DOUBLE INTEGRALS - PART 1

Learning Objectives:

- Understand the distinction between double integrals and iterated integrals.
- Appreciate that Fubini's Theorem links up the notions of double and iterated integrals

1. Iterated Integrals and Fubini's Theorem

An **iterated integral** is roughly the integration analogue of the composition of functions, examples of which are

$$\int_0^1\int_0^1 xy\,dxdy := \int_0^1 \Big(\int_0^1 xy\,dx\Big)dy,$$

$$\int_0^1\int_0^x \sin(x+y)\,dydx := \int_0^1 \Big(\int_0^x \sin(x+y)\,dy\Big)dx.$$

Let us evaluate the above iterated integrals for illustration.

Example 1.

$$\int_0^1\int_0^1 xy\,dxdy = \int_0^1 \Big(\int_0^1 xy\,dx\Big)dy$$

$$= \int_0^1 \Big(y\int_0^1 x\,dx\Big)dy = \int_0^1 y[x^2/2]_0^1\,dy$$

$$\int_0^1 y\cdot(1/2)\,dy = \frac{1}{2}\int_0^1 y\,dy = \frac{1}{4}.$$

We could have performed the integration the other way:

$$\int_0^1\int_0^1 xy\,dydx = \int_0^1 \Big(\int_0^1 xy\,dy\Big)dx$$

$$= \int_0^1 \Big(x\int_0^1 y\,dy\Big)dx = \int_0^1 x[y^2/2]_0^1\,dx$$

$$\int_0^1 x\cdot(1/2)\,dx = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{4}.$$

Notice that the answers are the same in both cases. This, as we shall see, is part of the content of Fubini's Theorem.

Example 2.

$$\int_0^1\int_0^x \sin(x+y)\,dydx = \int_0^1 \Big(\int_0^x \sin(x+y)\,dy\Big)dx$$

$$= \int_0^1 [-\cos(x+y)]_0^x\,dx = \int_0^1 (-\cos 2x + \cos x)\,dx$$

$$= [-\frac{1}{2}\sin 2x + \sin x]_0^1 = -\frac{1}{2}\sin 2 + \sin 1.$$

Example 3. *Calculate the iterated integral*

$$\int_2^4 \int_{-1}^1 (x^2 + y^2)\, dydx.$$

As can be seen from the above examples, the difficulty in evaluating iterated integrals is only as hard as the evaluation of one dimensional integrals.

Note also that the double integral is different from the doubly iterated integral. The symbol $\int\int_D$ in the double integral needs to be interpreted together as a whole. On the other hand, the symbol $\int_0^1 \int_0^1$ in the iterated integral can be interpreted as 'inner integration followed by outer integration'.

The general iterated integrals can be expressed as

$$\int_a^b \int_{f(x)}^{g(x)} F(x,y)\,dydx \ \text{ or } \ \int_c^d \int_{h(y)}^{k(y)} G(x,y)\,dxdy.$$

They may be visualized as follows:

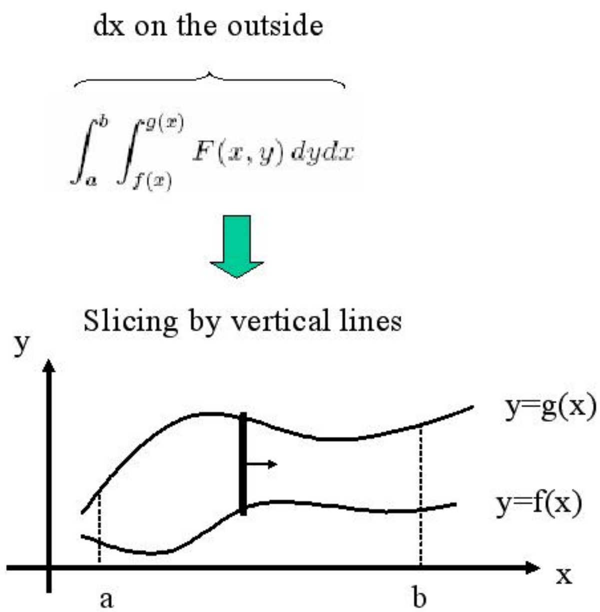

FIGURE 1. $\int_a^b \int_{f(x)}^{g(x)} F(x,y)\,dydx$

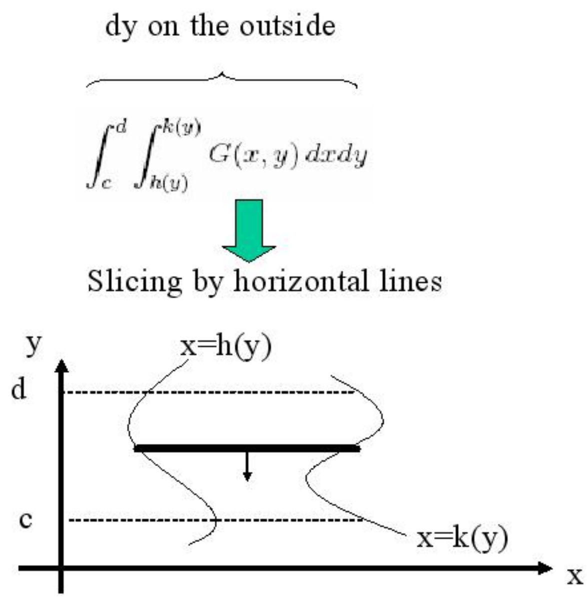

FIGURE 2. $\int_c^d \int_{h(y)}^{k(y)} G(x,y)\,dxdy$

What is the relationship between the iterated integral and the double integral? This is answered by the Fubini's Theorem.

Theorem 1 (Fubini's Theorem)**.** *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous. Then*

$$\int_a^b \int_c^d f(x,y)\,dydx = \int\int_{[a,b]\times[c,d]} f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dxdy.$$

The significance of Fubini's Theorem is that a higher dimensional integral may be evaluated in terms of lower dimensional integrals in the form of iterated integrals. Since the iterated integrals are intrinsically connected to the underlying region, regardless of the order of iteration, they are equal to each other.

Example 4. *Evaluate*

$$\int\int_{[0,\pi/2]\times[0,\pi/2]} \sin x \cos y\,dA.$$

Solution:

By the Fubini's Theorem,

$$\int\int_{[0,\pi/2]\times[0,\pi/2]} \sin x \cos y \, dA = \int_0^{\pi/2} \int_0^{\pi/2} \sin x \cos y \, dx dy$$
$$= \int_0^{\pi/2} \big(\int_0^{\pi/2} \sin x \cos y \, dx\big) dy = \int_0^{\pi/2} \big(\cos y \int_0^{\pi/2} \sin x \, dx\big) dy$$
$$= \int_0^{\pi/2} \big(\cos y \big(\int_0^{\pi/2} \sin x \, dx\big)\big) dy = \big(\int_0^{\pi/2} \sin x \, dx\big) \int_0^{\pi/2} \cos y \, dy$$
$$= [-\cos x]_0^{\pi/2} [\sin y]_0^{\pi/2} = 1 \times 1 = 1.$$

From the above argument, we distill the following useful rule for integration:

$$\int_a^b \int_c^d f(x) g(y) \, dx dy = \int_c^d f(x) \, dx \int_a^b g(y) \, dy.$$