(h^{-1})'(y) = \frac{1}{2}$.
When $x = 1$, $(h^{-1})'(y) = \frac{1}{5}$.
When $x = -1$, $(h^{-1})'(y) = \frac{1}{5}$.

10.6.2.1a For each of the following function on $\mathbb{R}$ to $\mathbb{R}$, find points of relative extrema, the intervals on which the function is increasing, and those on which it is decreasing

$$f(x) = x^2 - 3x + 5$$

$$f'(x) = 2x - 3 \begin{cases} \geq 0 \text{ if } x \geq \dfrac{3}{2} \\ \\ \leq 0 \text{ if } x \leq \dfrac{3}{2} \end{cases}$$

Therefore f is increasing on $[\frac{3}{2}, \infty)$ and decreasing on $(-\infty, \frac{3}{2}]$

10.6.2.4 Let $a_1, a_2, \dots, a_n$ be real numbers and let f be defined on $\mathbb{R}$ by

$$f(x) = \sum_{i=1}^{n} (a_i - x)^2, \quad x \in \mathbb{R}$$

. Find the unique point of relative minimum for f.

$$\begin{aligned} f(x) &= \sum_{i=1}^{n} (a_i - x)^2 = \sum_{i=1}^{n} (x - a_i)^2 \\ f'(x) &= \sum_{i=1}^{n} 2(x - a_i) = 0 \\ &\Rightarrow 2nx = 2\sum_{i=1}^{n} a_i \\ &\Rightarrow x = \frac{1}{n}\sum_{i=1}^{n} a_i \end{aligned}$$

As $f''\left(\sum_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} 2 = 2n > 0$, $x = \frac{1}{n}\sum_{i=1}^{n} a_i$ is a relative minimum for f.

10.6.2.6 Use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

By the Mean value Theorem,

$$|\sin x - \sin y| = |\cos(\xi)||x - y|$$

for some $\xi \in [x, y]$.

Hence $|\sin x - \sin y| < |x - y|$.

10.15.5.1: Use the chain rule to find dx/dt or dw/dt

Given: $z = x^2y + xy^2,\ x = 2 + t^4,\ y = 1 - t^3.$

$$\begin{aligned}
\Rightarrow z &= z(x, y) \\
x &= x(t) \\
y &= y(t) \\
\frac{dz}{dt} &= \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt} \\
&= (2xy + y^2) \cdot (4t^3) + (x^2 + 2xy)(-3t^2) \\
&= 4t^3(2xy + y^2) - 3t^2(x^2 + 2xy)
\end{aligned}$$

12.15.5.21: Use the chain rule to find the indicated partial derivatives

$$z = x^2 + xy^3,\ x = uv^2 + w^3,\ y = u + ve^w$$

Find $\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}, \dfrac{\partial z}{\partial w}$

$$\begin{aligned}
\Rightarrow \qquad z &= z(x, y) \\
x &= x(u, v, w) \\
y &= y(u, v, w)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} &= (2x + y^3)(v^2) + (3xy^2)(1) \\
&= v^2(2x + y^3) + 3xy^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} &= (2x + y^3)(2uv) + (3xy^2)(e^w) \\
&= 2uv(2x + y^3) + 3xy^2e^w
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} &= (2x + y^3)(3w^2) + 3xy^2(ve^w) \\
&= 3w^2(2x + y^3) + 3xy^2ve^w
\end{aligned}$$

12.15.5.52: Suppose $z = f(x,y)$ where $x = g(s,t)$ and $y = h(s,t)$.

a) *Show that*

$$\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial y^2}$$

b) *Find a similar formula for $\partial^2 z/\partial s \partial t$.*

$$z = f(x,y), x = g(s,t), y = h(s,t)$$

$$\Rightarrow \quad z = f(g(s,t), h(s,t))$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\begin{aligned}
\Rightarrow \frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} \\
&+ \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} \\
&= \left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} \\
&= \left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} \\
&= \left(\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y \partial x}\cdot\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} \\
&+ \left(\frac{\partial^2 z}{\partial x \partial y}\cdot\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2}\cdot\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 y}{\partial t^2} \\
&= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial y^2}
\end{aligned}$$

b)

$$
\begin{aligned}
\frac{\partial z}{\partial s \partial t} &= \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right) \\
&= \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t} \\
&+ \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t} \\
&= \left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t} \\
&+ \left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial s}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t} \\
&= \left(\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t} \\
&+ \left(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t} \\
&= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x \partial y}\left(\frac{\partial x}{\partial t}\frac{\partial y}{\partial s} + \frac{\partial x}{\partial s}\frac{\partial y}{\partial t}\right) \\
&+ \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial s}\right)\left(\frac{\partial y}{\partial t}\right) + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t}
\end{aligned}
$$

12.15.7.12: Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function

$$
\begin{aligned}
f(x,y) &= xy(1-x-y) = xy - x^2y - xy^2 \\
\frac{\partial f}{\partial x} &= y - 2xy - y^2 = 0 \\
\frac{\partial f}{\partial y} &= y - x^2 - 2xy = 0
\end{aligned}
$$

$$
\begin{aligned}
y(1-2x-y) &= 0 \\
x(1-x-2y) &= 0
\end{aligned}
$$

$$\begin{cases} y = 0 \\ x = 0 \\ \Rightarrow (x, y) = (0, 0) \end{cases} \quad \begin{cases} y = 0 \\ 1 - x - 2y = 0 \\ \Rightarrow (x, y) = (1, 0) \end{cases} \quad \begin{cases} 1 - 2x - y = 0 \\ x = 0 \\ \Rightarrow (x, y) = (0, 1) \end{cases} \quad \begin{cases} 1 - 2x - y = 0 \\ 1 - x - 2y = 0 \\ \Rightarrow x = y = \frac{1}{3} \end{cases}$$

$$\begin{aligned} D &= f_{XX} f_{YY} - f_{XY}^2 \\ &= (-2y)(-2x) - (1 - 2x - 2y)^2 \\ &= 4xy - (1 - 2x - 2y)^2 \end{aligned}$$

(x, y)	D	f_{XX}	Type
$(0, 0)$	-1		saddle point
$(1, 0)$	-1		saddle point
$(0, 1)$	-1		saddle point
$(1/3, 1.3)$	$\frac{1}{3}$	$-\frac{2}{3}$	local max

12.15.7.29: Find the absolute maximum and minimum values of f on the set D.

$$\begin{aligned} f(x, y) &= x^2 + y^2 + x^2 y + 4 \\ D &= \{(x, y) = |x| \leq 1, |y| \leq 1\} \end{aligned}$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x + 2xy = 0 \\ \dfrac{\partial f}{\partial y} = 2y + x^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x(1 + y) = 0 \\ 2y + x^2 = 0 \end{cases}$$

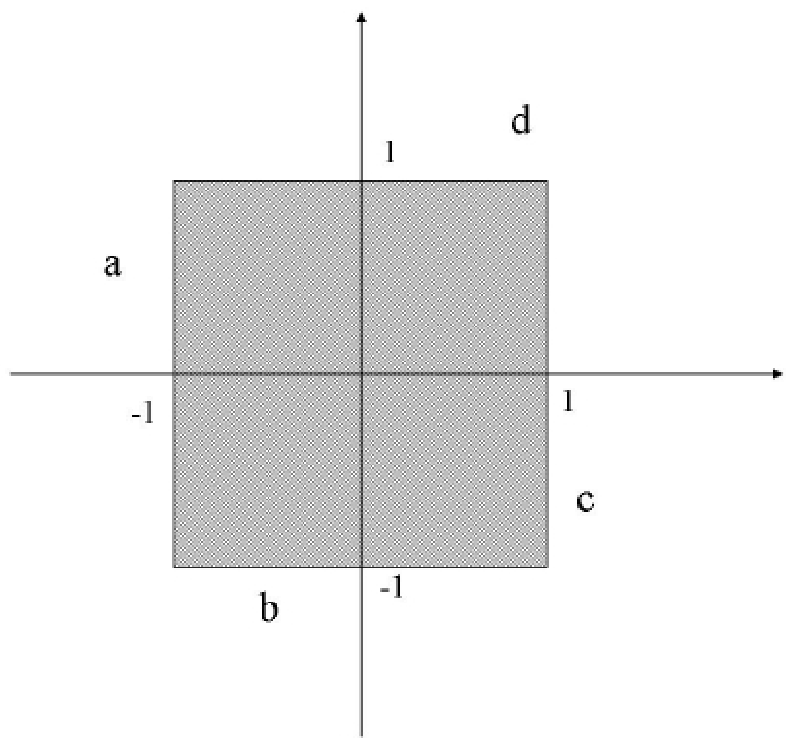
$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} y = -1 \\ x = \pm\sqrt{2} \end{cases}$$

Only $(0, 0) \in D$ is admissable

$$\left.\begin{array}{lcl} f_{XX} & = & 2+2y \\ f_{YY} & = & 2 \\ f_{XY} & = & 2x \end{array}\right\} \Rightarrow f_{XX}f_{YY} - f_{XY}^2 \begin{array}{cl} = & 2(2+2y) - 4x \\ = & 4(1+y-x^2) \\ = & 4 > 0 \text{ at } (0,0) \end{array}$$

Furthermore, $f_{XX}(0,0) = 2 > 0 \Rightarrow (0,0)$ is a local minimum.

Now let's restrict f to ∂D, the boundary of D.



It may be described as

$$\begin{array}{rl} a: & x=-1, \quad -1 \le y \le 1 \\ b: & y=-1, \quad -1 \le x \le 1 \\ c: & x=1, \quad -1 \le y \le 1 \\ d: & y=1, \quad -1 \le x \le 1 \end{array}$$

$$\begin{aligned} f|_a &= (-1)^2 + y^2 + (-1)^2 y + 4 \\ &= y^2 + y + 4 = (y+0.5)^2 + 4.75 \qquad -1 \le y \le 1 \end{aligned}$$

On side a, $f|_a \in [4.75, 7]$

$$\begin{aligned} f|_b &= x^2 + (-1)^2 + x^2(-1) + 4 \\ &= x^2 - x^2 + 5 = 5 \end{aligned}$$

On side b,$f|_b = 5$

$$\begin{aligned} f|_c &= 1^2 + y^2 + (1)^2 y + 4 \\ &= 5 + y + y^2 = 4\frac{3}{4} + \left(y + \frac{1}{2}\right)^2 \end{aligned}$$

On side c, $f|_c \in \left[\frac{19}{4}, \frac{19}{4} + \left(\frac{3}{2}\right)^2\right] = \left[\frac{19}{4}, 7\right]$

$$f|_d = x^2 + 1^2 + x^2(1) + 4 = 2x^2 + 5$$

On side d, $f|_d \in [5, 7]$.

Hence the absolute minimum of f on D is 7 and the absolute minimum of f on D is 4.

12.15.7.41: Find three positive numbers whose sum is 100 and whose product is a maximum

The problem may be cast as:

$$\text{Maximize } xyz$$
$$\text{given constraints } \begin{cases} x + y + z = 100 \\ x, y, z > 0 \end{cases}$$

Apply the Lagrange Multiplier Method.

Let $f(x, y, z) = xyz$
$h(x, y, z) = x + y + z = 100$

Thus we need to solve

$$\begin{cases} \nabla f + \lambda \nabla h = 0 \\ h = 0 \\ x, y, z > 0 \end{cases}$$

$$\text{Or} \quad \begin{cases} (yz, zx, xy) - \lambda(1,1,1) = 0 \\ x + y + z = 100 \\ x, y, z > 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} xy = yz = zx = \lambda \\ x + y + z = 100 \\ x, y, z > 0 \end{cases}$$

$$\begin{aligned} \Rightarrow xyz &= \lambda z = \lambda x = \lambda y \\ 3xyz &= \lambda x + \lambda y + \lambda z = \lambda(x+y+z) = 100\lambda \\ \Rightarrow 100^2\lambda^2 &= 9x^2y^2z^2 = 9(xy)(yz)(zx) = 9\lambda^3 \\ \Rightarrow \lambda^2(9\lambda - 100^2) &= 0 \\ \Rightarrow \lambda &= 0, \left(\frac{100}{3}\right)^2. \end{aligned}$$

But $\lambda = 0$ is not admissible, since $x, y, z > 0$.

$$\therefore \lambda = \left(\frac{100}{3}\right)^2$$

and $xyz = \frac{100}{3}\lambda = \left(\frac{100}{3}\right)^3$ (which occurs when $x = y = z = \frac{100}{3}$).