# LIMIT AND CONTINUITY OF FUNCTIONS - PART 1

Learning Objectives:

- Understand the concept of limit for single variable functions

## 1. LIMITS OF ONE-VARIABLE FUNCTIONS - PART 1

The following defines the limit of a function. Intuitively, the value of the function gets closer and closer to its limit as $x$ approaches the point under consideration.

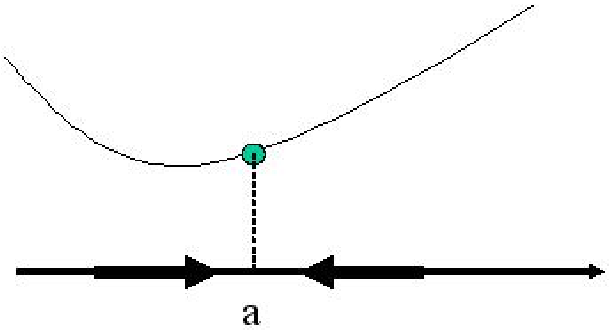


FIGURE 1. Limit of a function

**Definition 1.** *Suppose $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ is a function on $D$. We say that $f$ tends to $L$ as $x$ tends to $a$ if the following criterion is satisfied: for each $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$|f(x) - L| < \epsilon$$

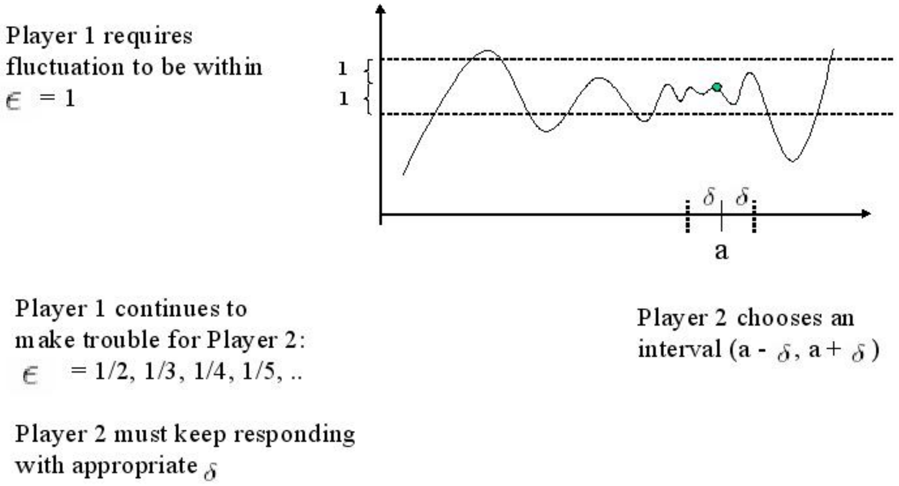
*whenever*

$$0 < |x - a| < \delta.$$

*In short, we write: $f(x) \to L$ as $x \to a$.*

*We may also write:*

$$\lim_{x \to a} f(x) = L.$$



Player 1 requires fluctuation to be within $\epsilon$ = 1

Player 1 continues to make trouble for Player 2: $\epsilon$ = 1/2, 1/3, 1/4, 1/5, ..

Player 2 must keep responding with appropriate $\delta$

Player 2 chooses an interval (a - $\delta$, a + $\delta$)

Figure 2. Game interpretation of the definition of limit

**Example 1.** *Let* $f(x) = 2x + 5$, $a = 1$, $\epsilon = 0.01$. *Given that* $L = 7$. *Find an appropriate* $\delta$ *to satisfy the requirement that the limit of* $f$ *at* 1 *is* 7.

*We require:*

$$|2x - 2| = |2x + 5 - 7| = |f(x) - L| < 0.01 = \epsilon.$$

*This is equivalent to:*

$$|x - 1| < 0.01/2 = 0.005.$$

*Hence, we may choose* $\delta = 0.005$.

**Example 2.** *Let* $f(x) = \sqrt{2x+1}$, $a = 4$, $\epsilon = 0.1$. *Given that* $L = 3$. *Find an appropriate* $\delta$ *to satisfy the requirement that the limit of* $f$ *at* 4 *is* 3.

*We require:*

$$|\sqrt{2x+1} - 3| = |f(x) - L| < 0.1 = \epsilon.$$

*Multiplying both sides by* $\sqrt{2x+1} + 3$, *the above is equivalent to:*

$$|2x + 1 - 9| = |\sqrt{2x+1} - 3||\sqrt{2x+1} + 3| < 0.1|\sqrt{2x+1} + 3|.$$

*This is equivalent to:*

$$|2x - 8| < 0.1|\sqrt{2x+1} + 3|,$$

*or*

$$|x - 4| < 0.05|\sqrt{2x+1} + 3|.$$

*In other words, this last inequality must be satisfied in order the requirement is met.*

*Now, if we impose:*

$$|x - 4| < 0.15 = 0.05 \times 3,$$

*the last inequality would be satisfied:*

$$|x - 4| < 0.05 \times 3 < 0.05|\sqrt{2x+1} + 3|.$$

*Hence, an appropriate choice is* $\delta = 0.15$.

**Example 3.** *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 0 & \textit{if } x \neq 0 \\ 1 & \textit{if } x = 0. \end{cases}$$

*Find the limit of $f$ as $x$ approaches 0.*

Here is another way to define the limit of a function, using the notion of sequences:

**Proposition 1.** *If for every sequence $a_1, a_2, a_3, \ldots$ with $a_n \neq a$ for all indices $n$ and which converges to $a$, we know that the sequence $f(a_1), f(a_2), f(a_3), \ldots$ converges to the same limit $L$, then*

$$\lim_{x \to a} f(x) = L.$$

*The converse is also true.*

*Proof.* Suppose $\lim_{x \to a} f(x) = L$.

Let $a_1, a_2, a_3, \ldots$ be a sequence such that $a_n \neq a$ for all indices $n$ and which converges to $a$.

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that whenever $x \in (a - \delta, a + \delta) \backslash \{a\}$, we have

$$|f(x) - L| < \epsilon.$$

For that $\delta$, there exists an index $N$ such that whenever $n \geq N$, we have

$$|a_n - a| < \delta.$$

This implies that

$$|f(a_n) - L| < \epsilon$$

whenever $n \geq N$.

Now assume that it is false that the limit of $f$ at $a$ is L, i.e. $\lim_{x \to a} f(x) \neq L$.

This implies that there exists some $\epsilon > 0$, such that for all $\delta > 0$, there exists some $\alpha \in (a-\delta, a+\delta)\backslash\{a\}$ for which

$$|f(\alpha) - L| \geq \epsilon.$$

For each $\delta = 1, 1/2, 1/3, \ldots$, pick the corresponding $\alpha_1, \alpha_2, \alpha_3, \ldots$.

Then the sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ converges to $a$ while

$$|f(\alpha_n) - L| \geq \epsilon.$$

□

Very often, we may find the limit without having to go through the definition.

**Example 4.** *Find* $\lim_{x\to 3} \frac{x^2-9}{x+3}$.

**Example 5.** *Find* $\lim_{x\to 0} x\sin(\frac{1}{x})$.

**Example 6.** *Find* $\lim_{x\to 1} \frac{x-1}{\sqrt[3]{x}-1}$.