LECTURE 2B: ANSWERS TO SUGGESTED EXERCISES

6.2.1 Evaluate each of the following limits (if it exists) and prove your answer:

d) $\lim_{n\to\infty} \left(\dfrac{1}{1+n}\right)^n$

e) $\lim_{n\to\infty} \left(2+\dfrac{1}{n}\right)^n$

6.2.1(d)

$$\lim_{n\to\infty} (1+n)^n = \infty$$

$$\text{since} \qquad n \le n^n < (1+n)^n$$

$$\text{and} \qquad \lim_{n\to\infty} n = \infty$$

$$\text{therefore} \qquad \lim_{n\to\infty} \frac{1}{(1+n)^n} = 0$$

6.2.1e)

$$\text{As} \qquad 2 + \frac{1}{n} > 2$$

$$\text{and} \qquad \lim_{n\to\infty} 2^n = \infty$$

$$\text{thus} \qquad \lim_{n\to\infty} \left(2 + \frac{1}{n}\right)^n = \infty$$

6.2.3) Prove that if $\lim_{n\to\infty} a_n = 0$ *and* $\{b_n\}$ *is bounded (but not necessarily convergent) then* $\lim_{n\to\infty} a_n b_n = 0$

Since $\{b_n\}$ is bounded, there exists some $M > 0$ such that $|b_n| < M$ for all $n \in \mathbb{Z}_+$

Let $\epsilon > 0$ be arbitrary.

There is an index N such if $i \geq N$, then

$$|a_i| < \frac{\epsilon}{M}$$

$$\Rightarrow \qquad |a_i b_i| < |a_i| M < \frac{\epsilon}{M} \times M = \epsilon$$

whenever $i \geq N$.

Hence $\lim_{n\to\infty} a_n b_n = 0$

6.2.4) Prove that if $\{a_n\}$ converges and $\{b_n\}$ diverges then $\{a_n + b_n\}$ diverges

Suppose $\{a_n + b_n\}$ converges.

Since $\{a_n\}$ converges and $b_n = (a_n + b_n) - a_n$, hence $\{b_n\}$ also converges. But this contradicts the assumption that $\{b_n\}$ diverges.

Hence the sequence $\{a_n + b_n\}$ must diverge.

6.2.5 Give examples of

a) *A convergent sequence $\{a_n\}$ and a divergent sequence $\{b_n\}$ such that $\{a_n b_n\}$ converges.*
b) *A pair of divergent sequences $\{a_n\}$ and $\{b_n\}$ with $\{a_n + b_n\}$ convergent*
c) *A pair of divergent sequences $\{a_n\}$ and $\{b_n\}$ with $\{a_n b_n\}$ convergent*

6.2.5

a)

$$a_n = \frac{1}{n^2}, \qquad b_n = n, \qquad a_n b_n = \frac{1}{n}$$

b)

$$a_n = n, \qquad b_n = -n \qquad a_n + b_n = 0$$

c)

$$a_n = (-1)^n, \qquad b_n = (-1)^n, \qquad a_n b_n = 1$$