

VECTOR CALCULUS - PART 1

Learning Objectives:

- Be familiar with the basic rules of vectors, the dot product theorem and the projection theorem

1. COORDINATES

The plane is the set \mathbb{R}^2 consisting of points (x, y) , where $x, y \in \mathbb{R}$. The *origin* is the distinguished point $(0, 0)$ denoted by O .

There are two basic methods to name points on the plane:

By means of *Cartesian Coordinates*:

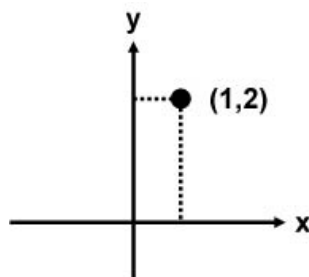


FIGURE 1. Cartesian Coordinates

Or by means of *Polar Coordinates*:

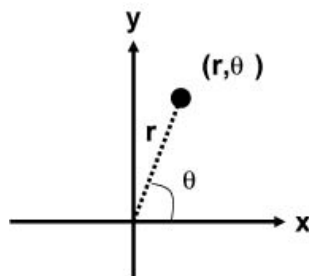


FIGURE 2. Polar Coordinates

When a point is stated in polar coordinates (r, θ) , one may ask for its cartesian coordinates:

$$x = r \cos \theta, \quad y = r \sin \theta.$$

By convention, the quantity θ is expressed in radians.

Vice versa, if a point is stated in cartesian coordinates, one may ask for its polar coordinates:

$$r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}.$$

Example 1. *The point $(10, 4)$ is in Cartesian coordinates. Find its polar coordinates.*

Example 2. *The point $(10, 4)$ is in polar coordinates. Find its Cartesian coordinates.*

2. VECTORS

Let us introduce some additional structures to the set \mathbb{R}^2 .

Addition:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

Scalar Multiplication:

$$\lambda(x, y) = (\lambda x, \lambda y)$$

With these structures endowed, we call \mathbb{R}^2 a *vector space*.

We may think of a vector as a quantity that has a magnitude and a direction. The vector (x, y) has magnitude

$$|(x, y)| = \sqrt{x^2 + y^2}$$

and direction indicated by the unit vector

$$\left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).$$

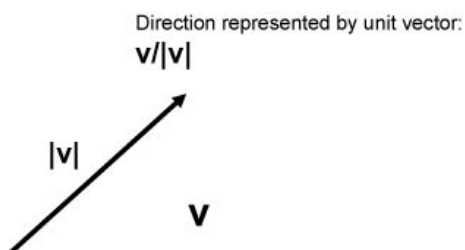


FIGURE 3. A Vector

Intuitively, a vector is a quantity that indicates the *relative position* of a point P relative to another point Q . For instance, the vector $(1, 3)$ indicates that the point $(1, 3)$ is $\sqrt{1^2 + 3^2}$ away from the point $(0, 0)$ (the origin) in the direction of $(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$.

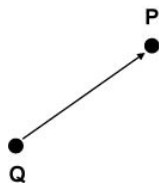


FIGURE 4. A vector indicates relative position

In general, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then the position of P relative to Q is indicated by the vector

$$\overrightarrow{QP} = (x_1 - x_2, y_1 - y_2) = (x_1, y_1) - (x_2, y_2) = \overrightarrow{OP} - \overrightarrow{OQ}.$$

Note that $\overrightarrow{PQ} = -\overrightarrow{QP}$.

The addition of vectors may be thought of as *successive displacements*.

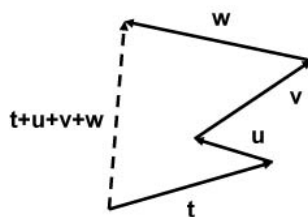


FIGURE 5. Addition of vectors as successive displacements

The scalar multiplication may be thought of as scaling/extension/contraction.

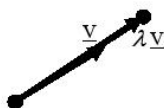


FIGURE 6. Scalar multiplication

The *standard basis vectors* are the vectors

$$\mathbf{i} = (1, 0), \quad \mathbf{j} = (0, 1).$$

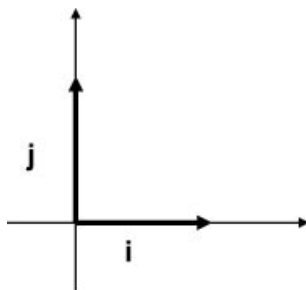


FIGURE 7. Scalar multiplication

They are thus named because every other vector can be expressed as a linear combination of them:

$$(x, y) = x\mathbf{i} + y\mathbf{j}.$$

The usefulness of vectors is the wealth of geometry that they can describe easily.

If P and Q are two points on the plane, and if

$$\overline{OP} + \overline{OQ} = \overline{OR},$$

then the points O, P, Q, R form a parallelogram.

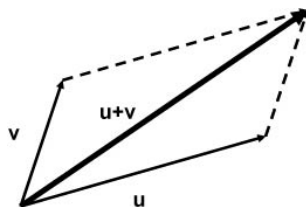


FIGURE 8. Addition of two vectors and the associated parallelogram

If P and Q are two points on the plane, and if

$$\frac{1}{2}(\overline{OP} + \overline{OQ}) = \overline{OR},$$

then the point R is the midpoint of PQ .

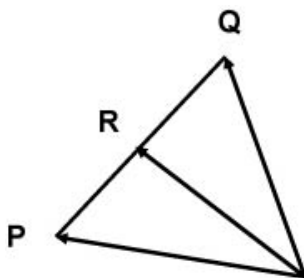


FIGURE 9. Midpoint is the average of two vectors

Example 3. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points on the plane. If the point $R = (x_3, y_3)$ lies on the line segment PQ , such that $PR : RQ = m : n$, prove that

$$x_3 = \frac{m}{m+n}x_2 + \frac{n}{m+n}x_1$$

and

$$y_3 = \frac{m}{m+n}y_2 + \frac{n}{m+n}y_1.$$

Proof. The fact that $PR : RQ = m : n$ may be equivalently stated as $PR : PQ = m : (m+n)$. Expressed in terms of vectors:

$$\overline{PR} = \frac{m}{m+n}\overline{PQ}.$$

Hence,

$$\begin{aligned} \overline{OR} - \overline{OP} &= \frac{m}{m+n}(\overline{OQ} - \overline{OP}) \\ \Rightarrow (m+n)\overline{OR} &= m\overline{OQ} + n\overline{OP} \\ \Rightarrow \overline{OR} &= \frac{m}{m+n}\overline{OQ} + \frac{n}{m+n}\overline{OP}. \end{aligned}$$

This implies the equations for x_3 and y_3 . □

The *dot product* \bullet of two vectors (x_1, y_1) and (x_2, y_2) is

$$(x_1, y_1) \bullet (x_2, y_2) = x_1x_2 + y_1y_2.$$

The *length* of a vector \mathbf{u} is the quantity

$$|\mathbf{u}| := \sqrt{\mathbf{u} \bullet \mathbf{u}}.$$

Notice that $\sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{u_1^2 + u_2^2}$. Hence if $\mathbf{u} = \overline{OP}$ and $P = (u_1, u_2)$, then the length of the vector \mathbf{u} is indeed the length of the line segment \overline{OP} .

Example 4. Given that $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, 4)$. Find the following quantities:

- (1) $\mathbf{u} \bullet \mathbf{v}$
- (2) $|\mathbf{u}|$
- (3) $|\mathbf{v}|$

The following propositions list the salient properties of the dot product.

Proposition 1. For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in \mathbb{R}^2 and any real number λ , the following hold:

- (1) $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
- (2) $\mathbf{u} \bullet \mathbf{0} = 0$
- (3) $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
- (4) $(\lambda\mathbf{u}) \bullet \mathbf{v} = \lambda(\mathbf{u} \bullet \mathbf{v})$

Proof. The justification is straightforward. We illustrate with one example: Equality 1 -

$$\mathbf{u} \bullet \mathbf{v} = (u_1, u_2) \bullet (v_1, v_2) = u_1v_1 + u_2v_2 = v_1u_1 + v_2u_2 = \mathbf{v} \bullet \mathbf{u}.$$

□

Example 5. Prove the remaining parts of Proposition 1.

Proposition 2. Suppose \overline{OP} is denoted by \mathbf{u} , \overline{OQ} by \mathbf{v} and the angle POQ is θ . Then

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta.$$

Corollary 1. Suppose P, Q are two points on the plane. Then $OP \perp OQ$ if and only if $\overline{OP} \bullet \overline{OQ} = 0$. (Remark: We regard the zero vector to be perpendicular to any other vector.)

Proof. From the proposition,

$$\begin{aligned} \mathbf{u} \bullet \mathbf{v} &= 0 \\ \Leftrightarrow |\mathbf{u}||\mathbf{v}| \cos \theta &= 0 \\ \Leftrightarrow \mathbf{u} = \mathbf{0}, \text{ or } \mathbf{v} = \mathbf{0}, \text{ or } \cos \theta &= 0. \end{aligned}$$

□

Example 6. Given that the slope of a line is $m \neq 0$. Show that the slope of another line that is perpendicular to the first is $-\frac{1}{m}$.

Proof. The vector representing the slope m is $(1, m)$. Suppose OP is perpendicular to OQ , where $P = (x, y)$ and $Q = (1, m)$. By the corollary,

$$(x, y) \bullet (1, m) = x + ym = 0.$$

This implies that

$$\frac{y}{x} = -\frac{1}{m}.$$

□

Proposition 3. Let \mathbf{u} and \mathbf{v} be two vectors. The projection of \mathbf{v} in \mathbf{u} is the vector

$$\text{Pr}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}} \mathbf{v}.$$

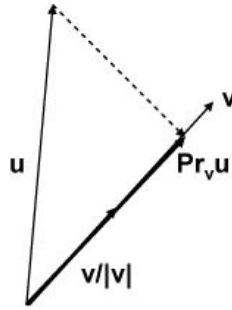


FIGURE 10. Projection of a vector in another

Proof. The unit vector in the direction of \mathbf{v} is $\frac{\mathbf{v}}{|\mathbf{v}|}$.

The projection of the vector \mathbf{u} in the direction of \mathbf{v} has length

$$|\mathbf{u}| \cos \theta.$$

Hence, the vector projection is

$$|\mathbf{u}| \cos \theta \frac{\mathbf{v}}{|\mathbf{v}|} = |\mathbf{u}| |\mathbf{v}| \cos \theta \frac{\mathbf{v}}{|\mathbf{v}|^2} = \mathbf{u} \bullet \mathbf{v} \frac{\mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}.$$

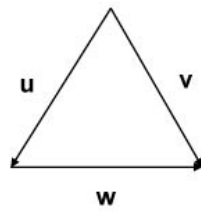
□

Example 7. Determine whether the following vectors are parallel, perpendicular or neither.

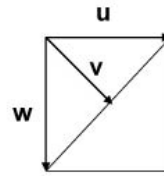
- (1) $(1, 2), (7, 14)$
- (2) $(2, 1), (7, -14)$
- (3) $(2, 1), (7, 14)$

Example 8. Determine if the triangle ABC is right-angled, where $A = (1, 2), B = (3, 4), C = (4, 3)$.

Example 9. If \mathbf{u} is a unit vector, find $\mathbf{u} \bullet \mathbf{v}$ and $\mathbf{u} \bullet \mathbf{w}$.



(A)



(B)

Example 10. Prove the parallelogram law:

$$|\mathbf{u} - \mathbf{v}|^2 + |\mathbf{u} + \mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2.$$