

**LECTURE 3: ANSWERS TO SUGGESTED EXERCISES**

*7.2.1.9: Determine a value  $\delta$  such that for the given values of  $L$  and  $\epsilon$ , the statement  $|f(x) - L| < \epsilon$  whenever  $0 < |x - \alpha| < \delta$  is valid. Sketch a graph of the given function*

7.2.1.9

Require:

$$\left| \frac{x^2 - 9}{x + 3} - (-6) \right| < \epsilon$$

whenever  $0 < |x - (-3)| < \delta$

ie

$$|x + 3| < 0.005$$

whenever  $0 < |x + 3| < \delta$

We may choose  $\delta = \epsilon = 0.005$

*7.2.2.7 Let  $f$  and  $g$  be continuous functions from  $\mathbb{R}^1$  to  $\mathbb{R}^1$ . Define*

$$F(x) = \max[f(x), g(x)]$$

*for each  $x \in \mathbb{R}^1$ . Show that  $F$  is continuous*

## 7.2.2.7

First, let us establish the following claim:

**Claim 1.**

$$\max(a, b) = \frac{a + b}{2} + \frac{|a - b|}{2}$$

*Proof:* if  $a \geq b$ , then  $|a - b| = a - b$  so

$$\frac{a + b}{2} + \frac{|a - b|}{2} = \frac{a + b}{2} + \frac{a - b}{2} = a$$

On the other hand, if  $a < b$ , then

$$\frac{a + b}{2} + \frac{|a - b|}{2} = \frac{a + b}{2} + \frac{b - a}{2} = b$$

Hence, the claim holds

**Claim 2.** The function

$$Q : x \mapsto |x|$$

is continuous

*Proof:* Let  $a \in \mathbb{R}, \epsilon > 0$ . Let's try to show that the function  $Q$  is continuous at  $a$ .

Case 1:  $a > 0$

We require  $|Q(x) - Q(a)| < \epsilon$  for  $x$  sufficiently close at  $a$ , ie we require

$$||x| - |a|| < \epsilon$$

if  $|x - a| < \delta$  for some  $\delta > 0$ .

In fact, let  $\delta = \min\left(\frac{a}{2}, \epsilon\right)$

Then if  $|x - a| < \delta$ , we have

$$\begin{aligned} x &\in \left(c - \frac{a}{2}, c + \frac{a}{2}\right) \\ &= \left(\frac{a}{2}, \frac{3a}{2}\right) \end{aligned}$$

which implies that  $x > 0$ . and  $||x| - |a|| = |x - a| < \delta \leq \epsilon$

Case 2:  $a < 0$

We require

$$||x| - |a|| < \epsilon \quad (\epsilon : \text{given})$$

whenever  $|x - a| < \delta$  ( $\delta : \text{to find}$ )

$$\text{Let } \delta = \min\left(\frac{|a|}{2}, \epsilon\right)$$

Then whenever  $|x - a| < \delta \leq \frac{|a|}{2}$ ,

$$\begin{aligned} x &\in \left(a - \frac{|a|}{2}, a + \frac{|a|}{2}\right) \\ &= \left(\frac{3a}{2}, \frac{a}{2}\right) \end{aligned}$$

since  $a < 0 \Rightarrow x < 0$

Therefore  $||x| - |a|| = |-x - (-a)| = |x - a| < \delta \leq \epsilon$

Case 3:  $a = 0$

We require  $||x| - |a|| = |x| < \epsilon$  whenever  $|x - a| = |x| < \delta$

Therefore it suffices to set  $\delta = \epsilon$

Now

$$\max(f(x), g(x)) = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}$$

Then functions

$$x \mapsto f(x)$$

$$x \mapsto g(x)$$

are continuous, hence  $x \mapsto f(x) + g(x)$  is continuous

hence  $x \mapsto \frac{1}{2}(f(x) + g(x))$  is continuous.

The function  $x \mapsto f(x) - g(x)$  is continuous,

hence  $x \mapsto \frac{|f(x) - g(x)|}{2}$  is continuous (composition of continuous functions is continuous).

Finally,  $\max(f(x), g(x)) = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}$  is continuous.

*7.2.3.5 Determine whether or not the function  $f$  is continuous at the given value of  $a$ . If it is not continuous, decide whether or not the function is continuous on the left or on the right. State the reasons for each step in the argument*

7.2.3.5

Let  $f(x) = x \left(1 + \left(\frac{1}{x^2}\right)\right)^{\frac{1}{2}}$  Then,

$$\begin{aligned} f(x) &= x \sqrt{\frac{x^2 + 1}{x^2}} \\ &= \frac{x}{|x|} \sqrt{x^2 + 1} \end{aligned}$$

If  $x > 0$ , then  $f(x) = \frac{x}{x} \sqrt{x^2 + 1} = \sqrt{x^2 + 1}$ . If  $x < 0$ , then  $f(x) = \frac{x}{-x} \sqrt{x^2 + 1} = -\sqrt{x^2 + 1}$ .

We have

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sqrt{x^2 + 1} = 1 \Rightarrow \text{Right limit exists}$$

and

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} -\sqrt{x^2 + 1} = -1 \Rightarrow \text{Left limit exists}$$

If  $f$  were defined such that  $f(0) = 1$ , then  $f$  would be right continuous. If  $f$  were defined such that  $f(0) = -1$ , then  $f$  would be left continuous.  $f$  can never be continuous regardless of how it's defined at 0.

7.2.4.10 Evaluate the limit or conclude that the function tends to  $\infty$ ,  $+\infty$ , or  $-\infty$

$$\begin{aligned} \lim_{x \rightarrow +\infty} (\sqrt{x^2 + 2x} - x) &= \lim_{x \rightarrow +\infty} (\sqrt{x^2 + 2x} - x) \cdot \frac{\sqrt{x^2 + 2x} + x}{\sqrt{x^2 + 2x} + x} \\ &= \lim_{x \rightarrow +\infty} \frac{(x^2 + 2) - x^2}{\sqrt{x^2 + 2x} + x} = \lim_{x \rightarrow +\infty} \frac{2x}{\sqrt{x^2 + 2x} + x} \\ &= \lim_{x \rightarrow +\infty} \frac{2x}{\sqrt{x^2 + 2x} + x} \cdot \frac{1/x}{1/x} \\ &= \lim_{x \rightarrow +\infty} \frac{2}{\sqrt{1 + \frac{2}{x}} + 1} \\ &= \frac{2}{1 + 1} = 1 \end{aligned}$$

7.2.4.11 Suppose  $f(x) \rightarrow +\infty$  and  $g(x) \rightarrow -\infty$  as  $x \rightarrow +\infty$ . Find examples of functions  $f$  and  $g$  with these properties and such that

- $\lim_{x \rightarrow \infty} [f(x) + g(x)] = +\infty$
- $\lim_{x \rightarrow \infty} [f(x) + g(x)] = -\infty$
- $\lim_{x \rightarrow \infty} [f(x) + g(x)] = A$  where  $A$  is an arbitrary real number

- $f(x) = x^2$      $g(x) = -x$
- $f(x) = x$      $g(x) = -x^2$
- $f(x) = x$      $g(x) = -x$

8.3.3.1 For each of the following statements, determine whether it is true or false and justify your answer.

- If the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, then  $f(\mathbb{R}) = \mathbb{R}$ .
- For any function  $f : [0, 1] \rightarrow \mathbb{R}$ , its image  $f([0, 1])$  is an interval.

- c) For any continuous function  $f : D \rightarrow \mathbb{R}$ , its image  $f(D)$  is an interval.
- d) For a continuous strictly increasing function  $f : [0, 1] \rightarrow \mathbb{R}$  its image is the interval  $[f(0), f(1)]$

## 8.3.3.1

- a) False. Counterexample:  $f : \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow 0$
- b) False. Counterexample:

$$f : [0, 1] \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} 0 & \text{if } x \in Q \\ 1 & \text{otherwise} \end{cases}$$

- c) False: Counterexample:

$$D = [-1, 0] \cup [1, 2]$$

$$f : D \rightarrow \mathbb{R}$$

$$f(x) = x$$

Then  $f(D) = [-1, 0] \cup [1, 2]$  is not an interval even though  $f$  is continuous.

- d) Let us prove that  $f([0, 1]) = [f(0), f(1)]$

**Claim 3.**  $f([0, 1]) \subseteq [f(0), f(1)]$

*Proof:* Let  $x \in [0, 1] \Rightarrow 0 \leq x \leq 1$ .

Since  $f$  is increasing, thus  $f(0) \leq f(x) \leq f(1)$

ie  $f(x) \in [f(0), f(1)]$

**Claim 4.**  $f([0, 1]) \supseteq [f(0), f(1)]$

*Proof:* Let  $y \in [f(0), f(1)]$ . By the Intermediate Value Theorem, since  $f$  is continuous, there exists  $x \in [0, 1]$  such that  $f(x) = y$

8.3.5.2 Define  $f(x) = \sqrt{x}$  for all  $x$ . Verify the  $\epsilon - \delta$  criterion for continuity at  $x = 4$  and  $x = 100$ . Hint: First show that for  $x \geq 0, x_0 > 0$ ,

$$|\sqrt{x} - \sqrt{x_0}| \leq |x - x_0| / \sqrt{x_0}$$

Let  $f(x) = \sqrt{x}$ ,  $x \geq 0$ . Let  $\epsilon > 0$  be given.

Case 1: Continuity at 4.

Require:  $|f(x) - f(4)| = |\sqrt{x} - 2| < \epsilon$  whenever  $|x - 4| < \delta$ .

Now from  $|\sqrt{x} - 2| < \epsilon$ , we obtain

$$|x - 4| = |(\sqrt{x} + 2)(\sqrt{x} - 2)| < |\sqrt{x} + 2|\epsilon$$

This suggests we may set

$$\delta = \min(1, (\sqrt{3} + 2)\epsilon)$$

Let's check: suppose  $|x - 4| < \delta$ ,

Then  $|x - 4| < 1$  ( $\Rightarrow x \in (3, 5)$ )

and  $|x - 4| < (\sqrt{3} + 2)\epsilon$ .

This implies that

$$|\sqrt{x} - 2| = \frac{|x - 4|}{|\sqrt{x} + 2|} < \frac{(\sqrt{3} + 2)\epsilon}{|\sqrt{3} + 2|} = \epsilon$$

Case 2: Continuity at 100.

Now set  $\delta = \min(1, (\sqrt{99} + 10)\epsilon)$ .

Then whenever  $|x - 100| < \delta$ , we have

$$|x - 100| < 1 \quad (\Rightarrow x \in (99, 101))$$

and

$$|x - 100| < (\sqrt{99} + 10)\epsilon$$

and so

$$|\sqrt{x} - 10| = \frac{|x - 100|}{|\sqrt{x} + 10|} < \frac{(\sqrt{99} + 10)\epsilon}{|\sqrt{99} + 10|} = \epsilon$$

12.15.2.8 and 12.15.2.16: Find the limit, if it exists, or show that the limit does not exist

12.15.2.8

Take the limit along the x-axis:  $\{(x, 0) : x \in \mathbb{R}\}$

$$\lim_{(x,0) \rightarrow (0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Now take the limit along the y-axis:  $\{(0, y) : y \in \mathbb{R}\}$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2} = \lim_{y \rightarrow 0} \frac{\sin^2 y}{y^2} = \lim_{y \rightarrow 0} \left( \frac{\sin y}{y} \right)^2 = 1$$

Hence the limit does not exist.

12.15.2.16

Set  $x = y^4$

Then  $\frac{xy^4}{x^2 + y^8} = \frac{y^8}{y^8 + y^8} = \frac{1}{2} \Rightarrow$  taking limit  $(x, y) \rightarrow (0, 0)$  along the curve  $x = y^4$  the limit is  $\frac{1}{2}$ .

Set  $x = 0$ .

Then  $\frac{xy^4}{x^2 + y^8} = \frac{0}{y^8} = 0 \Rightarrow$  taking limit  $(x, y) \rightarrow (0, 0)$  along the curve  $x = 0$  the limit is 0.

Hence the limit does not exist