LECTURE 1B: ANSWERS TO SUGGESTED EXERCISES

Question 1.1.1

a) B

b) A

c) E

d) B

e) G

f) H

g) G

h) H

i) A

j) H

k) A

l) C

Question 1.1.4

a) *Prove that* $\{1, 2, 3\} = \{1, 1, 2, 3\}$

b) *Recall the notion for power sets; the power set of a set* A *is written* $P(A)$*. How many elements are there in the set* $P(\{1, 2, 3\})$

c) *Determine the cardinality of* $P(\emptyset)$

d) *Determine the cardinality of* $P(P(\emptyset))$

e) *Determine the cardinality of* $P(P(P(P(P(P(\emptyset))))))$

a) The elements of $\{1,2,3\}$ are 1,2,3.
The elements of $\{1,1,2,3\}$ are 1,2,3.
$\Rightarrow \{1,2,3\} = \{1,1,2,3\}$

b) $\#\mathrm{P}(\{1,2,3\}) = 2^{\#\{1,2,3\}} = 2^3 = 8$

c) $\#\mathrm{P}(\emptyset) = 2^0 = 1$

d) $\mathrm{P}(\mathrm{P}(\emptyset)) = \mathrm{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
$\therefore \mathrm{P}(\mathrm{P}(\emptyset)) = 2$

e)

$$\underbrace{\mathrm{P}(\underbrace{A}_{m})}_{2^m}$$

$$\underbrace{\mathrm{P}(\underbrace{\mathrm{P}(\underbrace{\mathrm{P}(\underbrace{\emptyset}_{1})}_{2})}_{2^2=4})}_{2^4=16}$$

Ans: 2^{65536}

Question 1.1.7 *Let* $f : \mathbb{Z} \mapsto \mathbb{Z}$ *be defined by* $f(x) = x^2 \quad \forall x \in \mathbb{Z}$ *and let* $g : \mathbb{Z} \mapsto \mathbb{Z}$ *be defined by* $g(x) = x + 1 \quad \forall x \in \mathbb{Z}$.

a) *Give the formulas which define the maps* $f \circ g$ *and* $g \circ f$*, carefully distinguishing which is which.*

b) *Let* $n \in \mathbb{N}$ *Give the formulas which define the maps* f^n *and* g^n.

c) *Which of the maps* f*,* g*,* f^2 *and* g^2 *are bijections?.*

Question 1.1.7

a)

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(x+1) = (x+1)^2 \end{aligned}$$

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(x^2) = x^2 + 1 \end{aligned}$$

b)

$$\begin{aligned} f(x) &= x^2 \\ f^2(x) &= f(f(x)) = f(x^2) = (x^2)^2 = x^4 \\ f^3(x) &= f(f^2(x)) = f(x^4) = (x^4)^2 = x^8 \\ f^n(x) &= x^{2^n} \end{aligned}$$

$$\begin{aligned} g(x) &= x + 1 \\ g^2(x) &= g(g(x)) = g(x+1) = x + 2 \\ g^n(x) &= x + n \end{aligned}$$

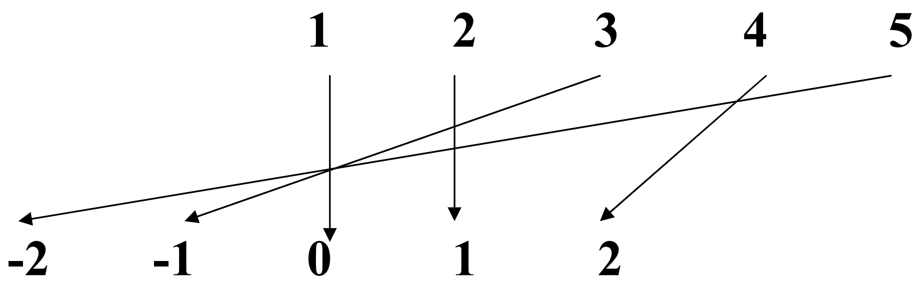
c) Bijective: g, g^2 ; Not bijective: f, f^2

Question 1.1.9

a) *Define a bijective map from* $\mathbb{N}$ *to* $\mathbb{Z}$

b) *Define a surjective map from* $\mathbb{Z}$ *to* $\mathbb{Z}$ *which is not a bijection.*

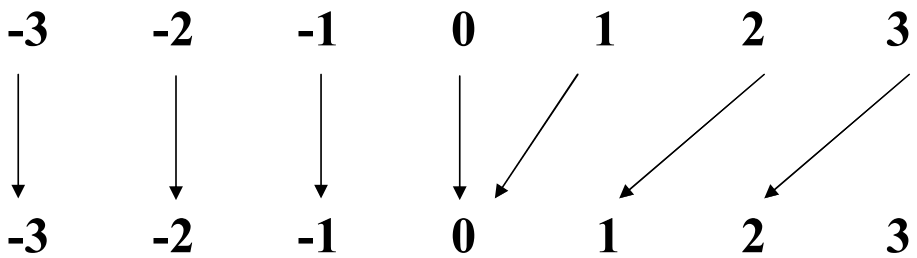
Question 1.1.9



a) $f : \mathbb{N} \to \mathbb{Z}$ defined by the rule

$$\begin{cases} 1 & \mapsto & 0 \\ 2n & \mapsto & n \\ 2n-1 & \mapsto & -n \end{cases}$$

is a bijection



b) $g : \mathbb{Z} \to \mathbb{Z}$ defined by the rule

$$\begin{cases} 0 & \mapsto & 0 & \\ n & \mapsto & n & \text{if } n < 0 \\ n & \mapsto & n-1 & \text{if } n > 0 \end{cases}$$

is not a bijection but is a surjection.

Question 1.1.11) *Let* $S = \{1, 2, 3, \ldots, n\}$

a) *How many maps are there from* S *to* S *?*

b) *How many surjective maps are there from* S *to* S*?*

c) *How many injective maps are there from* S *to* S*?*

d) *How many bijective maps are there from* S *to* S*?*

Question 1.1.11

a) A map $f : S \mapsto S$ assigns an element $x \in S$ to an element $y \in S$. For each $x \in = \{1, 2, 3, \ldots, n\}$,, there are n devices to assign to x. Thus there are $\overbrace{n \times n \times n \times \ldots n}^{n} = n^n$ such maps.

b) A surjection must be a bijection since domain and codomain are both finite and of the same size. Thus there are $n!$ surjections.

c) A injection must be a bijection since domain and codomain are both finite and of the same size. Thus there are $n!$ injections.

d) A bijection from S to S is a permutation of the elements $1, 2, 3, \ldots, n$. Thus there are $n!$ bijections.

Question 1.1.15 *In this question we discuss a map* $f : A \longrightarrow B$

a) *Suppose that there is a map* $g : B \longrightarrow A$ *such that* $f \circ g = Id_B$ *Prove that* f *is surjective.*

b) *Suppose that there is a map* $h : B \longrightarrow A$ *such that* $h \circ f = Id_A$. *Prove that* f *is injective*

c) *Suppose now that the hypotheses of parts (a) and (b) hold simultaneously. Prove that* f *is bijective and that* $g = h$

1.1.15a)

$$f \circ g = \mathrm{Id}_B$$

means that

$$f(g(x) = x \text{ for all } x \in \mathbb{R}$$

Let $y \in \mathbb{R}$ be arbitrary.

Since $f(g(y)) = y$, the element $g(y) \in A$ is mapped to y by f. Since y is arbitrary, f is surjective.

b)

$$h \circ f = \mathrm{Id}_A$$

means that

$$h(f(z)) = z \text{ for all } z \in A$$

.

Suppose

$$f(a) = f(b)$$

Then

$$a = h(f(a)) = h(f(b)) = b$$

This implies that f is injective.

c)

(a) and (b) holds $\Leftrightarrow$ f is injective and surjective. We have $h(f(g(x))) = h(x)$ and $h(f(g(x))) = g(x)$, hence $g = h$

Question 2.1.6 *Prove that if f and g are odd functions then the product fg is an even function. Prove also that the product of two even functions is even, and that the product of an even and an odd function is also an odd function.*

2.1.6 f and g having are odd means that

$$\begin{aligned} f(-x) &= -f(x) \\ g(-x) &= -g(x) \quad \forall x \end{aligned}$$

$$\begin{aligned} (fg)(x) &= f(x)g(x) \\ &= (-f(x))(-g(x)) \\ &= f(-x)g(-x) \\ &= (fg)(-x) \end{aligned}$$

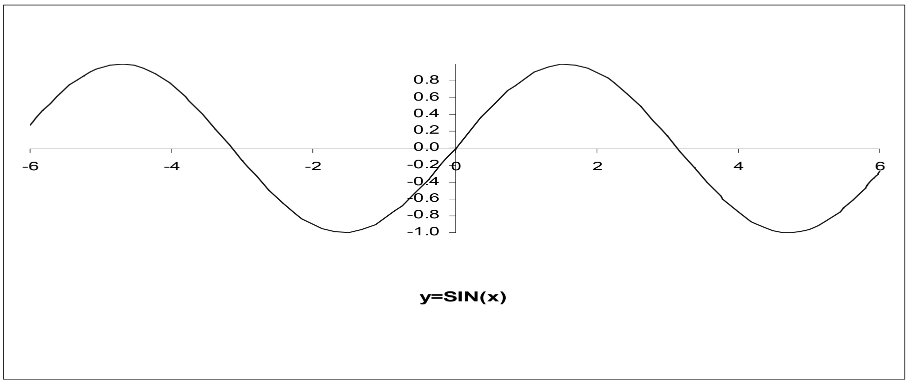
hence fg is an even function.

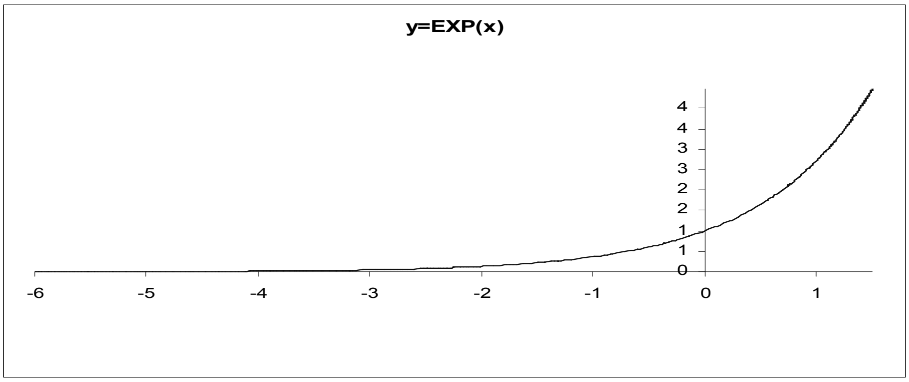
Suppose f is even and g is odd:

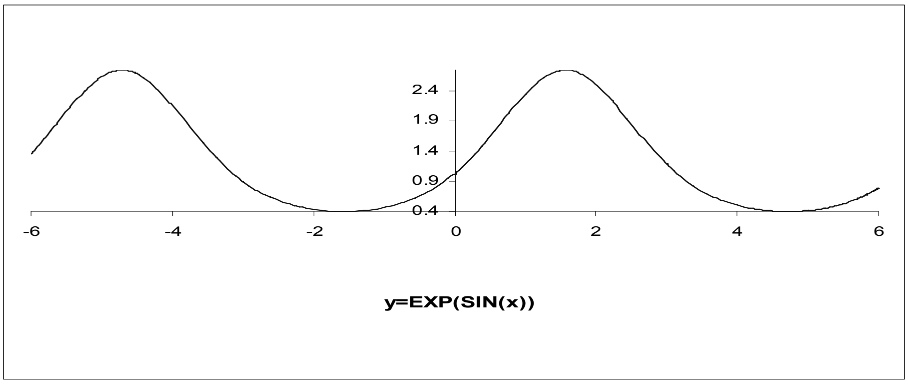
$$\begin{aligned} f(-x) &= f(x) \\ g(-x) &= g(x) \\ (fg)(-x) &= f(-x)g(-x) \\ &= f(x)(-g(x)) \\ &= -f(x)g(x) \\ &= -(fg)(x) \end{aligned}$$

hence, fg is odd.

Question 2.1.8 *Sketch the graph of* $y = e^{\sin x}$*, using the method of Example 1.10, giving a clear explanation of your reasoning*







Question 2.1.22 *Simplify the following expressions as far as possible*

a) $\cosh(\ln x)$
b) $\coth(\ln x)$
c) $\dfrac{\cosh(\ln(x)) - \sinh(\ln(x))}{\cosh(\ln(x)) + \sinh(\ln(x))}$

2.1.22a)

$$\cosh(\ln(x)) = \frac{e^{\ln x} + e^{-\ln x}}{2} = \frac{x + \frac{1}{x}}{2}$$

2.1.22b)

$$\coth(\ln x) = \frac{\cosh(\ln x)}{\sinh(\ln x)} = \frac{x + \frac{1}{x}}{x - \frac{1}{x}} = \frac{x^2 + 1}{x^2 - 1}$$

2.1.22c)

$$\frac{\cosh(\ln(x)) - \sinh(\ln(x))}{\cosh(\ln(x)) + \sinh(\ln(x))} = \frac{\left(x + \frac{1}{x}\right) - \left(x - \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right) + \left(x - \frac{1}{x}\right)} = \frac{\frac{2}{x}}{2x} = \frac{1}{x^2}$$

10) <u>Q30</u>
a) VI b) V c) I d) IV e) II f) III

11) <u>Q53-58</u>
53) B III 54) C II 55) F V

56) A VI 57) D IV 58) E I