LECTURE 4A: ANSWERS TO SUGGESTED EXERCISES

10.6.1.1a) Use the definition to find the derivative of $f(x) = x^3$

10.6.1.1a)

$$f(x) = x^3$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3h^2x + 3hx^2 + h^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{3h^2x + 3hx^2 + h^3}{h} \\ &= \lim_{h\to 0} 3hx + 3x^2 + h^2 \\ &= \lim_{h\to 0}(3hx) + \lim_{h\to 0}(3x^2) + \lim_{h\to 0} h^2 \\ &= 0 + 3x^2 + 0 = 3x^2 \end{aligned}$$

10.6.1.2 Show that $f(x) = x^{1/3},\ x \in \mathbb{R},\ x \neq 0$ *is not differentiable at* $x = 0$

Let $f(x) = x^{1/3}$

Then $\dfrac{f(0+h) - f(0)}{h} = \dfrac{h^{1/3} - 0}{h} = h^{-2/3}$.

As $h \to 0$, $h^{-2/3} \to \infty$. Thus the limit and derivative does not exist.

10.6.1.5 Differentiate and simplify

a) $f(x) = \dfrac{x}{1+x^2}$

b) $g(x) = \sqrt{5-2x+x^2}$

c) $h(x) = \left(\sin x^k\right)^m$ for $m, k \in \mathbb{N}$

d) $k(x) = \tan(x^2)$ for $|x| < \sqrt{\pi/2}$

a)

$$f'(x) = \frac{(1+x^2)-(x)(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

b)

$$g'(x) = \frac{1}{2}\frac{(-2-2x)}{\sqrt{5-2x+x^2}} = \frac{x-1}{\sqrt{5-2x+x^2}}$$

c)

$$\begin{aligned} h'(x) &= m\left(\sin(x^k)\right)^{m-1}\cos(x^k)(kx^{k-1}) \\ &= mkx^{k-1}\cos(x^k)\left(sin(x^k)\right)^{m-1} \end{aligned}$$

d)

$$k'(x) = \sec^2(x^2)\cdot(2x) = 2x\sec^2(x^2)$$

10.6.1.7 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c and that $f(c) = 0$. show that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

$$\begin{aligned} \frac{g(c+h)-g(c)}{h} &= \frac{|f(c+h)|-|f(c)|}{h} \\ &= \frac{|f(c+h)|}{h} \end{aligned}$$

Suppose $f'(c) = 0$

Then $\left|\frac{f(c+h)}{h}\right| = \left|\frac{f(c+h) - f(c)}{h}\right| \to |f'(c)| = 0$
as $h \to 0$.

Since

$$-\left|\frac{f(c+h)}{h}\right| \leq \frac{|f(c+h)|}{h} \leq \left|\frac{f(c+h)}{h}\right|$$

therefore, as $h \to 0$, $\frac{|f(c+h)|}{h} \to 0$ as well.

$$\Rightarrow \lim_{h\to 0} \frac{g(c+h) - g(c)}{h} = 0$$

ie, $g'(c) = 0$ and g is differentiable at c.

Now suppose $g(x)$ is differentiable at c.

$$\Rightarrow \lim_{h\to 0} \frac{g(c+h) - g(c)}{h} \text{ exists}$$

$$\text{ie} \quad \lim_{h\to 0} \frac{|f(c+h) - f(c)|}{h} = \lim_{h\to 0} \frac{|f(c+h)|}{h} \text{ exists}$$

$$\Leftrightarrow \lim_{h\to 0^+} \frac{|f(c+h)|}{h} \text{ and } \lim_{h\to 0^-} \frac{|f(c+h)|}{h} \text{ both exist and are equal}$$

But note that $\lim_{h\to 0^+} \frac{|f(c+h)|}{h} \geq 0$ while $\lim_{h\to 0^-} \frac{|f(c+h)}{h} \leq 0$.

Hence, $\lim_{h\to 0^+} \frac{|f(c+h)|}{h} = \lim_{h\to 0^-} \frac{|f(c+h)}{h} = \lim_{h\to 0} \frac{|f(c+h)|}{h} = 0$

$$\Rightarrow \left|\frac{f(c+h)}{h}\right| = 0$$

Since

$$-\left|\frac{f(c+h)}{h}\right| \leq \frac{f(c+h) - f(c)}{h} = \frac{f(c+h)}{h} \leq \left|\frac{f(c+h)}{h}\right|$$

Therefore $\lim_{h\to 0} \dfrac{f(c+h)}{h} = 0$ as well.

But $f'(c) = \lim_{h\to 0} \dfrac{f(c+h) - f(h)}{h} = \lim\limits_{h\to 0} \dfrac{f(c+h)}{h}$. Hence $f'(c) = 0$.

12.15.3.5-6 Determine the signs of the partial derivatives for the function f whose graph is shown

5.

a) $f_x(1,2)$
b) $f_y(1,2)$

6.

a) $f_x(-1,2)$
b) $f_y(-1,2)$
c) $f_{xx}(-1,2)$
d) $f_{yy}(-1,2)$

5.

a) +
b) -

6.

a) -
b) -
c) +
d) -

12.15.3.15 Find the first partial derivatives of the function $z = xe^{3y}$.

$$z = xe^{3y}$$
$$\frac{\partial z}{\partial x} = e^{3y}$$
$$\frac{\partial z}{\partial y} = x(3e^{3y}) = 3xe^{3y}$$

12.15.3.57

$$f(x,y) = 3xy^4 + x^3y^2$$

$$f_x = 3y^4 + 3x^2y^2 \quad f_y = 12xy^3 + 2x^3y$$
$$f_{xx} = 6xy^2 \quad f_{yy} = 36xy^2 + 2x^3$$
$$f_{xxy} = 12xy \quad f_{yyy} = 72xy$$

12.15.3.66 Level curves are shown for a function f. *Determine whether the following partial derivatives are positive or negative at the point* P.

12.15.3.66

a) -
b) +
c) +
d) -
e) +