BASIC FACTS ABOUT DIFFERENTIATION - PART 1

Learning Objectives:

- Be familiar with the basic rules to manipulate derivatives and partial derivatives, particularly, the Chain Rule
- Theorems for differentiable functions on $[a, b]$: Rolle's Theorem and Mean Value Theorem

1. SOME USEFUL THEOREMS ON THE DERIVATIVE

First we write down the theorems spelling out the relationship between the derivative and the arithmetic operations.

Theorem 1. *Let I be an open interval and $a \in I$. Let $f, g : I \to \mathbb{R}$ be differentiable at a.*

1. *The sum function $f + g$ is differentiable at a and*
$$(f+g)'(a) = f'(a) + g'(a).$$
2. *The product function fg is differentiable at a and*
$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$
3. *If $g(x) \neq 0$ for all $x \in I$, then the reciprocal function $\frac{1}{g}$ is differentiable at a and*
$$(\frac{1}{g})'(a) = -\frac{g'(a)}{g(a)^2}.$$

Example 1. *A polynomial function $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : \mathbb{R} \to \mathbb{R}$ is differentiable. This follows from Example 5 and the prior Theorem. In fact,*

$$\begin{aligned} f'(x) &= \frac{d}{dx}\big(a_0\big) + \frac{d}{dx}\big(a_1x\big) + \frac{d}{dx}\big(a_2x^2\big) + \cdots + \frac{d}{dx}\big(a_nx^n\big) \\ &= a_1\frac{d}{dx}\big(x\big) + a_2\frac{d}{dx}\big(x^2\big) + \cdots + a_n\frac{d}{dx}\big(x^n\big) \\ &= a_1 + 2a_2x + \cdots + na_nx^{n-1}. \end{aligned}$$

Example 2. *Prove, using Theorem 1, the 'quotient rule':*

$$\frac{d}{dx}\big(\frac{f}{g}(x)\big) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

The following theorem concerns how differentiation behaves with inverse functions.

Theorem 2. *Assume the following:*

1. $x_0 \in (a,b)$
2. $f : (a,b) \to \mathbb{R}$ *is strictly monotone and continuous*
3. f *is differentiable at* x_0
4. $f'(x_0) \neq 0$

Denote the image of (a,b) *under* f *by* J.

Then the inverse function $f^{-1} : J \to \mathbb{R}$ *is differentiable at* $y_0 = f(x_0)$ *and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

If we write $y = f(x)$, then the theorem can be written suggestively as

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Example 3. *Let* n *be a natural number. Show that*

$$\frac{d}{dx}x^{1/n} = \frac{1}{n}x^{1/n-1}$$

for all $x > 0$.

Solution:

Let $y = x^{1/n}$. Then $y^n = x$. We have

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{ny^{n-1}} = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{1/n-1}.$$

Example 4. *The function* $f : (0,\pi/2) \to (0,1)$ *with* $f(x) = \sin x$ *has an inverse* $g : (0,1) \to (0,\pi/2)$ *with* $g(y) = \sin^{-1} y$. *Find* $g'(y)$.

The following theorem concerns how differentiation behaves with composition.

Theorem 3 (Chain Rule)**.** *Let I be an open interval and $a \in I$. Let $f : I \to \mathbb{R}$ be a function differentiable at a. Let J be an open interval containing $f(a)$ and assume that $f(I) \subset J$. Let $g : J \to \mathbb{R}$ be a function differentiable at $f(a)$. Then the composition $g \circ f$ is differentiable and*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Example 5. *Show that*

$$\frac{d}{dx} \sin(\sin x) = (\cos \sin x) \cos x.$$

Proof. Let $f(x) = g(x) = \sin x$. By the Chain Rule,

$$\frac{d}{dx} \sin(\sin x) = (g \circ f)'(x) = g'(f(x))f'(x) = (\cos \sin x) \cos x.$$

□

Example 6. *Find*

$$\frac{d}{dx} \sin(1 + x^2).$$

The next two theorems are qualitative statements concerning differentiable functions, unlike the prior computational ones.

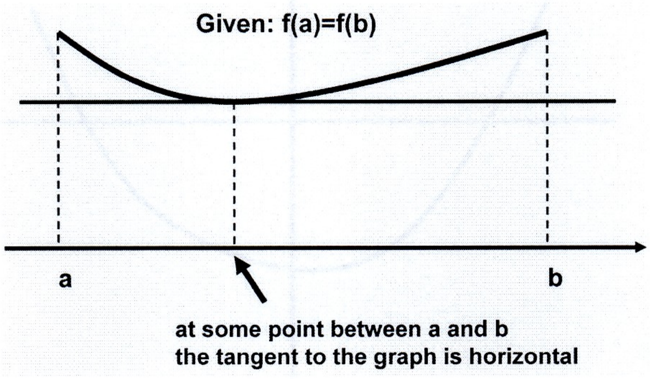

Figure 1. Rolle's Theorem

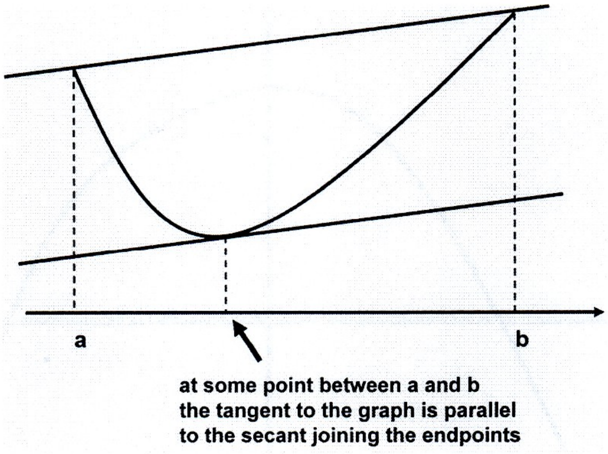

Figure 2. Mean Value Theorem

Theorem 4 (Rolle)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous and the its restriction to (a,b) is differentiable. If $f(a) = f(b)$, then there is some $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. As $f(a) = f(b)$, one of the following possibilities must hold:

(1) The function f is a constant
(2) There is some point in (a,b) at which f is less than $f(a)$
(3) There is some point at (a,b) at which f is greater than $f(a)$

In Case 1, $f' \equiv 0$.

By the Extreme Value Theorem, there is some point $c \in (a,b)$ at which f attains its minimum in Case 2 and maximum in Case 3.

At a maximum point, we have $f'(c) = 0$. (Strictly speaking, this needs to be justified.) ☐

Theorem 5 (Mean Value Theorem)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous and the its restriction to (a,b) is differentiable. There is some $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. To apply Rolle's Theorem, we need to move one end down (or up).

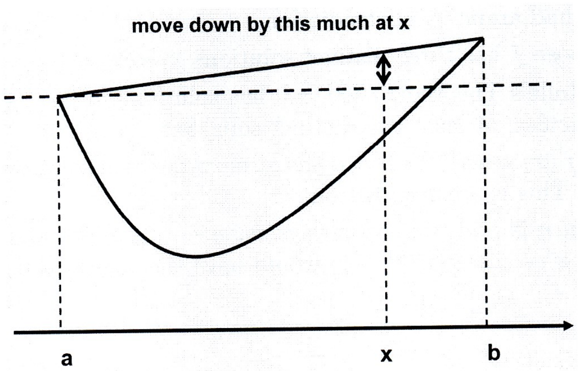

FIGURE 3. Shifting the graph

The slope of the secant is $\frac{f(b)-f(a)}{b-a}$.

At $x = a$, we want to shift by 0. Hence at $x \in [a, b]$, we want to shift by

$$\frac{f(b) - f(a)}{b - a}(x - a).$$

So, let's define

$$g(x) := f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

We have $g(a) = f(a) = g(b)$.

Hence, the function g satisfies the conditions in Rolle's Theorem, which then implies that there is some $c \in (a, b)$ such that $g'(c) = 0$. In other words,

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and the proof is done.

$\square$

Example 7. *For any numbers a and b and an even number n, show that the following equation has at most two distinct solutions:*

$$x^n + ax + b = 0, \quad x \in \mathbb{R}.$$

Is it true if n is odd?

Proof. Let $f(x) = x^n + ax + b : \mathbb{R} \to \mathbb{R}$.

We shall argue by contradiction.

Suppose f has three distinct solutions $x_1 < x_2 < x_3$.

By Rolle's Theorem, $f'(x) = 0$ has solutions in $[x_1, x_2]$ and $[x_2, x_3]$, in particular, at least two distinct solutions.

But $f'(x) = nx^{n-1} + a = 0$ has at most one real solution since $(n-1)$ is odd. This is a contradiction.

When n is odd, we can always set $a = -1$, $b = 0$ and the function $f(x) = x^n - x = x(x^{n-1} - 1)$ would have the solutions $0, \pm 1$. $\square$

2. THE CHAIN RULE

Since partial derivatives are defined componentwise, as if the other variables are held constant, all the usual rules of differentiation in the calculus of one variable: linearity of partial differentiation, product rule, quotient rule, etc. continues to hold.

Nevertheless the **chain rule** in the higher dimensions deserves a closer look. This rule tells us how to find the partial derivatives of the composition of functions.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$, $g, h : \mathbb{R} \to \mathbb{R}$ and $G, H : \mathbb{R}^3 \to \mathbb{R}$ are all differentiable functions. Then the chain rule says that

$$\frac{d}{dt} f(g(t), h(t)) = \frac{\partial f}{\partial x} g'(t) + \frac{\partial f}{\partial y} h'(t).$$

$$\frac{\partial}{\partial x} g(f(x, y)) = g'(f(x, y)) \frac{\partial f}{\partial x}.$$

$$\frac{\partial}{\partial Y} f(G(X, Y, Z), H(X, Y, Z)) = \frac{\partial f}{\partial x}\frac{\partial G}{\partial Y} + \frac{\partial f}{\partial y}\frac{\partial H}{\partial Y}.$$

In general, let $f = f(y_1, y_2, \ldots, y_n) : \mathbb{R}^n \to \mathbb{R}$ and $g_i = g_i(x_1, x_2, \ldots, x_m) : \mathbb{R}^m \to \mathbb{R}$. Let the function $F : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\begin{aligned} &F(x_1, x_2, \ldots, x_m) \\ &= f(g_1(x_1, x_2, \ldots, x_m), g_2(x_1, x_2, \ldots, x_m), \ldots, g_n(x_1, x_2, \ldots, x_m)). \end{aligned}$$

Then for each $i = 1, 2, \ldots, m$, we have

$$\frac{\partial F}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}\frac{\partial g_j}{\partial x_i}.$$

Example 8. *If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f are differentiable, show that g satisfies the equation*

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

Proof. Write $f = f(x, y)$. We have

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial}{\partial s}(s^2 - t^2) + \frac{\partial f}{\partial y}\frac{\partial}{\partial s}(t^2 - s^2) = 2s\frac{\partial f}{\partial x} - 2s\frac{\partial f}{\partial y},$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial}{\partial t}(s^2 - t^2) + \frac{\partial f}{\partial y}\frac{\partial}{\partial t}(t^2 - s^2) = -2t\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y},$$

hence

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

□

Example 9. *Suppose* $R(u, v, w) = \log(u^2 + v^2 + w^2)$, $u = x + 2y$, $v = 2x - y$, $w = 2xy$, *find* $\frac{\partial R}{\partial x}(1, 1)$ *and* $\frac{\partial R}{\partial y}(1, 1)$.

Example 10. *Suppose* $u = u(x, y)$ *is infinitely differentiable,* $x = e^s \cos t$, $y = e^s \sin t$. *Show that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right).$$

Example 11. *Suppose* f *is infinitely differentiable and homogeneous of degree* $n \in \mathbb{N}$, *i.e.* $f(tx, ty) = t^n f(x, y)$. *Show that*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$