LECTURE 2A: ANSWERS TO SUGGESTED EXERCISES

Question 5.1 *If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational*

5.1)

Suppose r is rational, $\neq 0$ and x is irrational. We may write $r = \dfrac{a}{b}$ $(a, b \in \mathbb{Z}, a, b \neq 0)$

Let's prove by contradiction.

Suppose $r + x$ is rational.

$$\Rightarrow r + x = \frac{p}{q} \quad (p, q \in \mathbb{Z}, q \neq 0)$$

$$\text{Then } x = \frac{p}{q} - r = \frac{p}{q} - \frac{a}{b} = \frac{bp - aq}{bq}$$

which is rational.

But this contradicts the assumption that x is irrational.

Suppose $rx = \dfrac{p}{q}$ is rational. $(p, q \in \mathbb{Z}, q \neq 0)$

Then $x = \dfrac{(\frac{p}{q})}{r} = \dfrac{pb}{qa}$

which is rational.

But this contradicts the assumption that x is irrational.

Question 5.2 *Prove that there is no irrational number whose square is 12*

5.2)

Suppose $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3} = \dfrac{p}{q}$ and $\sqrt{3} = \dfrac{p}{2q}$ is rational.

Let's write $\sqrt{3} = \dfrac{a}{b}$ where a, b are 2 integers that have no common factor other than 1. (i.e. the fraction is expressed in lowest terms)

Then

$$3 = \frac{a^2}{b^2}$$
$$\Rightarrow 3b^2 = a^2$$

This implies that a is a multiple of 3 (since 3 is a prime number)

Let $a = 3k$
Then

$$3b^2 = a^2 = (3k)^2 = 9k^2$$
$$\Rightarrow b^2 = 3k^2$$

This implies that b is a multiple of 3 too. But this contradicts the assumption that a and b do not have any common factors other than 1.