BASIC MATHEMATICS - PART 2

Learning Objectives:

- Appreciate the fact that the set of real numbers is suitable for real analysis because it is dense and complete
- Understand the notions of infimum and supremum

Suggested Exercises:

(1) 4.1
(2) 4.2

1. Numbers

We may classify numbers into the following sets:

- Natural numbers: $\mathbb{N}$
- Integers: $\mathbb{Z}$
- Rational numbers: $\mathbb{Q}$
- Real numbers: $\mathbb{R}$
- Complex numbers: $\mathbb{C}$

As we go down the above list, the structure of the numbers becomes richer.

From $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$, the arithmetic operations become *closed*.

- $\mathbb{N}$ is not closed under $-$. The additive closure of $\mathbb{N}$ is $\mathbb{Z}$
- $\mathbb{Z}$ is not closed under $\div$. The multiplicative closure of $\mathbb{Z}$ is $\mathbb{Q}$

From $\mathbb{Q}$ to $\mathbb{R}$ to $\mathbb{C}$, the enrichment is *completion*.

- The set of rational numbers consists of all numbers of the form
$$a_n a_{n-1} \cdots a_1 . a_{-1} a_{-2} \cdots a_{-m} \overline{b_1 b_2 \cdots b_p}.$$
The set of real numbers enlarges this set by including all decimal expansions as being valid.
- The set of complex numbers is obtained from the real numbers and the imaginary number i: any complex number is of the form $a + bi$. The remarkable *Fundamental Theorem of Algebra* states that any polynomial equation
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$
with $a_n \neq 0$ has a complex number solution.

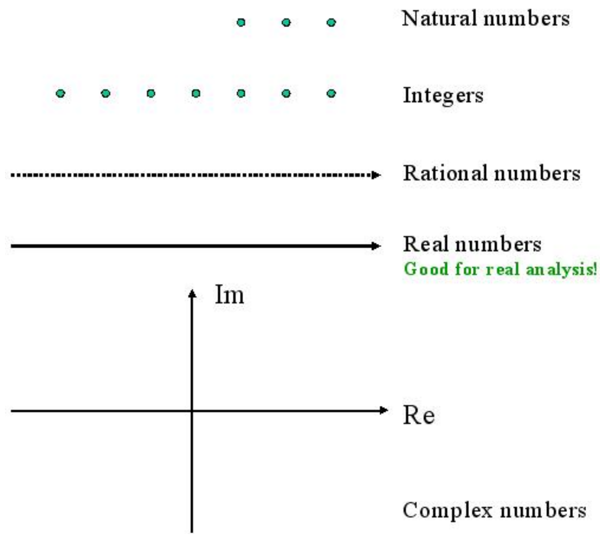

FIGURE 1. The common sets of numbers in mathematics

For our study of real analysis, real numbers are the most important.

This is because $\mathbb{R}$ possesses certain good properties that $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ don't. Intuitively: there isn't enough rational numbers to draw a continuous curve on the plane - there are too many 'gaps' among the rational numbers.

The good properties of $\mathbb{R}$ that will allow us to 'draw continuous curves' are:

- density
- completeness

Let us first define the notion of density.

Definition 1. *A subset A of real numbers is said to be dense if the following property holds: for every pair $x, y \in A$, there exists $z \in A$ such that*

$$z \in (x, y).$$

Example 1. *Which of the following subsets of $\mathbb{R}$ are dense?*

(1) $\mathbb{N}$
(2) $\mathbb{Z}$
(3) $\mathbb{Q}$
(4) $\mathbb{R}$

The notion of completeness is harder to grasp. It is done so in several steps below.

Definition 2. *Let A be a subset of $\mathbb{R}$. A number $z \in \mathbb{R}$ is said to be an upper bound of A if the following condition holds: for all $x \in A$, we have*

$$x \leq z.$$

A number $y \in \mathbb{R}$ is said to be a lower bound if the following condition holds: for all $x \in A$, we have

$$y \leq x.$$

If A has at least one upper bound, it is said to be bounded above.

If A has at least one lower bound, it is said to be bounded below.

Example 2. *Give a lower bound and/or an upper bound to the following sets if they exist:*

(1) $\mathbb{N}$
(2) $\mathbb{Q}$
(3) $[1, 2]$
(4) $(1, 2)$

Definition 3. *A subset A of real numbers is said to be complete if the following property holds: for every subset $B \subseteq A$ that is bounded above, the set*

$$M(B) := \{x \in A : x \text{ is an upper bound of } B\}$$

has a minimum element.

Example 3. *Does each of the following sets have a maximum/minimum element?*

(1) $\mathbb{R}$
(2) $(0, 1]$
(3) $[0, 1)$
(4) $(0, 1)$

Example 4. *Let* $A = \mathbb{Q}$. *Let*

$$B = \{1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421,\ 1.414214, \ldots\}$$

consist of the truncated decimal expansions of $\sqrt{2}$. *Then*

$$M(B) = \mathbb{Q} \cap [\sqrt{2}, \infty).$$

We will see later that $\sqrt{2}$ *is not a rational number. Hence the set* $M(B)$ *does not have a minimum and* $\mathbb{Q}$ *is not complete,*

Theorem 1 (Completeness of $\mathbb{R}$)**.** *The set of real numbers* $\mathbb{R}$ *is complete.*

What this entails is that on $\mathbb{R}$, we may coin the following concepts:

Definition 4. *The supremum of* S, *denoted by* $\sup S$ *is defined as follows:*

- *If* $S = \emptyset$, *then* $\sup S = -\infty$.
- *If* S *is not bounded above, then* $\sup S = \infty$.
- *If* S *is bounded above, then* $\sup S$ *is the minimum among all the upper bounds of* S.

Example 5. *What are they?*

(1) $\sup \emptyset$
(2) $\sup[0, 1]$
(3) $\sup(0, 1)$
(4) $\sup \mathbb{N}$
(5) $\sup\{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{2^2}, \ldots\}$

Definition 5. *The infimum of S, denoted by* $\inf S$ *is defined as follows:*

- *If $S = \emptyset$, then* $\inf S = \infty$.
- *If S is not bounded below, then* $\inf S = -\infty$.
- *If S is bounded above, then* $\inf S$ *is the maximum among all the lower bounds of S.*

Example 6. *What are they?*

(1) $\inf \emptyset$
(2) $\inf[0, 1]$
(3) $\inf(0, 1)$
(4) $\inf \mathbb{N}$
(5) $\inf\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

The concepts of maximum/minimum and supremum/infimum are concerned with pinpointing the biggest/smallest elements of sets. The peculiar property of real numbers forces us to make this distinction between the the two concepts. The technical advantage of the concept of supremum/infimum over that of maximum/minimum is that the former *always exists* and *coincides* with the latter when it exists.

We left as issue unsettled from before.

Proposition 1. *The number $\sqrt{2}$ is not rational.*

Proof. Suppose on the contrary that

$$\sqrt{2} = \frac{p}{q}$$

is rational. We may as well assume that p and q do not have any common factor other than 1.

Rearranging the equation,

$$2q^2 = p^2.$$

Since the LHS is even, so must the RHS. This implies that

$$p = 2k$$

is even.

Plugging into the above,

$$2q^2 = 4k^2.$$

This implies that

$$q^2 = 2k^2.$$

Now the RHS is even. This implies that

$$q = 2l$$

is even.

But this contradicts our assumption that p and q do not have any common factor apart from 1. □

The proof technique here is called *proof by contradiction*. There are several techniques of proof in mathematics such as *direct proof* and *proof by cases*.