EVALUATION OF MOMENT RISK: CAN THE SHARPE RATIO MAKE THE CUT?[1]

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Abstract

Traditional tests of financial risk for optimal portfolio choice based on Sharpe ratios are inherently ensconsed in the normality assumption of the return distribution besides independence. Such tests are not strictly valid for financial data that are known to be leptokurtic, and often show persistence in levels or volatility. We propose a smooth total moment risk measure with directional components that address the drawbacks of such procedures for practical implementation and inference. Our illustration of the proposed test on hedge fund indices with other existing measures show promising future for the new risk measure that has known tabulated distributions.

Keywords: Sharpe Ratio, smooth test, score test, higher-order moments, dependence test, hedge-funds

1 Introduction

Financial risk assessment exercises and inference based on parametric measures like the Sharpe Ratio and Mean-Variance analysis ignore higher order moments of the return distribution, and possibly a non-linear structure (Agarwal and Naik, 2004, see Fama, 1970 or Campbell, Lo and MacKinlay, 1997, for review). Traditional tests for optimal portfolio choice like those on Sharpe ratios are inherently dependent on the normality assumption of the return distribution besides independence of the sample drawn (Sharpe, 1966, 1994, Jobson and Korkie, 1980, Memmel, 2003). Hence, such tests are not strictly valid for financial data that are known to be leptokurtic (heavier tails than the normal distribution), and for time series of data that often show persistence in volatility (e.g. stocks and mutual funds) or in levels (e.g. hedge funds, see Getmansky, 2004, Getmansky, Lo and Makarov, 2003). It has been suggested that bootstrap or resampling based tests on robust measures of (Studentized) Sharpe ratio can address the problem of leptokurtosis and dependent structure using Heteroscedasticity and Autocorrelation Consistent (or HAC) type estimators (Andrews, 1991, Ledoit and Wolf, 2008).

We can identify at least four drawbacks of such procedure for practical implementation besides the computational complexity. First, "...for certain applications the Sharpe ratio is not the most appropriate performance measure; e.g. when returns are far from normally distributed or autocorrelated..." (Ledoit and Wolf, 2008, p. 851, see Getmansky, 2004). Second, it is well established that bootstrap-based methods might not capture the true dependent structure of the return distribution that can be obtained by a reasonably "close" parametric specification or for certain limited dependent variable (truncated or censored) distributions (see Hall, P., Horowitz, J., L. and Jing, B., Y., 1995). Third, tests based purely on the function of the first two moments like the Sharpe Ratio fail to account for restrictions or differences in higher order moments (like model selection including the assumption of normality) jointly. Tests based purely on comparing two Sharpe ratios of competing asset classes might ignore variations in higher order moments that among other things constitute the estimation error of the Sharpe ratios. Finally, as a tool for financial risk assessment Sharpe ratios are estimated based on past data to forecast future risk adjusted returns.

A graphical test of Density Forecast Evaluation using probability integral transforms discussed by in Diebold, Gunther and Tay (1998) was formalized analytically in Ghosh and Bera (2006) as a variant of Neyman's smooth test for parametric models. They explicitly looked at the dependent structure of the model besides the fat tails to explore model selection issues along with testing. It has been empirically observed that although financial returns data of stocks and mutual funds do rarely show persistence or autocorrelation in levels, but they do often show persistence in higher order moments like volatility. On the other hand, hedge funds and private equity funds tend to show some persistence in levels as well (Lo, 2001, Brooks and Kat 2002; Agarwal, V., and N.Y. Naik, 2004, Malkiel and Saha 2005; Getmansky, 2004, Getmansky, Lo, and Makarov, 2003, Kalpan and Schoar 2005; Ghosh, 2008).

If we have the return data given by $R_1, R_2, ..., R_T$ then the population Sharpe ratio is

$$SR = (\mu_R - R_f) / \sigma_R \tag{1}$$

where μ_R, σ_R^2 and R_f are the population mean, population variance of the Return distribution and the existing risk free rate, respectively. The corresponding sample counterpart or the estimated Sharpe ratio is

$$\widehat{SR} = (\hat{\mu}_R - R_f) / \hat{\sigma}_R \tag{2}$$

where $\hat{\mu}_R = \frac{1}{T} \sum_{t=1}^{T} R_t$ and $\hat{\sigma}_R^2 = \frac{1}{T-1} \sum_{t=1}^{T} (R_t - \hat{\mu}_R)^2$ are the unbiased sample mean and variance estimates. We observe that if we assume the data to be independent and identically normally distributed then we can test the hypothesis $H_0 : \mu_R = R_f$ against $H_1 : \mu_R \neq R_f$, the test statistic is

$$tstat = \frac{(\hat{\mu}_R - R_f)}{\hat{\sigma}_R / \sqrt{T}} = \sqrt{T} \frac{(\hat{\mu}_R - R_f)}{\hat{\sigma}_R} = \sqrt{T}\widehat{SR}$$

where $\hat{\sigma}_R^2 = \frac{1}{T-1} \sum_{t=1}^{T} (R_t - \hat{\mu}_R)^2$ is an unbiased estimator of the population variance. Incidentally, the distribution of $\widehat{SR} = (\hat{\mu}_R - R_f) / \hat{\sigma}_R = \frac{tstat}{\sqrt{T}}$ is nothing new, in fact, it was first proposed by Student (1908) himself, and only later Fisher (1925) formulated the test statistic tstat and defined the Student's t distribution with $(T - 1)$ degrees of freedom. However, this test is crucially dependent on the parametric assumption that the underlying distribution is normal, and that the data in independently and identically distributed.

There are several objectives of the current paper. First we explore the incredible and most definitely incomplete list of competing risk adjusted performance (RAP) measures. We identify their similarities and differences, and if possible, statistical properties.

Second, as part of this project we would explore the probability distribution of a proposed measure of risk adjusted returns when estimated from a return distribution based on the smooth test methodology. Being a score test such tests could be modified to be invariant to non-linear transformations of moments like the Sharpe ratio.

Our third objective in this project is to relax the normality assumption and propose a test based on the Sharpe ratio that is robust to violations of the *iid* assumptions under general weak dependence through mixing conditions, and testing them jointly (see White, 1984). Our proposed score test that would address the leptokurtic and time series dependent structure not explicitly addressed in previous literature (see Leung and Wong, 2007, Ledoit and Wolf, 2008).

Finally, we look at the hedge fund indices proposed in Diez de Los Rios and Garcia (2008) and test for equity market neutrality and sensitivity to the market and global hedge fund indeices Patton (2009). We also compare the nature of other hedge fund strategies based on the proposed *smooth moment risk* measures (SMR) incorporating dependence.

The rest of the paper is arranged as follows. In Section 2 we review a list of RAP measures and compare and contrast them. We setup a time series framework in Section 3, and apply theorems proposed by Rosenblatt (1952) and Ghosh and Bera (2005), to propose a series of tests that are themselves test statistics with tabulated distributions. We explore an empircal example of hedge fund indices with standard risk measures and evaluate the effectiveness of the Sharpe ratio in Section 4. We conclude in Section 5 and provide a proof of a theorem proposed in Ghosh and Bera 2005, and give cases as examples of dependence functions as illustrations in Appendices 5.0.1, 5.0.2 and 5.0.3.

2 Review of Risk Adjusted Performance Measures

Alternative investments like hedge funds suffer from severe information assymetry as they are usually not under the purview of regulatory bodies like the Association of Investment Management and Research (AIMR) and compliance with AIMR-Portfolio Presentation Standards (AIMR-PMS) and more recently instituted Global Investment Performance Standards (GIPS) that since early 1990's are aimed to protect individual investors against predataory practices. Since Alfred Winslow Jones formed the first hedge fund in 1949, he managed to operate in almost complete secrecy for 17 years. Nearly 50 years later LTCM (Long Term Capital Management) whose specatcular collapse and bailout brought the attention back to Hedge Fund operational secrecy and risk measures (Lhabitant 2006). However, we are just more brutally reminded the need for performance standards after Bernie Madoff's hedge fund, Ascot Partners turned out to be a 50 billion dollar Ponnzi scheme in 2008 (http://www.forbes.com/2008/12/12/madoff-ponzi-hedge-pf-ii-in_rl_1212croesus_inl.html).

Risk as a concept is often individual or target specific, application or theory specific, uncertainty or risk aversion specific, and measures of risk also reflects such dichotomies. This however leads to conflicts in ranking of portfolios by measures of riskiness, as the measures are often non-affine or non-linear transformation, or sometimes not even functions of each other. We provide here a brief description of the standard measures of risk and risk adjusted returns following Lhabitant (2006) and others.

In general, return of an individual asset in period t is composed of two parts gains and losses (Bernardo and Ledoit, 2000). So, in symbols we can write,

$$R_t = G_t I\left\{R_t \geq 0\right\} - L_t I\left\{R_t < 0\right\}, \tag{3}$$

where G_t and L_t are absolute values of gains and losses made by the fund in period t, respectively, and $I\left\{A\right\}$ is an indicator function that takes a value 1 when A has occurred. It is also worth noting that we cannot observe G_t in a period of loss and L_t in a period of gain. The average return $\bar{R}$ is the difference of the average of gains G_t which is left-censored below at $zero$ and average of losses L_t that is right-censored at $zero$. One measure to address this is to look the average gains $\bar{G}$ in periods of gains and average losses $\bar{L}$ in the periods of losses. The gain-to-loss ratio is average gains over average losses, $\bar{G}/\bar{L}$, is commonly used by fund managers. While it reflects the average amount of gains made in good times to the average losses in bad times, it however hides the number of periods of losses or gains. This is a naive measure of risk that does not reflect the overall riskiness of the portfolio. A very high (or low) value of gains-to-loss ratio does however reflect whether you expect some very high gains (or losses) in very few periods and might affect the volatility discussed later.

One common form of measure of risk is the volatility measure *Mean Absolute Deviation (MAD)* defined as

$$MAD = E\left|R_t - \bar{R}\right| = \frac{1}{T}\sum_{t=1}^{T}\left|R_t - \bar{R}\right|, \tag{4}$$

that measures the average L_1 distance from the mean return. From an optimization perspective, its better to use the deviation from median return than the mean as a measure of central tendency as MAD is minimized at the median. This particularly useful when there the data is skewed (like in many hedge funds) where mean and median are different. It is also effective when there extreme observations or returns, the median is not affected by movements in extreme tails unlike the mean.

A more common measure of volatility is the *sample variance* defined by

$$\hat{\sigma}_R^2 = E\left(R_t - \bar{R}\right)^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(R_t - \bar{R}\right)^2, \tag{5}$$

or it's positive square root $\hat{\sigma}_R$ termed as standard deviation (of the same unit as return) both of which are taken to be measures of absolute risk or volatility as it uses all the observations to arrive at the statistics. Variance offers lots of advantages as a measure of volatility (or volatility square) including in optimization, and in particular, under the assumption of normality together with the mean its forms a *sufficient statistic* (contains all possible available information) to characterize the entire distribution of returns.

This also brings us back to the assumption of normality, a symmeteric bell shaped distribution, that often is shown not to charecaterize the return distribution of financial assets in particular alternative assets like hedge funds. We can perform the celebrated Jarque-Bera (1983) test to verify normality before we embark on making inference based on the mean and volatility alone. Higher order moments like skewness and kurtosis seem to play a much more important role in determination of riskiness of a portfolio that just mean and variance as Markowitz have promulgated in his celebrated work (Markowitz, 1952).

The symmetric treatment of positive and negative deviations from a benchmark be it *zero* (see G_t and L_t in gain-to-loss ratio) or the mean (or median) return in standard deviation (or mean absolute deviation) makes more sense in symmetric distributions where both positive and negative are treated equally. The case for an asymmetric treatment of positive and negative returns have solid foundations from the standpoints of economic and statistical theory, empirical evidence besides behavioral finance.

First, in particular, hedge funds use dynamic trading strategies that are often assymetric like stop losses, actively managed leverage and options trading (Lhabitant 2006). Second, individual risk averse investors and institutions aspires to adopt investment strategies that essentially limit their downside risk be it from a benchmark or an average return. Volatility measures that we have discussed so far fails to take this into account. Finally, statistically return distribution show evidence of dispersion in higher order moments like skewness and kurtosis from the Gaussian distribution, which is completely identified by the first two moments. Hence, statistical inference based on normality will fail to differentiate the risk profile of individuals or institutions who have divergent higher order moments or will have very low or no power against such divergence. All these reasons lead risk managers delve into measures of downside risk discussed below.

Generalizing a concept of variance (or standard deviation) let's define a concept of *semi-variance* (or *semi-deviation*). Suppose we have a prespecified benchmark or target rate R^*,

$$Downside\ risk = \frac{1}{T}\sqrt{\sum_{t=1}^{T} d_t^2 I\left\{d_t < 0\right\}}, \tag{6}$$

where $d_t = R_t - R^*$ and $I\left\{d_t < 0\right\} = 1$ if $d_t < 0; = 0$ otherwise. When we replace R^* by the mean return we get the semi-deviation or *below-mean standard deviation* (Markowitz, 1959). As mentioned before, setting $R^* = 0$, makes $d_t = L_t$ discussed in the context of equation (3). On the other hand if R^* is replaced by a moving target like the treasury bill rate (risk free rate) or the returns to a benchmark like S&P 500, we get a *below-target semi-deviation* often of interest to institutional investors.

Other measures inspired by downside risk concerns include:

1. The *downside frequency* or the frequency of occurence below a target R^* (i.e., $\sum_{t=1}^{T} I\left\{d_t < 0\right\}$);

2. The *gain standard deviation* or standard deviation conditional of a gain period $\sqrt{\frac{1}{T_G-1}\sum_{t=1}^{T_G}\left(G_t - \bar{G}\right)^2}$, where $T_G = \sum_{t=1}^{T} I\left\{R_t \geq 0\right\}$, suppose we arrange the gains to be the first T_G period and remaining $T_L = T - T_G$ periods of losses;

3. The *loss standard deviation* or standard deviation conditional of a loss period $\sqrt{\frac{1}{T_L-1}\sum_{t=T_G+1}^{T}\left(L_t - \bar{L}\right)^2}$.

4. An estimated shortfall risk measure, the *shortfall probability* is defined with the target R^* as

$$\widehat{Risk} = P\left(\widehat{R_t < R^*}\right) = \frac{1}{T}\sum_{t=1}^{T} I\left\{d_t < 0\right\} = \frac{downside\ frequncy}{T}. \tag{7}$$

5. *Value-at-Risk* is defined at the maximum amount of capital that one can lose over a period of time say one month at a certain confidence level, say $100(p)\%$. In other words, its the $100\,(1-p)\,\%$ percentile of the distribution of profit and loss percentage distribution. In notations, it can be estimated by

$$\widehat{VaR_p} = \underset{R}{Min}\left\{R : P\left(\widehat{R_t \leq R}\right) \geq p\right\} = \underset{R}{Min}\left\{R : \sum_{t=1}^{T} I\left\{(R_t - R) < 0\right\} \geq Tp\right\}.$$

The calculations simplifies substantially if the original return distribution is normal, then using normal probabilities it is easy to find the $100\,(1-p)^{th}$ quantile, it is simply $VaR_p = \mu_R + \xi_{1-p}\sigma_R$, where $P\left(Z \leq \xi_{1-p}\right) = 1-p$.

6. Any period to period drop can be taken as a *drawdown statistic* during a holding period, however, a maximal loss in percentage terms over a period (highest minus the lowest) is called the *maximum drawdown*. Maximum drawdown is really the range of percentage returns over a period of time. In notations, we can derive as

$$\begin{aligned} Max.drawdown = \max\{&\max\left(G_t\right) + \max\left(L_t\right), \\ &\max(G_t) - \min(G_t), \max(L_t) - \min\left(L_t\right)\}. \end{aligned} \tag{8}$$

It is clear that the gain standard debiation looks at the conditional volatility of upside or gain period, while loss standard deviation measures conditional volaitility of downside. Despite the obvious advantages of adoption downside risk measures have a slow "penetration rate" among practitioners mainly due to assumptions of normality (then downside risk is just a constant times the volatility) and the variety of different options to choose from. Many of the other risk measures used are directly or indirectly related to the ones discussed. Even though it is a simple measure of probability of achieving a target, shortfall probability misses the expected value of the underperformance (i.e. expected shortfall) or its variation.

Value-at-Risk is one of the most commonly used (and possibly, abused) measure of risk management in the financial and banking industry, its quick interpretation does betray a sense of oversimplification, and possible pitfalls under conditions of extremes or rare events, and often non-normality. Besides, Value-at-Risk is not a "coherent measure of risk," as it is not sub-additive like any other quantiles (Artzner et al, 1999). That is to say, the value-at-risk associated with a combined portfolio of two assets might have a higher risk than individual ones. So an investor can reduce value-at-risk of a portfolio by simply holding several smaller portfolios.

Another example of a downside measure statistic is one of "disappointment aversion" from not selling high or known as *drawdown statistics* defined as the drop from the historical maximal points. The *duration of drawdown* or *recovery time* is the time taken to recover from a drawdown to come back to original level. Though it is not usually required some hedge fund managers voluntarily disclose their maximum drawdowns, it is a tangible and intuitive measure of regret felt by investors. There are two major concerns. First, because of smoothing longer the duration or measurement period there will be less severe "spikes" hence lower maximum drawdown. Second, if there is a longer series there will be more observations hence range i.e. the maximum drawdown will be larger. Hence care should be exercized before comparing such measures.

There are various statistics that are related to benchmarks chosen by investors listed below. They are related to performance capture (*capture indicator* is the average ratio of funds returns and benchmark's returns; *up capture indicator* is the ratio of funds average return and benchmark's returns in up periods; *down capture indicator* is the ratio of funds average return and benchmark's returns in down periods), conditional probabilities (*up number ratio* is the conditional probability that fund was up when benchmark was up; *down number ratio* is the conditional probability that fund was down when benchmark was down; *up percentage ratio* is the conditional probability that fund outperformed when benchmark was up; *down percentage ratio* is the conditional probability that fund outperformed when benchmark was down; ratio of negative months over total months gives the unconditional probability of getting a negative month) and odds ratio (*percent gain ratio* is the ratio of periods fund was up over benchmark was up).

Finally, a measure of systematic (according to CAPM) risk is the *market beta* which reflects the sensitivity of the fund with respect to the market index. This provides a measure of systematic risk.

Tracking Error is a commonly used measure of fidelity to a benchmark defined by

$$TE_{Diff} = \frac{1}{T}\sum_{t=1}^{T}(R_t - R_t^*) \text{ or } TE_{MAD} = \frac{1}{T-1}\sum_{t=1}^{T}|R_t - R_t^*|, \qquad (9)$$

where there is a time varying benchmark R_t^*, could be just a simple mean difference or a standard or mean absolute deviation form (Rudolf et al 1999).

Two funds with same mean return and different risk characterestics are easy to compare, less risk is better. Similarly, for two funds with the same risk but different mean return, more return is better. However, the problem comes in when an investor has to decide between two funds, one with a lower value of both risk and return than the other. Risk adjusted performance measures (RAPM) helps an investor to incorporate the risk return tradeoff in the decision making process. However, alternate investment vehicles like hedge funds does share some common features and some peculiarities that help us choose some RAPMs more often than others (Sharma, 2004). While on one hand there is no "one measure fits all" RAPM for investment and risk profiles, there is usually a measure that addresses specific risk apetite and direction. The indispensability of the "entire" risk distribution is gradually sinking in.

Sharpe (1966) introduced the ratio, "excess return per unit of volatility" that has stood the test of time, defined by

$$SR_P = \frac{\mu_P - R_f}{\sigma_P},$$

and is still one of the most commonly used RAPM. The attractiveness of the Sharpe ratio stems from the "leverage" invariant measure, all funds with different portfolio weights would have the same Sharpe ratio but that possibly could be taken as a criticism as well. However, the Sharpe Ratio depends on the total risk and is not related to the market index (and hence the systematic risk related to the market) which might not be be well defined (Roll, 1977). Graphically, in the mean-volatility space, Sharpe ratio is the slope of the line joining teh risk free rate and the point representing the fund. However, there is a possibility there might not be any risk free rate, hence Sharpe (1994) generalized the definition to a benchmark portfolio return R_B, and defined the generalized version

$$\text{Information Ratio}_P = \frac{\mu_P - R_B}{TE_P} = \frac{\mu_P - R_B}{\sigma\left(R_P - R_B\right)},$$

where $TE_P = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(R_{Pt} - R_{Bt}\right)^2}$ is the tracking error that gives the original Sharpe ratio when $R_{B,t} = R_f$. It can be interpreted as a portfolio that takes a long position on the fund and a short position on the benchmark portfolio. Statistically, as seen before in (2), Sharpe Ratio can be interpreted as $tstat/\sqrt{T}$ for testing equality of the average return with the risk free rate, $H_0 : \mu_P = R_f$ vs. $H_1 : \mu_P \neq R_f$.

Sharpe ratio has been used to test between two portfolio using the method suggested by Jobson and Korkie (1981) who tested $H_0 : SR_1 = SR_2$ vs $H_1 : SR_1 \neq SR_2$

and used

$$Z = \frac{\sigma_1\mu_2 - \sigma_2\mu_1}{\sqrt{\theta}} \xrightarrow{d} N(0,1),$$

where the asympotic variance of the numerator is

$$\theta = \frac{1}{T}\left[2\sigma_1^2\sigma_1^2 - 2\sigma_1\sigma_2\sigma_{12} + \frac{1}{2}(\mu_1\sigma_2)^2 + \frac{1}{2}(\mu_2\sigma_1)^2 - \frac{\mu_1\mu_2}{\sigma_1\sigma_2}\sigma_{12}^2\right].$$

This however gives an asymptotic distribution that has low power for small samples, as Jorion (1985) noted at 5% level the power could be as low as 15%. One of the main problems in the test proposed by Jobson and Korkie (1981) is the assumption of normality that is often not entirely justified in financial asset returns in particular hedge funds.

Gibbons et. al. (1989) suggested a test for *ex-ante* portfolio efficiency using maximum Sharpe ratio as an estimator of risk adjusted returns. It has been used in the literature to test for the effect of additional assets to the universe. The test statistic is given

$$W = \left[\frac{\sqrt{1+SR_2^2}}{\sqrt{1+SR_1^2}}\right]^2 - 1 \equiv \psi^2 - 1,$$

where SR_2 is the ex-post price of risk or maximum Sharpe ratio and SR_1 is the Sharpe ratio of the portfolio. This would have a Wishart distribution (genealization of χ^2 under the null) and has been widely used in the literature. A more tractable statistic is given by

$$F = \frac{T(T+N-1))}{N(T-2)}W \sim F_{N,T-N-1},$$

under the null hypothesis where T is the number of returns observed and N is the number of assets originally present (Morrison, 1976).

Lo (2002) finds that tests based on the Sharpe ratio crucially depend on the normality, and assumptions of independence and identically distributed.

The other measures of risk adjusted returns are based on the CAPM model

$$E(R_P) = R_f + \beta[E(R_M) - R_f] \Longrightarrow E(R_P) - R_f = \beta_P[E(R_M) - R_f],$$

gives the securities market line (SML) where R_P and R_M are respectively the percentage returns on the portfolio P and on the market portfolio M, R_f denotes the riskfree rate, β_P is the beta of the portfolio P with respect to market portfolio M, and $E(.)$ denotes the expectation operator. The time-series *market model* that assigns *ex-post* excess return for individual asset i in time t is given in terms of risk premium as

$$R_{it} = \alpha_i + R_f + \beta_i \left(R_{Mt} - R_f\right) + \varepsilon_{it},$$

where R_{it}, R_{Mt} and ε_{it} are the returns of individual asset and the market model in period t. For indivudual i, and α_i, β_i are individual firm specific effects and risk free rate R_f.

According to the Sharpe-Lintner one factor CAPM model, while the standard deviation or volatility σ_P gives a measure of the total or absolute risk, the systematic risk is given by the regression slope coeffcient in the market model or β_P. Hence, while the Sharpe ratio discussed before gives a measure of the return with respect to unit volatility, a measure of the return for unit systematic risk ($\beta_P \neq 0$) is (Treynor, 1965; Treynor and Black, 1973)

$$\text{Treynor ratio}_P = \frac{\alpha_P}{\beta_P} = \frac{\left(R_P - R_f\right)}{\beta_P}.$$

Treynor ratio is directly related to the CAPM slope β_P and is appropriate for a well diversified portfolio, hence will be affected by the critique that the market index might not be well defined (Roll, 1977). Srivastava and Essayyad (1994) proposed an extension of the Treynor ratio that combines beta's of different portfolio and as a combination of the CAPM model and the mean-lower partial moment CAPM as a combined index might be more efficient.

Another problem of the Sharpe ratio is that the volatility measure used is random, hence there is estimation error involved in it (Lo, 2002). The Double Sharpe Ratio was proposed to accomodate for the estimation error, and was defined as

$$DSR_P = \frac{SR_P}{\sigma\left(SR_P\right)},$$

where $\sigma\left(SR_P\right)$ is the estimated standard error of the Sharpe Ratio using bootstrap methods.

There are some alternate measures of volatility and downside risk that has been used to replace the denominator like VaR of a particular confidence level expressed as a percentage. Generalized Sharpe Ratio based on incremental VaR (Dowd, 2000) and similar method with the *benchmark VaR* (or BVaR) (Dembo 1997) has been proposed. It was notice that both Sharpe and Information Ratio may lead to spurious ranking of mutual funds when excess returns are negative. Israelson (2005) proposed the modified Sharpe ratio

$$SR_P^{\text{mod}} = \frac{\mu_P - R_f}{\sigma_P^{(\mu_P - R_f)/\left|\mu_P - R_f\right|}}$$

that coincides with the Sharpe ratio when the excess return is positive. Similarly the information ratio can also be modified.

Jensen's *alpha* for a portfolio P is defined as the abnormal return of the portfolio over and above the expected return under the CAPM model

$$\text{Jensen's } \alpha_P = R_P - E\left(R_P\right) = \left(R_P - R_f\right) - \beta_P\left(R_M - R_f\right),$$

gives the difference between the observed and predicted risk premia (Jensen, 1968). We can perform statistical tests on Jensen's α using the standard t-tests assuming normality of the errors in the market model. Unlike the Sharpe and the Treynor ratio's Jensen's α can be expressed as an excess return and expressed in basis points, it also suffers from Roll's (1977) criticism as it depends on the market index. It has also been brought to the attention that form money managers who practised market timing, Jensen's α might not be a good measure as it can turn negative and fails to address the manager's performance. It has been modified to accomodate for varying beta as well as for higher moments of returns minus risk-free rate (Treynor and Mazuy, 1966, Merton, 1981; Henriksson and Merton, 1981; Henriksson, 1984). This model was particularly useful to check market timing ability incorporating non-linearities in the CAPM framework (Jensen, 1972, Bhattacharya and Pfleiderer, 1983).

There were other extensions of Jensen's α like Black's zero-beta model where there is no risk-free rate (Black, 1972), adjusting for the impact of taxes liabilities (Brennan, 1970), considering total risk σ_P as opposed to just market risk β_M (Elton and Gruber, 1995). However, the total risk measure called *Total Risk Alpha* along with Jensen's alpha can be manupulated using leverage, as opposed to Sharpe and Treynor ratios Jensen's α is not leverage invariant (Scholtz and Wilkens, 2005, Gressis, Philippatos and Vlahos, 1986).

One of the issues of all these three Sharpe Ratio, Jensen's alpha and Treynor Ratio is whether they will generate the same ranking of riskiness across funds or portfolios. For portfolios which are dominated by systematic risk compared to diversifiable non-systematic tisk it is expected that the ranks of funds in terms of riskiness will give you similar rankings. However, in funds like hedge funds they are expected to generate very different rankings when the measure of risk is changed and the rankings will be similar only under very restrictive conditions (Lhabitatnt 2006, p. 467).

CAPM is a single factor model where the only systematic risk is assumed to come from the market, this has been generalized to multi-factor models like the APT model. There are some generalizations to the standard measures like extension of the Treynor ratio to a case of multifactor model by using orthonormal basis in the directions of risk (Hubner 2005). However, as discussed before, hedge funds are uniquely placed which focusses more on non-systematic or total risk, hence, Sharpe Ratios and generalizations discussed are more commonly used.

The main drawback of the Sharpe ratio for an average investor was that although it gives the excess return from risk free rate, it gives per unit of volatility σ_P that is not well understood. M^2 measure was proposed to put all returns in excess of the risk free rate in terms of the same volatility, say the market or benchmark volatility σ_M (Modigliani and Modigliani, 1997; Modigliani, 1997). They suggested de-leveraging (or leveraging) using the risk free rate forming a portfolio P^* of the portfolio and treasury bills (with R_f and no volatility) to equate the Sharpe ratios, *i.e.*,

$$\frac{R_P - R_f}{\sigma_P} = \frac{R_{P^*} - R_f}{\sigma_M} \Longrightarrow M^2 = R_{P^*} = \frac{\sigma_M}{\sigma_P}(R_P - R_f) - R_f,$$

hence for this *risk-adjusted performance* (RAP) measure similar to Sharpe ratio the fund with the highest M^2 will have the highest resturn for any level of risk. The resulting ranking would be similar as Sharpe ratio of a portfolio on which M^2 is based is not affected by leverage with the risk free asset. Here the term σ_M/σ_P is called the *leverage factor*. This measure is based on the total risk hence suitable for any investors including those holding undiversified portfolio. Scholtz and Wilkens (2005) suggests a measure that is a *market risk adjusted performance measure* (MRAP) that accounts for the market risk rather than total risk, similar to the Treynor Ratio.

Muralidharan (2000) suggested the M^3 measure that corrects for the unaccounted for correlation in M^2. Lobosco (1999) developed the *Style RAP* (SRAP) and Muralidhar (2001) also developed the *SHARAD* measure is an extension of the M^3 measure that is adjusted for style specific investment benchmark (Sharpe, 1992). There were two further measures that were proposed GH1 and GH2 that also uses the leveraging-deleveraging approach of M^2 (Graham and Harvey, 1997). First measure (GH1) matches the volatility of the fund using the market and the T-bills, then take the difference between the fund's return and the return on the matched market portfolio to evaluate whether the fund underperforms (or outperforms) the market if it's negative (or positive). The second measure, GH2, takes an alternate route and uses the fund and uses leverage or deleverage of the risk-free rate to replicate the volatility of the market. Hence GH2 is given by the difference of return on the the portfolio that matches the market's volatility and the retrun on the market, hence positive means outperformance. In essense, the Graham-Harvey measures GH1 and GH2 (that refuses to take volatility of risk free asset to be zero) are similar to Jensen's alpha and M^2 respectively. Similar in essense to the GH measures Cantaluppi and Hug (2000) proposed a measure of risk that is called the *efficiency ratio* that gives the best possible performance by a certain portfolio with respect to the efficient frontier.

There are several measures that are based on downside risk listed below.

1. Define MAR as the minimum acceptable return and DD_P is the downward deviation below MAR, (Sortino and van der Meer, 1991)

$$\text{Sortino Ratio}_P = \frac{R_P - MAR}{DD_P} = \frac{E\left(R_P\right) - MAR}{\sqrt{\frac{1}{T}\sum_{\substack{t=0 \\ R_{Pt}<MAR}}^{T}\left(R_{Pt} - MAR\right)^2}},$$

   which can be compared if the value of MAR is the same for the funds. Sortino and Price (1994) proposed the *Fouse Index* $=\mu_P - B\delta^2$ based on Expected Utility Theory where B is the degree of Risk Aversion and δ is downside risk with MAR.

2. Another variant of the Sortino ratio is to replace the denominator by the "upside potential ratio" that is the ratio of the upside potential and the downside risk (Sortino, van der Meer, Plantinga, 1999).

3. Sharpe ratio has strong foundation in the underlying theory of normality in mean-variance analysis, in particular, on the assumption of *independent and identically distributed* returns (Lo, 2002). Ziemba (2005) calculates the Sharpe Ratio with downside variance defined before as loss deviation σ_{x-}(divided by T-1),

$$SR_P^- = \frac{\mu_P - R_f}{\sqrt{2}\sigma_{x-}},$$

   which is similar and converges to Sortino ratio under normality or symmetry.

4. When portfolios are non-normal standard mean-variance analysis do it suffice to capture the risk distribution of the portfolio, and higher order moments like skewness and kurtosis need to be considered. If a three moment CAPM is

assumed with a quadratic return process Hwang and Satchell (1998) proposed a new performance measure is proposed based on higher order moments.

5. *Omega measure* is closely associated with downside risk, lower partial moments, gain-loss functions, breakdown of normality assumptions and need for higher order moments (Keating and Shadwick, 2002). It is simple to define as for certain MAR
$$\Omega\left(MAR\right)=\frac{\int_{MAR}^{b}\left(1-F\left(x\right)\right)dx}{\int_{a}^{MAR}F\left(x\right)dx},$$
defined on (a,b) of possible returns and cumulative distribution function $F\left(.\right).$ The ranking based on the omega measure is expected to be different from Sharpe ratio, alphas and VaR. The *Kappa measure* generalizes Sortino ratio and Omega measures (Kaplan and Knowles, 2004).

6. Sterling ratio also considers drawdowns to measure risk defined as
$$\text{Sterling}_P=\frac{R_P-R_f}{\overline{drawdown}}\left(or=\frac{R_P-R_f}{\max .drawdown},alternative\right),$$
where $\overline{drawdown}$ is the average of the "high" drawdowns during the period.

7. Burke ratio looks at the average L_2-distance defined as the square root sum of squares of the drawdowns instead of the average or the maximum (Burke , 1994)
$$\text{Burke}_P=\frac{R_P-R_F}{\sqrt{\sum_{i=1}^{N}\left(drawdown_i\right)^2}}.$$

3 Testing moments of time series and Sharpe Ratio

We can look at the two sample version of comparison of Sharpe Ratio to discuss the benefits of the proposed methodology. Following Ledoit and Wolf (2008), consider the following problem with the returns from two investment strategies say, R_{1t} and R_{2t}, $t=1,2,...,T$. First we consider a strictly stationary distribution, hence, the covariance (and higher order moments) structure remain a bivariate distribution that is Ergodic with
$$\mu=\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}\text{ and }\sum=\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}.$$

To test $H_0 : SR_1 - SR_2 \equiv \frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma 2} = 0$ against $H_1 : SR_1 - SR_2 \equiv \frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma 2} \neq 0$, we can use function of the parameter vector of the form $g\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2\right)$ $= f\left(\mu_1, \mu_2, \mu_{12}, \mu_{22}\right) = f\left(\theta\right)$, where μ_{ij} is the j^{th} raw moment of the i^{th} asset return distribution. For ease of exposition we will simply write $\mu_{i1} = \mu_i, i = 1, 2$. The hypotheses becomes $H_0 : f\left(\mu_1, \mu_2, \mu_{12}, \mu_{22}\right) = 0$ vs. $H_0 : f\left(\mu_1, \mu_2, \mu_{12}, \mu_{22}\right) \neq 0$ where $f\left(\mu_1, \mu_2, \mu_{12}, \mu_{22}\right) = \frac{\mu_1}{\sqrt{\mu_{12} - \mu_1^2}} - \frac{\mu_2}{\sqrt{\mu_{22} - \mu_2^2}}$. Under the assumptions of stationarity with the appropriate mixing conditions, existence of at least the fourth moment and normality, and a consistent estimator of the parameter vector we can use the delta method to $\sqrt{T}\left(f\left(\hat{\theta}\right) - f\left(\theta\right)\right) \rightarrow N\left(0, \nabla' f\left(\theta\right)\Omega\nabla f\left(\theta\right)\right)$where $\sqrt{T}\left(\hat{\theta} - \theta\right) \rightarrow N\left(0, \Omega\right)$ where Ω is an unknown symmetric positive semi definite matrix.

Further, we can estimate Ω by a heteroscedasticity and autocorrelation consistent (HAC) estimator with an appropriate kernel like Bartlett kernel (Andrews, 1991, Andrews and Monahan, 1992, Newey and West, 1994). As with other tests using the HAC estimator, for small or moderately big samples the inference the test have high size distortion, hence the true null hypothesis would be rejected too often (Andrews, 1991, Andrews and Monahan, 1992).

We would propose a score based test that will give at several advantages over Wald-type test that is commonly used. First, unlike the Wald test it will be invariant to the specification of the different functional form (see functions g(.) and f(.) above, or several other equivalent forms). Second, it will adjust for size distortion by appropriately controlling the same sizes and parameter estimation error in serially dependent structure like GARCH type disturbance (see Ghosh and Bera, 2006). Third, we will jointly test normality like the Jarque-Bera statistic which is also a ratio of excess skewness and kurtosis terms. Finally, the test will be an Locally Most Powerful Unbiased test and in general optimal test as it will be function of sample score statistics (Bera and Bilias, 2001). We would compare the test with the existing tests of comparing multiple Sharpe ratios and other measures of risk like the Omega ratio commonly used used for evaluating risk of hedge funds accounting for dependence.

Sharpe ratio and other measures if risk adjusted performance crucially depend on the assumptions made on the return process particularly the assumptions of normality and hence, linearity and symmetry, and independence (Lo, 2002, Getmansky, Lo and Makarov, 2003). Further more, the existence variation of higher order moments, nonlinearity and significant probability of extreme or "iceberg" risk further complicates the simplifying assumptions for testing with Sharpe ratio alone (Bernardo and Ledoit, 2000, Brooks and Kat, 2002, Agarwal and Naik, 2004, Sharma, 2004, Malkiel and Saha, 2005, Diez de los Rios and Garcia, 2009). The need for a more robust test using measures like the Sharpe ratio has been highlighted in several papers (Ledoit and Wolf, 2008, Zakamouline and Koekebakker, 2009). It has also been noted that tests based on specific moments like the Sharpe ratio is prone to manipulation (Leland, 1999, Spurgin, 2001). Goetzmann, Ingersoll, Spiegel, Welch (2002) observes that "...the best static manipulated strategy has a truncated right tail and a fat left tail."

All these accumulated evidence and a plethora of new risk adjusted performance measures addressing some shortcomings of previous ones (partially discussed in Section 2) brings to our attention the need for a few important developments. First, a statistical inference framework that identifies the distributional differences among returns of funds, particularly in the directions of several moments. Second, a joint test that identifies the nature of dependence structure of the return series that aids the testing, and hence estimation of moment based measures with minimal computational complexity. Third, an inference framework that is robust to existence of higher moments on the return distribution ("iceberg risk" as defined by Osband, 2002). Finally, a test that limits the vagaries of simulation based inference due to issues with unspecified dependence structure and block length selection. In the literature we are aware of, GMM based method has been used to address most of these concerns except that it still suffers from the estimation of the variance covariance matrix (Lo, 2002, Getmansky, Lo and Makarov, 2003). Ledoit and Wolf's (2008) first procedure uses asymptotic inference with a HAC type robust covariance estimator (Andrews, 1991, Andrews and Manohan, 1992). Their second procedure address finite sample issues using a simulation based "studentized time series bootstrap." We propose a smooth test framework that addresses at least three of theses concerns and partially address the fourth one. One main advantage of the procedure is the orthogonality of moment and dependence directions and the score test framework reduces the estimation complexity of the covariance matrix under the null.

Let $(X_1, X_2, ..., X_n)$ has a joint probability density function (PDF) $g(x_1, x_2, ..., x_n)$. Define $\tilde{X}_1 = \{X_1\}$, $\tilde{X}_2 = \{X_2 | X_1 = x_1\}$, $\tilde{X}_3 = \{X_3 | X_2 = x_2, X_1 = x_1\}$, ..., $\tilde{X}_n = \{X_n | X_{n-1} = x_{n-1},\ \ X_{n-2} = x_{n-2}\ \ ..., X_1 = x_1\}$. Then we have

$$g(x_1, x_2, ..., x_n) = f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1) ... f_{X_n|X_{n-1}X_{n-2}...X_1}(x_n|x_{n-1}, x_{n-2}, ..., x_1).$$

The above result can immediately be seen using the Change of Variable theorem that gives

$$\begin{aligned} P\left(Y_i \leq y_i, i=1,2,...,n\right) &= \int_0^{y_1}\int_0^{y_2}...\int_0^{y_n} f\left(x_1\right)dx_1 f\left(x_2|x_1\right)dx_2...f\left(x_n|x_1,...,x_{n-1}\right)dx_n \\ &= \int_0^{y_1}\int_0^{y_2}...\int_0^{y_n} dt_1 dt_2...dt_n \qquad (10) \\ &= y_1 y_2 ... y_n. \end{aligned}$$

Hence, $Y_1, Y_2, ..., Y_n$ are *IID* $U(0,1)$ random variables. Let's recall the following theorem from Rosenblatt(1952).

Theorem 1 *Let* $(X_1, X_2, ..., X_n)$ *be a random vector with absolutely continuous density function* $f(x_1, x_2, ..., x_n)$. *Then, if* $F_i(.)$ *denotes the distribution function of the* i^{th} *variable* X_i, *the* n *random variables defined by*

$$\begin{aligned} Y_1 = F_1\left(X_1\right), Y_2 = F_2\left(X_2|X_1 = x_1\right), \\ ..., Y_n = F_n\left(X_n|X_1 = x_1, X_2 = x_2, ..., X_{n-1} = x_{n-1}\right) \end{aligned}$$

are IID $U(0,1)$.

Furthermore, using Theorem 1 if we define $(Y_1, Y_2, ..., Y_n)$ as conditional cumulative distribution functions (CDF) of $(X_1, X_2, ..., X_n)$ or the probability integral transforms (PIT) evaluated at $(x_1, x_2, ..., x_n)$,

$$Y_1 = F_{X_1}\left(x_1\right), Y_2 = F_{X_2|X_1}\left(x_2|x_1\right), ..., Y_n = F_{X_n|X_{n-1}X_{n-2}...X_1}\left(x_n|x_{n-1}, x_{n-2}, ..., x_1\right)$$

are then distributed as *IID* $U(0,1)$. Suppose now, under null hypothesis H_0 of the true specification of the model CDF $F(.)$ or PDF $f(.)$, $(Y_1, Y_2, ..., Y_n) = (U_1, U_2, ..., U_n)$ where $U_t \sim U(0,1)$, $t = 1, 2, ...n$, so the joint PDF is

$$\begin{aligned} h\left(y_1, y_2, ..., y_n|H_0\right) &= h_1\left(y_1\right)h_2\left(y_2|y_1\right)...h_n\left(y_n|y_{n-1}, y_{n-2}, ..., y_1\right) \\ &= 1.1....1 = 1. \end{aligned}$$

Under the alternative H_1, $Y_i's$ are neither uniformly distributed nor are they *IID*. Let us suppose the conditional density function of Y_t depends on p lag terms, that is to say,

$$
\begin{aligned}
h\left(y_t|y_{t-1}, y_{t-2}, ..., y_1\right) &= h\left(y_t|y_{t-1}, y_{t-2}, ..., y_{t-p}\right) \\
&= c\left(\theta,\phi\right)\exp\left[\sum_{j=1}^{k}\theta_j\pi_j\left(y_t\right)+\sum_{l=1}^{q}\phi_l\delta_l\left(y_t, y_{t-1}, ..., y_{t-p}\right)\right],
\end{aligned}
\tag{11}
$$

where we have assumed for now $k \geq q$.

For simplicity, we start with $p = 1$, this could be more general than it sounds in one-step-ahead forecasts as we can test pairwise dependence including models like AR(1), ARCH(1) etc.

Theorem 2 (Ghosh and Bera, 2005) *If the conditional density function under the alternative hypothesis is given by equation (11) and $p = 1$, the augmented smooth test statistic is given by*

$$
\hat{\Psi}_k^2 = \begin{bmatrix} U'U + U'BEB'U - V'EB'U \\ -U'BEV + V'EV \end{bmatrix} = U'U + \left(V - B'U\right)' E\left(V - B'U\right)
$$

has a central χ^2 distribution with $k+q$ degrees of freedom where U is a $k-$vector of components $u_j = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\pi_j\left(y_t\right),\ j = 1, ..., k,\ V$ is a $q-$vector of components $v_l = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\delta_l\left(y_t, y_{t-1}\right),\ l = 1, ..., q,\ B = E\left[\pi\delta\right],\ D = E[\delta\delta]$ are components of the information matrix defined in equation (44) in Ghosh and Bera (2005) and $E = \left(D - B'B\right)^{-1}$.

Proof. See Appendix and Ghosh and Bera, 2005. ■

As an illustration of Theorem 2, let us now consider a very simple example of the smooth test for autocorrelation for

$$
y_t - \mu = \rho\left(y_{t-1} - \mu\right) + \sigma_t\varepsilon_t \tag{12}
$$

where $E\left(\varepsilon_t\right) = 0$, $V\left(\varepsilon_t\right) = 1$, $\sigma_t = \sigma$ and $a_1 = \frac{1}{\sqrt{12}}$. We define, if $m_1 = E\left(y_{t-1}\right)$,

$$
\delta_1\left(y_t, y_{t-1}\right) = \left(y_t - 0.5\right)\left(y_{t-1} - m_1\right) = \frac{1}{\sqrt{12}}\pi_1\left(y_t\right)\left(y_{t-1} - m_1\right) = a_1\pi_1\left(y_t\right)\left(y_{t-1} - m_1\right). \tag{13}
$$

Then, we can denote $v_1 = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\delta_1\left(y_t, y_{t-1}\right) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(y_t - 0.5\right)\left(y_{t-1} - m_1\right)$. Given information set $\Omega_t = \left\{y_{t-1}, y_{t-2}, ...\right\}$, applying the Law of Iterative Expectation,

$$\int_0^1 a_1 \pi_j (y_t)(y_t - 0.5)(y_{t-1} - m_1) dy_t$$
$$= (y_{t-1} - m_1) \int_0^1 \pi_j (y_t) \pi_1 (y_t) \, dy_t = \begin{cases} a_1 (y_{t-1} - m_1) & j = 1 \\ 0 & j \neq 1 \end{cases},$$
$$\Rightarrow E\left[E\left[\int_0^1 \pi_j (y_t)(y_t - 0.5)(y_{t-1} - m_1) \, dy_t | \Omega_t\right]\right] = \begin{cases} a_1 E[y_{t-1} - m_1] = 0 & j = 1 \\ 0 & j \neq 1 \end{cases}. \tag{14}$$

Applying the Law of Iterative Expectation once again, defining $\sigma^2 = E(y_{t-1} - m_1)^2$,

$$\int_0^1 ((y_t - 0.5)(y_{t-1} - m_1))^2 \, dy_t$$
$$= a_1^2 (y_{t-1} - m_1)^2 \int_0^1 \pi_1^2 (y_t) \, dy_t$$
$$= a_1^2 (y_{t-1} - m_1)^2$$
$$\Rightarrow E\left[E\left[\int_0^1 ((y_t - 0.5)(y_{t-1} - m_1))^2 \, dy_t | \Omega_t\right]\right] = a_1^2 E[y_{t-1} - m_1]^2 = a_1^2 \sigma^2. \tag{15}$$

Hence, it follows that

$$\begin{aligned} E[\pi\delta] &= \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \end{pmatrix}' = B \\ E[\delta\delta] &= a_1^2 E[y_{t-1} - m_1]^2 = D, \end{aligned} \tag{16}$$

which in turn gives the information matrix

$$\mathcal{I} = n \begin{bmatrix} 1 & 0'_{k-1} & 0 \\ 0_{k-1} & I_{k-1} & 0_{k-1} \\ 0 & 0'_{k-1} & a_1^2 \sigma^2 \end{bmatrix} \tag{17}$$

where I_p is the identity matrix of order p and 0_p is a p^{th} order vector of $0's$. In order to evaluate the inverse of the information matrix in (17) we use the following results:

$$\begin{aligned}
D - B'B &= a_1^2 \left[E\left(y_{t-1}^2\right) - \left(E\left(y_{t-1}\right)\right)^2 \right] = a_1^2 \sigma^2, \\
U'BEB'U &= a_1^2 u_1^2 \mu^2 / \left(a_1^2 \sigma^2\right) = 0, \\
V'EB'U &= v_1 u_1 \mu / \left(a_1^2 \sigma^2\right) = 0, \\
V'EV &= v_1^2 / \left(a_1^2 \sigma^2\right).
\end{aligned} \tag{18}$$

Hence, using (47) we have a correction term as an LM test for autocorrelation (Breusch, 1978)

$$\begin{aligned}
\Psi_{k+1}^2 &= \sum_{j=1}^{k} u_i^2 + \frac{1}{\left(a_1^2 \sigma^2\right)} \left[v_1^2\right] \\
&= \sum_{j=1}^{k} u_i^2 + \frac{\left(v_1\right)^2}{\left(a_1^2 \sigma^2\right)} = \sum_{j=1}^{k} u_i^2 + \frac{12 \left(v_1\right)^2}{\sigma^2} \overset{a}{\sim} \chi_{k+1}^2 \text{ under } H_0.
\end{aligned} \tag{19}$$

The sample counterpart of the second expression in (19) is

$$12 \left(\frac{\sqrt{\frac{1}{n}} \sum_{t=2}^{n} \left(y_t - 0.5\right) \left(y_{t-1} - m_1\right)}{\sqrt{\frac{1}{n-1} \sum_{t=1}^{n} \left(y_t - \overline{y}\right)^2}} \right)^2 \overset{a}{\sim} \chi_1^2. \tag{20}$$

It is evident that this will give us an asymptotic test for autocorrelation of the first order in a *global* sense. To further illustrate this technique, let us consider a test for ARCH (1) type alternative with mean equation (12),

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 \varepsilon_{t-1}^2 \tag{21}$$

For testing ARCH(1) dependence, define

$$\delta_2 \left(y_t, y_{t-1}\right) = \left(y_{t-1}^2 - m_2\right) \left(y_t^2 - \frac{1}{3}\right) = \left(y_{t-1}^2 - m_2\right) \left(a_1 \pi_1 \left(y_t\right) + a_2 \pi_2 \left(y_t\right)\right) \tag{22}$$

where $a_1 = \frac{1}{\sqrt{12}}$, $a_2 = \frac{1}{6\sqrt{5}}$, $a_3^2 = a_1^2 + a_2^2 = \frac{4}{45}$ and $m_j = E\left(y_{t-1}^j\right)$ for notational convenience.

The joint smooth test statistic incorporating an ARCH(1) type effect where $v_l =$

$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \delta_l \left(y_t, y_{t-1}\right)$,

$$\hat{\Psi}^2_{k+1} = \sum_{j=1}^{k} u_j^2 + \left(a_3^2 E\left(y_{t-1}^2 - m_2\right)^2\right)^{-1} [v_2]^2$$
$$\sim \chi^2_{k+1}(0). \qquad (23)$$

Similarly, we can obtain a joint test incorporating *leverage effect* (a negative correlation between past returns and future volatilities) with an appropriate function like

$$\delta_3\left(y_t, y_{t-1}\right) = \left(y_{t-1} - m_1\right)\left(y_t^2 - \frac{1}{3}\right) = \left(y_{t-1} - m_1\right)\left(a_1\pi_1\left(y_t\right) + a_2\pi_2\left(y_t\right)\right)$$
$$\Longrightarrow v_3 = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \delta_3\left(y_t, y_{t-1}\right),$$

that yields the test statistic defining $m_{11} = E\left(y_{t-1} - m_1\right)^2$,

$$\hat{\Psi}^2_{k+1} = \sum_{j=1}^{k} u_j^2 + \left(a_3^2 m_{11}\right)^{-1} [v_3]^2$$
$$\sim \chi^2_{k+1}(0). \qquad (24)$$

The joint test of both leverage effect and ARCH(1) type effects is more involved but can be derived from the shortcut matrix formula for the correction term given

$$E = \frac{1}{\Delta} \begin{bmatrix} a_3^2 m_{22} & -a_3^2 m_{12} \\ -a_3^2 m_{12} & a_3^2 m_{11} \end{bmatrix},$$

where $m_{ij} = E\left(y_{t-1}^i - m_i\right)\left(y_{t-1}^j - m_j\right)$ and $\Delta = a_3^4\left(m_{11}m_{22} - m_{12}^2\right)$,hence,

$$\frac{a_3^2}{\Delta}\left[v_2^2 m_{22} + v_3^2 m_{11} - 2v_3 v_2 m_{12}\right] \sim \chi_2^2. \qquad (25)$$

Similarly, a joint test of AR(1) and ARCH(1) effects can be shown to be function of the crossed and central moments of y_{t-1}, besides the score functions $u_j's$ and $v_l's$ follows a χ_2^2.

Unfortunately, the choice of the *dependency function* $\delta_l\left(y_t, y_{t-1}\right), l = 1, 2, ..., q$ (a moment condition to capture the dependent structure) involves a trade-off. On one hand, the smaller the number q there are fewer parameters to estimate, however, there will be a loss of power owing to the types of dependencies that are ignored; on the other, if q is large we will suffer from a curse of dimensionality as there will be several parameters to be estimated based on the same data. In the following examples we illustrate how to incorporate more general dependence structures like ARMA(1,1), GARCH (1,1) and several ARCH parameters, and not to increase the dimensionality of the problem substantially under certain regularity conditions.

Suppose we want to incorporate an ARMA (1,1) error term of the following form [Bera and Ra (1994), Andrews and Ploberger (1996)]

$$y_t - (\phi + \xi)\, y_{t-1} = \varepsilon_t - \phi\varepsilon_{t-1}, \tag{26}$$

where ε_t is $IID\ \mathcal{N}(0, \sigma_\varepsilon^2)$, ϕ and $(\phi + \xi) \in (-1, 1)$ and $t = 1, 2, ..., n$. Here, to test for white noise we can test $H_0 : \xi = 0$ against $H_1 : \xi \neq 0$. It is worth noting that under H_0, the parameter ϕ becomes unidentified, hence we have a nuisance parameter under the null which is often termed as the Davies' problem (Davies 1977, 1987). We will start of assuming that the parameter ϕ is fixed and then relax that assumption to do the test. Define the dependency function for some fixed and finite $r\ (\leq t-1)$,

$$\delta_1 (y_t, y_{t-1}, ..., y_1) = (y_t - 0.5) \sum_{s=1}^{r} \phi^{s-1} (y_{t-s} - m_1). \tag{27}$$

Hence, if ϕ is a known constant and for some finite r, as shown in the Addendum A subsection 5.0.2, the smooth test statistic incorporates the LM test similar to Andrews and Ploberger (1996),

$$\begin{aligned} \hat{\Psi}_{k+1}^2 &= \sum_{j=1}^{k} u_j^2 + v_1^2 \left(a_1^2 m_{11} \frac{1-\phi^r}{1-\phi} \right)^{-1} \\ &\sim \chi_{k+1}^2 \text{ where } v_1 = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (y_t - 0.5) \sum_{s=1}^{r} \phi^{s-1} (y_{t-s} - m_1). \end{aligned} \tag{28}$$

It is worth noting that putting $\phi = 0$ we get back the test using AR(1) terms. In order to test for dependence alone of ARMA(1,1) form we can simply look at the second expression in (54) and follow a test procedure like the suggested in Bera and Ra (1994) or Andrews and Ploberger (1996). See Ghosh and Bera (2005) for details.

We would finally suggest a procedure inspired by Engle (1982, 1983) where we considered a weighted ARCH type alternative. The conditional variance function suggested by Engle (1982, 1983) was

$$h_t = \alpha_0 + \alpha_1 \sum_{s=1}^{r} w_s \left(u_{t-s}^2 - m_2\right), \text{ where } w_s = \frac{(r+1)-s}{\frac{1}{2}r\,(r+1)} \text{ for some fixed } r. \quad (29)$$

We explicitly derive the RS test for testing $H_1 : \alpha_1 = 0$ against $H_1 : \alpha_1 \neq 0$ which tests whether there is an ARCH term against a constant variance in Addendum A, subsection 5.0.3. From our results, if r is a known constant, the smooth test statistic

$$\hat{\Psi}_{k+1}^2 = \sum_{j=1}^{k} u_j^2 + v_2^2 \left(\frac{4\,(3+r)}{3r\,(r+1)} a_3^2 m_{22}\right)^{-1} \sim \chi_{k+1}^2,$$

where $v_2 = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left(y_t^2 - \frac{1}{3}\right) \sum_{s=1}^{r} w_s \left(y_{t-s}^2 - m_2\right)$.

Unfortunately, this formulation also suffer from the Davies' problem through the choice of r, though to a lesser degree. We can choose the r through maximization of some likelihood based information criterion or model selection. For details, see Ghosh and Bera (2005).

Now suppose we wantto test for moments of the return distribution of hedge funds (say, a hedge fund global index) and test for independence and identical distribution. Using the augmented smooth test of density forecast evaluation , we would be able to individually test all the moments of the distribution, say upto order $k = 4$ (Ghosh and Bera, 2005). This can be used for testing both in-sample (estimation and testing on the same sample) and out-of-sample (split sample and estimate using one, and test with the other). We would also be able to test jointly whether moments of the distribution are the same. In particular, as a corrolary to Theorem 2 we can formulate a test for comparison of the a modified version of Sharpe Ratio composed of the probability integral transforms and any other moment based test. The attractive feature of the smooth test is that as individually each of the $\hat{u}_j^2$ are asymptotically distributed as χ_1^2, we can construct ratios of variables that will be distributed as $F-$distribution. One further note, as the Rao score or LM test is transformation invariant unlike the Wald test, we would be able to run the test whether in any equivalent way. We summarize this is the following theorem.

For the one sample version the test is straightforward. Suppose, we assume the distribution of the index of hedge funds as $F_t(.)$, this can be generated using the Cumulative Distribution Function (CDF) of returns of a relevant index fund (say, the Hedge Fund Index). First, suppose we want to test $H_0 : \mu_R = R_f$ vs. $H_1 : \mu_R \neq R_f$, we can use $k = 2$ or $k = 4$ if you want to hold higher moments fixed. For simplicity, in our Theorem 2, we simply replace $q = 0$. We however, would be testing in the direction of normalized Legendre polynomials of the probability integral transform of the original variable X_t, hence the test is $H_0 : \theta_1 = 0$ vs. $H_1 : \theta_1 \neq 0$.Since these are orthonormal polynomials $\pi_j(.)$'s, we do not need to recalculate the joint test statistic $\hat{\Psi}_k^2$, with additional directions of departure, simply adding $\hat{u}_j^2$ for extra j would suffice. For completeness, we defined the probability integral transform $y_t = \int_{-\infty}^{x_t} f_t(x)\,dx,\ t = 1, ..., T$ where $f_t(.)$ is the probability density function (PDF) of the relevant index. As the probability integral transform is a monotonic transformation, the directional results true in y_t are also true in the original variable x_t. Further j^{th} order normalized Legendre polynomials are $\pi_0(y) = 1$, $\pi_1(y) = \sqrt{12}\left(y - \frac{1}{2}\right)$, $\pi_2(y) = \sqrt{5}\left(6\left(y - \frac{1}{2}\right)^2 - \frac{1}{2}\right)$, $\pi_3(y) = \sqrt{7}\left(20\left(y - \frac{1}{2}\right)^3 - 3\left(y - \frac{1}{2}\right)\right)$, $\pi_4(y) = 210\left(y - \frac{1}{2}\right)^4 - 45\left(y - \frac{1}{2}\right)^2 + \frac{9}{8}$, etc. Hence, the moments we are testing are in orthogonal directions of the normalized Legendre polynomials of the probability integral transform.

Hence, we can define the smooth test statistic $\hat{\Psi}_{F,k}^2$ for each value of $k = 1, 2, 3, 4$ that provides the aggregated level of risk from each moment of the distribution upto that k as the *Smooth Total Moment Risk* $(STMR_F^{(k)})$ measure with respect to the benchmark distribution $F_t(.)$,

$$STMR_F^{(k)} = \hat{\Psi}_{F,k}^2 = \sum_{j=1}^{k} u_{F,j}^2 \sim \chi_k^2, \text{ where } u_{F,j} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \pi_j(y_t)$$

In particular, as we are interested in the amount of risk associated with i^{th} moment in the presense of higher order moments upto k, we define a new measure the i^{th} order *Smooth Moment Risk* $\left(SMR_{F,i}^{(k)}\right)$ with respect to $F(.)$ as

$$SMR_{F,i}^{(k)} = \frac{u_{F,i}^2}{\sum_{\substack{j=1 \\ j \neq i}}^{k} u_{F,j}^2 / (k-1)} \sim F_{1,k-1}$$

has a central F distribution with 1 degree of freedom in numerator and $k - 1$ degree of freedom in denominator asymptotically. For $k = 2$, this can give the overall risk associated with the first moment direction. For higher values of k, we can identify the levels of return risk from higher order moments. The main advantage of these smooth moment risk measures are they are themselves test statistic with tabulated asymptotic distributions.

These can be generalized to include different flexible dependent structures like

AR(1) or ARMA(1,1) as dicussed before, to get the *Dependence Smooth Total Moment Risk* ($DSTMR_F^{(k+q)}$) with benchmark distribution $F_t(.)$

$$DSTMR_F^{(k)} = \hat{\Psi}_{F,k}^2 = U'U + (V - B'U)' E (V - B'U) \overset{a}{\sim} \chi_{k+q}^2,$$

where U,V, B and E are as defined in Theorem 2 and proof.

Similarly, the different dependence functions can be tested with the i^{th} *Dependence Smooth Moment Risk* ($DSMR_F^{(k+q)}$) (like the *Autocorrelation Smooth Moment Risk*, *Leverage Smooth Moment Risk*, *ARCH smooth moment risk* etc.) as

$$DSMR_{F,i}^{(k)} = \frac{Correction_{F,i}}{DSTMR_F^{(k)}/k} \overset{a}{\sim} F_{1,k},$$

where $Correction_{F,i}$ is the correction factor that itself has a χ_1^2 distribution asymptotically. These can be extended to a variety of tests that are targeted at particular moment risk premia.

4 Empirical Applications: Checking for Evidence for Market Neutrality in Equity Market Neutral Funds

We address the issue of distributional test of neutrality of equity neutral hedge funds using a equity market neutral index fund provided in Diez de Los Rios and Garcia (2009). In particular, we want to compare the equity neutral fund index (C4 in their Table 1) with the global index they created. The data provided is monthly between Jan 1996 till March 2004 (99 observations). We would compare some standard risk measures and our smooth moment risk measures across the board. We wish to address the issue raised in Patton (2008) about whether Equity Neutral Funds are truly neutral with this index returns.

We first estimate the parametric distribution of the global hedge fund database using the smooth test technique starting with the naive model with the empirical distribution function (EDF), then gradually increase the level of complexity (reported in Table 1). We observe that there is substantial difference of all the moments in particular, the second, third and fourth moments from the market index fund (here we are using the Value Weighted S&P 500 returns from WRDS database). We further update the model using an ARMA specification, but it gives the same qualitative results, although now only the second and fourth moment are signficant ($u_1^2 = 0.42$ or $u_2^2 = 109$). We introduce conditional heteroskcadisticity along with MA(1) term. We also introduce leverage effect in the model by using a linear GJR-GARCH model. The overall smooth total moment risk (STMR) declines slightly with higher level of compexity in the model,. and is statistically distinguisble from the equity market index particularly in the directions of the second and fourth moments. The first smooth moment risk (SMR_1) shows that the MA(1) t-GARCH(1,1) gives very small and statistically insignificant measures in presense of 4 moment directions for all these models F. It appears that the main dispersion is coming from the second moment risk (SMR_2), however, none of them turn out to be statistically significant at 5% level in the presense of higher order moments. This implies that there is significant influence of higher order moment directions like skewness and kurtosis that affects the returns dispersion. If however we use only moment moment directions there will be overwhelming evidence that the second moment direction is strongly significant in determining Equity Neutral Hedge Fund index returns. So based on this evidence we cannot support the claim that Equity Neutral hedge Fund index seems to be fairly independent of the market risks both in returns and in volatility. We also calculate the augmented smooth test jointly for autoregressive and ARCH type errors that gives the dependent smooth total moment risk ($\mathrm{DSTMR}^{(6)}$), which shoes a very similar pattern as the $\mathrm{STMR}^{(4)}$ and hence dependence across the moments does not seem to have an affect either.

F(.)	$STMR^{(4)}$	$\hat{u}_1^2$	$\hat{u}_2^2$	$\hat{u}_3^2$	$\hat{u}_4^2$	$\mathrm{SMR}_1^{(4)}$	$\mathrm{SMR}_2^{(4)}$	$DSTMR^{(6}$
EDF	201.86***	4.81**	104.6***	22.83***	69.62***	0.07	3.23	201.88***
	(0.00)	(0.03)	(0.00)	(0.00)	(0.00)	(0.81)	(0.17)	(0.00)
ARMA(1,1)	194.11***	0.42	109***	2.09	82.6***	0.01	3.84	194.88***
	(0.00)	(0.52)	(0.00)	(0.15)	(0.00)	(0.92)	(0.14)	(0.00)
MA(1)-	183.64***	0.36	105.89***	1.88	75.5***	0.01	4.09	183.8***
$t-$ GARCH (1,1)	(0.00)	(0.55)	(0.00)	(0.17)	(0.00)	(0.93)	(0.13)	(0.00)
MA(1)-t-	172.49***	0.48	102.18***	2.45	67.38***	0.01	4.36	172.62***
GJRGARCH (1,1)	(0.00)	(0.09)	(0.00)	(0.12)	(0.00)	(0.93)	(0.13)	(0.00)

*** *significant at* 1% *level.*** *significant at* 5% *level.*

Table 1.Smooth Moment Risk and components (p-values are in parenthesis).

We use the MA(1)-t-GARCH(1,1) as a benchmark distribution of the market index (Value weighted returns), and evaluate all the 10 hedge fund indices. We wanted to evaluate how the market index affects Hedge Funds in our sample, in particular with respect to the Equity Market Neutral Index (Patton, 2008). We observe strong overall statistically significant difference or significant $STMR^{(k)}$ almost all hedge funds indices except Emerging Markets and marginally for Dedicaed Short Bias funds. This is expected, as Emerging Market funds often include all asset classes, hence it is very much like a mutual fund in that respect. We find Equity Market Neutral Funds to be quite strongest in significance in smooth moment risk coming from all moment directions ($STMR = 183$). This does confirm the doubt about overall market neutrality of such funds (Patton 2009). If we lookly closely enough, none of the significance is coming in teh direction of the return level ($\hat{u}_1^2$) but mostly, from the second moment dispersion ($\hat{u}_2^2$) except emerging market funds. Convertible Arbitrage and Fixed income arbitrage from the index, particularly in the direction of the second moment. This does assure us that hedge funds indeed does "hedge" or change teh variability of the return distribution compared tto an equity fund. There is however a very strong influence on higher moment directions that causes the F-statistics in the form of both first and second Smooth moment risk (SMR) measures. They show that comparatively there is insignificant effect in the direction of the first risk moment (SMR_1) for all funds. Further, only Long-Short Equity that thrives on volatility, and Managed Future funds have a higher contribution of volatility compared to other moments (SMR_2). We also looked at the level of dependence in terms of autoregressive smooth moment risk ($DSMR^{(4)}$) and found no residual dependence in that direction.

We also report the Sharpe-Lintner CAPM based measures like the Beta (β) and Jensen's alpha (α). As expected the Market Neutral Hedge Fund does show close to "Beta neutrality," as it is close to zero, but so is Global Macro and Fixed Income Arbitrage. The highest beta is for the Emrrging Market fund that is really an international mutual fund, and the lowest one is on Dedicated Short Bias that thrives on betting against the market. From the smooth total moment risk standpoint (STMR), *Beta* does not replicate the same ordering. This is expected as beta is based on inherent normality assumption of CAPM that assumes away dispersion risk in higher order moments. In fact, systematic risk from beta can be take to be the risk associated with market, hence those funds which play the market like emerging market and dedicated short are most sensitive, while equity neutral strategy is not. Higher Jensen's alpha also do not price higher order moments hence are not dependent on STMR. Using Spearman's rank correlation and Pearson's product moment correlation (not reported here) we see that STMR is negatively correlated with Beta, moderately correlated with alpha and quite strongly correlated with the Sharpe Ratio.

	$\text{STMR}^{(4)}$	$\hat{u}_1^2$	$\text{SMR}_1^{(4)}$	$\hat{u}_2^2$	$\text{SMR}_2^{(4)}$	$\text{AR}^{(4)}$	$\text{DSMR}^{(4)}$	Beta	alpha
Under $H_0 \sim$	χ^2_4	χ^2_1	$F_{1,3}$	χ^2_1	$F_{1,3}$	χ^2_1	$F_{1,4}$	t_{97}	t_{97}
C1 Cnvrt. Arb	156.12^+	0.05	0.00	93.91^+	4.53	0.32	0.01	0.34	4.63
C2 Fxd. Inc. Arb.	176.42^+	0.49	0.01	98.76^+	3.82	0.89	0.02	0.06	2.68
C3 Evnt Driven	115.52^+	0.00	0.00	77.5^+	6.12	0.12	0	0.49	2.39
C4 Eqt. Neutral	183.64^+	0.36	0.01	105.89^+	4.09	0.06	0	0.05	3.34
C5 Lng-Shrt Eqt.	28.6^+	0.16	0.02	27.4^+	68.5^+	1.09	0.15	0.98	1.72
C6 Global Macro	83.75^+	0.79	0.03	59.71^+	7.45	0.72	0.03	0.07	2.05
C7 Emrgng Mkts.	5.04	0.91	0.66	1.4	1.15	3.07	2.44	1.69	-3.44
C8 Ded Shrt Bias	11.77^*	3.48	1.26	5.78^*	2.89	0.14	0.05	-1.77	7.83
C9 Mngd Fut.	32.36^+	1.00	0.1	28.56^+	22.55^*	1.03	0.13	-0.11	2.8
C10 Fnd of Fnd	94.26^+	0.61	0.02	67.17^+	7.44	0.01	0.00	0.53	-0.57
Global Index	74.13^+	0.11	0.11	58.4^+	11.14^*	0.01	0.00	0.67	0.32

Table 2: Hedge Fund Styles and Smooth Risk Moments with R_f=3.775% ($^+$ *signif.at* 5%,* *signif.at* 1

From tests with the market index we would like to explore the relationship with the Global Hedge Fund Index (Diez de los Rios and Garcia, 2009). Table 3 also provides the Sharpe ratio for all the hedge fund indices (using $R_f = 3.775$, given in Table 1 of Diez de los Rios and Garcia 2009). Both the Arbitrage Funds (C1 and C2) shows a substantial risk exposure measured by STMR compared to the Global Hedge Fund Index. Both these are in the specific direction of volatility as shown in $\hat{u}_2^2$, however due to the presense of significant higher order moments their contribution measured by Smooth Moment Risk of first and second order are not statistically significant at 5%. This implies that the arbitrage funds probably strategize on oppurtunities that possibly assymetric , and in the tails of the return distribution. Further, we find that event driven fund, long short equity and global macro shows very little dispersion in moment risk from the global index as they form a majority of the funds out there at that period. However, equity market neutral funds have has a strong deviation in the direction of the first moment though overall it is similar to the global index. Short bias and Managed Futures funds shows affects of overall dependence and variation in volatility risk from global hedge fund. Fund of

Funds is very similar and indistuishable from the Global index. Sharpe ratio gives an indication of the level of risk assuming underlying normality. Hence funds that have higher order moment exposure like Arbitrage funds and dependence like Managed Futures and volatility dynamics like dedicated short are not adequately treated by the Sharpe Ratio.

	$\text{STMR}^{(4)}$	$\text{DSTMR}^{(6)}$	$\hat{u}_1^2$	$\text{SMR}_1^{(4)}$	$\hat{u}_2^2$	$\text{SMR}_2^{(4)}$	$\text{AR}^{(4)}$	$\text{DSMR}^{(4)}$	SR
Under $H_0 \sim$	χ_4^2	χ_6^2	χ_1^2	$F_{1,3}$	χ_1^2	$F_{1,3}$	χ_1^2	$F_{1,4}$	$T^{-\frac{1}{2}}t$
C1 Cnvrt. Arb	13.6***	17.63***	1.14	0.27	10.38***	9.67	3.77	1.11	1.63
C2 Fxd. Inc. Arb.	33.3***	34.76***	0.38	0.03	11.98***	1.69	0.59	0.07	0.73
C3 Evnt Driven	4.61	7.99	0.07	0.05	2.69	4.2	0.74	0.64	1.02
C4 Eqt. Neutral	6.33	7.15	5.1**	12.44**	0.26	0.13	0.00	0.00	1.29
C5 Lng-Shrt Eqt.	4.65	6.09	1.07	0.9	1.08	0.91	0.39	0.34	0.78
C6 Global Macro	5.89	8.32	2.32	1.95	3.43	4.18	1.06	0.72	0.4
C7 Emrgng Mkts.	5.55	6.89	2.22	2	0.18	0.1	0.58	0.42	0.45
C8 Ded Shrt Bias	8.78	12.9***	2.24	1.03	4.21**	2.76	2.47	1.13	-0.18
C9 Mngd Fut.	7.96	17.61***	0.81	0.34	5.61***	7.16	8.24***	4.14	0.23
C10 Fnd of Fnd	2.81	4.57	1.01	1.68	0.03	0.03	0.05	0.07	0.47
Global Index									0.69

Table 3: Hedge Fund Styles and Smooth Risk Moments with R_f=3.75% (*** $signif.at$ 5%, ** $signif.at$

5 Conclusion

Financial risk evaluation had attracted substantial attention of late both in the academic community and outside with the growing financial crisis that might have had its genesis in faulty methodology. Rampant use measures like Value-at-Risk as expressed by practitioners, consultants and field experts described in the New York Times as Risk Mismanagement (http://www.nytimes.com/2009/01/04/magazine/04risk-t.html?em), fails to prevent the effect of "black swans" or very rare events like market crashes or meltdown. Our objective in this paper is to look at the instruments of risk assessment like the Sharpe Ratio that are commonly used and make it more robust in cases of extreme uncertainty or misinformation that leads to noisy data (see Garcia, Renault and Tsafack, 2005). One way of achieving that would be to account for the higher order moments of adjusted return distributions.

We would like evaluate the effectiveness of the forecast models for risk management using out-of sample performance (see Santos, 2008). Hence, out-of-sample forecast evaluation of Sharpe ratio or risk adjusted return distributions using "in-sample" bootstrapped confidence intervals might not be optimal in case there are structural breaks. Further more, the commonly used risk measures like the Sharpe ratio or Value-at-Risk might not be "a coherent measure of risk" (Artzner et. al, 1999, Garcia, Renault and Tsafack, 2005). Under certain weak parametric specifications the entire distributions of the Sharpe ratios can be compared themselves using smooth type tests. Distributional tests of Sharpe Ratio is still in its infancy particualrly accomdating for higher order moments and dependence. An exploration of comparisions of various measures proposed particularly in an inference context is an object of ongoing and future research.

Finally, another possible aspect of returns data is selection biases like survivorship and other non-linearities particularly for Private Equity and Hedge Fund data (Agarwal, and Naik, 2004,. Diez de los Rios, A. and R. Garcia, 2005). Ideally, a model selection and testing procedure should be robust against such problems of truncation or censoring (See Ghosh, 2008). We would explore the robustness properties of the proposed test procedure in the presence of survivorship and other selection biases (Cakici and Chatterjee, 2008, Carlson and Steinman, 2008).

Appendix A1 (Proof of Theorem 2)

5.0.1 Proof of Theorem 2

Proof. In order to test for uniformity and as well as for dependence, one would test $H_0 : \theta_1 = \theta_2 = ... = \theta_k = 0; \phi_1 = \phi_2 = ... = \phi_q = 0$ against the alternative $H_1 : \theta_j \neq 0$ for at least one j or $\phi_l \neq 0$ for at least one l. However, we have not specified the forms of the functions $\pi_j(.)$ and $\delta_l(.)$. The log-likelihood function is

$$\begin{aligned}
\sum_{t=1}^{n} \ln\left(h\left(y_t|y_1, y_2, ..., y_{t-1}\right)\right) &= \sum_{t=1}^{n} \ln f\left(y_t, y_{t-1}\right) \\
&= \sum_{t=1}^{n} \ln c\left(\theta, \phi\right) + \sum_{t=1}^{n}\sum_{j=1}^{k} \theta_j \pi_j\left(y_t\right) + \sum_{t=1}^{n}\sum_{l=1}^{q} \phi_l \delta_l\left(y_t, y_{t-1}\right) \\
&= n \ln c\left(\theta, \phi\right) + \sum_{j=1}^{k} \theta_j \sum_{t=1}^{n} \pi_j\left(y_t\right) + \sum_{l=1}^{q} \phi_l \sum_{t=1}^{n} \delta_l\left(y_t, y_{t-1}\right) \\
&= \ln L = l, \ \ say. \qquad (30)
\end{aligned}$$

So, if we use $\theta=(\theta_1,\theta_2,...,\theta_k)'$ and $\phi=(\phi_1,\phi_2,...,\phi_q)'$ then under the null hypothesis H_0

$$\left.\frac{\partial l}{\partial \theta_j}\right|_{\theta=0,\phi=0} = n\left.\frac{\partial \ln c(\theta,\phi)}{\partial \theta_j}\right|_{\theta=0,\phi=0} + \sum_{t=1}^{n}\pi_j(y_t)$$

$$\Rightarrow \frac{1}{\sqrt{n}}\left.\frac{\partial l}{\partial \theta_j}\right|_{\theta=0,\phi=0} = \sqrt{n}\left.\frac{\partial \ln c(\theta,\phi)}{\partial \theta_j}\right|_{\theta=0,\phi=0} + \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\pi_j(y_t). \quad (31)$$

Similarly, we have

$$\left.\frac{\partial l}{\partial \phi_l}\right|_{\theta=0,\phi=0} = n\left.\frac{\partial \ln c(\theta,\phi)}{\partial \phi_l}\right|_{\theta=0,\phi=0} + \sum_{t=1}^{n}\delta_l(y_t,y_{t-1})$$

$$\Rightarrow \frac{1}{\sqrt{n}}\left.\frac{\partial l}{\partial \phi_l}\right|_{\theta=0,\phi=0} = \sqrt{n}\left.\frac{\partial \ln c(\theta,\phi)}{\partial \phi_l}\right|_{\theta=0,\phi=0} + \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\delta_l(y_t,y_{t-1}). \quad (32)$$

Further, if we take derivative twice and evaluate at $H_0:\theta=0,\phi=0$, from (30)

$$\left.\frac{\partial^2 l}{\partial \theta_i\partial \theta_j}\right|_{\theta=0,\phi=0} = n\left.\frac{\partial^2 c(\theta,\phi)}{\partial \theta_i\partial \theta_j}\right|_{\theta=0,\phi=0}, \quad (33)$$

$$\left.\frac{\partial^2 l}{\partial \phi_l\partial \theta_j}\right|_{\theta=0,\phi=0} = n\left.\frac{\partial^2 c(\theta,\phi)}{\partial \phi_l\partial \theta_j}\right|_{\theta=0,\phi=0}, \quad (34)$$

$$\left.\frac{\partial^2 l}{\partial \phi_i\partial \phi_l}\right|_{\theta=0,\phi=0} = n\left.\frac{\partial^2 c(\theta,\phi)}{\partial \phi_i\partial \phi_l}\right|_{\theta=0,\phi=0}. \quad (35)$$

Since (11) is a density function under H_1, we have for each value of y_{t-1}

$$c(\theta,\phi)\int_0^1 \exp\left[\sum_{j=1}^{k}\theta_j\pi_j(y_t)+\sum_{l=1}^{q}\phi_l\delta_l(y_t,y_{t-1})\right]dy_t = 1. \quad (36)$$

Evaluating the identity in (36) at $\theta_j=0$, $j=1,...,k$ and $\phi_l=0$, $l=1,...,q$, $c(0,0)=1$. Also, if we differentiate (36) and evaluate at $\theta=0$, $\phi=0$ the following results are obtained:[1]

$$(i) \qquad \left.\frac{\partial c(\theta,\phi)}{\partial \theta_j}\right|_{\theta=0,\phi=0} + c(0,0)\int_0^1 \pi_j(y_t)\,dy_t = 0$$

$$\Rightarrow \left.\frac{\partial c(\theta,\phi)}{\partial \theta_j}\right|_{\theta=0,\phi=0} = 0, \text{ since } \int_0^1 \pi_j(y_t)\,dy_t = 0,\ j \neq 0. \tag{37}$$

$$(ii) \qquad \left.\frac{\partial c(\theta,\phi)}{\partial \phi_l}\right|_{\theta=0,\phi=0} + c(0,0)\int_0^1 \delta_l(y_t, y_{t-1})\,dy_t = 0$$

$$\Rightarrow \left.\frac{\partial c(\theta,\phi)}{\partial \phi_l}\right|_{\theta=0,\phi=0} = -\int_0^1 \delta_l(y_t, y_{t-1})\,dy_t = 0. \tag{38}$$

$$(iii) \qquad \frac{\partial^2 c(\theta,\phi)}{\partial \theta_i \partial \theta_j} + \frac{\partial c(\theta,\phi)}{\partial \theta_j}\int_0^1 \pi_i(y_t)\,dy_t +$$

$$\frac{\partial c(\theta,\phi)}{\partial \theta_i}\int_0^1 \pi_j(y_t)\,dy_t + c(\theta,\phi)\int_0^1 \pi_i(y_t)\,\pi_j(y_t)\,dy_t = 0$$

$$\Rightarrow c_{\theta_i\theta_j} + c_{\theta_j}.0 + c_{\theta_i}.0 + \int_0^1 \pi_i(y_t)\,\pi_j(y_t)\,dy_t = 0$$

$$\Rightarrow c_{\theta_i\theta_j} = -\delta_{ij}, \tag{39}$$

where $\delta_{ij} = 1$ if $i = j$; $\delta_{ij} = 0$ if $i \neq j$, $c_{\theta_i\theta_j} \equiv \left.\frac{\partial^2 c(\theta,\phi)}{\partial \theta_i \partial \theta_j}\right|_{\theta=0,\phi=0}$ and $c_{\theta_j} = \left.\frac{\partial c(\theta,\phi)}{\partial \theta_j}\right|_{\theta=0,\phi=0}$. Similarly, it can be shown that

$$(iv) \qquad \frac{\partial^2 c(\theta,\phi)}{\partial \phi_l \partial \theta_j} + \frac{\partial c(\theta,\phi)}{\partial \theta_j}\int_0^1 \delta_l(y_t, y_{t-1})\,dy_t +$$

$$\frac{\partial c(\theta,\phi)}{\partial \phi_l}\int_0^1 \pi_j(y_t)\,dy_t + c(\theta,\phi)\int_0^1 \pi_j(y_t)\,\delta_l(y_t, y_{t-1})\,dy_t = 0$$

$$\Rightarrow c_{\phi_l\theta_j} = -\int_0^1 \pi_j(y_t)\,\delta_l(y_t, y_{t-1})\,dy_t, \text{ where } c_{\phi_l\theta_j} = \left.\frac{\partial^2 c(\theta,\phi)}{\partial \phi_l \partial \theta_j}\right|_{\theta=0,\phi=0}. \tag{40}$$

Finally, using the same procedure we can obtain

$$(v) \qquad c_{\phi_i\phi_l} = \left.\frac{\partial^2 c(\theta,\phi)}{\partial \phi_i \partial \phi_l}\right|_{\theta=0,\phi=0} = -\int_0^1 \delta_i(y_t, y_{t-1})\,\delta_l(y_t, y_{t-1})\,dy_t. \tag{41}$$

[1] For (ii) we can choose δ_l appropriately to make $\int_0^1 \delta_l(y_t, y_{t-1})\,dy_t = 0$, this can be achieved by using $\tilde{\delta}_l(y_t, y_{t-1}) = \delta_l(y_t, y_{t-1}) - \int_0^1 \delta_l(y_t, y_{t-1})\,dy_t$ if indeed $\int_0^1 \delta_l(y_t, y_{t-1})\,dy_t \neq 0$.

Using $(i)-(v)$, the score functions under the null are given by

$$\frac{\partial l}{\partial \theta_j} = \sum_{t=1}^{n} \pi_j(y_t), \; j = 1, ..., k,$$
$$\frac{\partial l}{\partial \phi_l} = \sum_{t=1}^{n} \delta_l(y_t, y_{t-1}), \; l = 1, ..., q. \tag{42}$$

The information matrix under H_0, $\mathcal{I}$ is given by

$$\mathcal{I} = -\left.\left[\begin{array}{cc} E\left[\frac{\partial^2 l}{\partial\theta\partial\theta'}\right] & E\left[\frac{\partial^2 l}{\partial\phi_l\partial\theta_j'}\right] \\ E\left[\frac{\partial^2 l}{\partial\phi_l\partial\theta_j'}\right]' & E\left[\frac{\partial^2 l}{\partial\phi_i\partial\phi_l}\right] \end{array}\right]\right|_{\theta=0,\phi=0}, \tag{43}$$

where given I_k is the $k \times k$ identity matrix

$$E\left[-\frac{\partial^2 l}{\partial\theta\partial\theta'}\right] = nI_k,$$
$$E\left[-\frac{\partial^2 l}{\partial\phi_l\partial\theta_j'}\right] = n\left(E\left[\int_0^1 \pi_j(y_t)\,\delta_l(y_t, y_{t-1})\,dy_t\right]\right)_{j=1,...,k;\; l=1,...,q}$$
$$= E[\pi\delta],$$
$$E\left[-\frac{\partial^2 l}{\partial\phi_i\partial\phi_l}\right] = n\left(E\left[\int_0^1 \delta_i(y_t, y_{t-1})\,\delta_l(y_t, y_{t-1})\,dy_t\right]\right)_{j=1,...,k;\; l=1,...,q}$$
$$= E[\delta\delta]. \tag{44}$$

So, using the well-known results of the Rao score test, defining $u_j = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\pi_j(y_t)$, $j = 1, ..., k$ and $v_l = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\delta_l(y_t, y_{t-1})$, $l = 1, ..., q$,

$$\left[\begin{array}{c} \sqrt{n}U \\ \sqrt{n}V \end{array}\right]' \left[\begin{array}{cc} nI_k & nE[\pi\delta] \\ nE[\pi\delta]' & nE[\delta\delta] \end{array}\right]^{-1} \left[\begin{array}{c} \sqrt{n}U \\ \sqrt{n}V \end{array}\right] \sim \chi^2_{k+q}(0)$$
$$\Rightarrow \left[\begin{array}{c} U \\ V \end{array}\right]' \left[\begin{array}{cc} I_k & E[\pi\delta] \\ E[\pi\delta]' & E[\delta\delta] \end{array}\right]^{-1} \left[\begin{array}{c} U \\ V \end{array}\right] \sim \chi^2_{k+q}(0), \tag{45}$$

where $U = (u_1, u_2, ..., u_k)'$, $V = (v_1, v_2, ..., v_q)'$ and $\chi^2_d(0)$ means a central χ^2 distribution with d degrees of freedom. Simplifying the notation further, and defining $B = E[\pi\delta]$, $D = E[\delta\delta]$,from results on block matrices we have

$$\left[\begin{array}{cc} I_k & B \\ B' & D \end{array}\right]^{-1} = \left[\begin{array}{cc} I_k + BEB' & -BE \\ -EB' & E \end{array}\right] \tag{46}$$

where $E = (D - B'B)^{-1}$. From (45) and (46),

$$\begin{aligned}
&\begin{bmatrix} U \\ V \end{bmatrix}' \begin{bmatrix} I_k & E[\pi\delta] \\ E[\pi\delta]' & E[\delta\delta] \end{bmatrix}^{-1} \begin{bmatrix} U \\ V \end{bmatrix} \\
&= \begin{bmatrix} U'U + U'BEB'U - V'EB'U \\ -U'BEV + V'EV \end{bmatrix} \overset{a}{\sim} \chi^2_{k+l}. \qquad (47)
\end{aligned}$$

As E is non-singular there exists a non-singular matrix L, such that $E = LL'$. Substituting this in equation (47), we can rewrite as

$$\begin{aligned}
&U'BEB'U - V'EB'U - U'BEV + V'EV \\
&= U'BLL'B'U - V'LL'B'U - U'BLL'V + V'LL'V \\
&= (L'V)' L'V - (L'V)' L'B'U \\
&- (L'B'U)' L'V + (L'B'U)' L'B'U \\
&= (L'V - L'B'U)' (L'V - L'B'U) \\
&= (L'(V - B'U))' (L'(V - B'U)) \\
&= (V - B'U)' LL' (V - B'U) \\
&= (V - B'U)' E (V - B'U) \qquad (48)
\end{aligned}$$

From (47) this gives

$$U'U + (V - B'U)' E (V - B'U) \overset{a}{\sim} \chi^2_{k+l}. \qquad (49)$$

■

5.0.2 Case 1: (Fixed ϕ) Test for Weighted Autoregressive Terms

In our usual formulation with $q = 1$, $\delta_1(y_t, y_{t-1}, ..., y_1) = (y_t - 0.5)\sum_{s=1}^{r} \phi^{s-1}(y_{t-s} - m_1)$, we can obtain

$$v_1 = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (y_t - 0.5) \sum_{s=1}^{r} \phi^{s-1} (y_{t-s} - m_1) \qquad (50)$$

as the score function related to δ_1. Furthermore, given the model in (26), $E(y_t) = \mu = 0$,

$$E\left[\int_0^1 \pi_j(y_t)\,\delta_1(y_t, y_{t-1}, ..., y_1)\,dy_t\right] = E\left[\int_0^1 \pi_j(y_t)\,a_1\pi_1(y_t)\sum_{s=1}^{r}\phi^{s-1}(y_{t-s}-m_1)\,dy_t\right]$$
$$= \begin{cases} a_1 E\left[\sum_{s=1}^{t-1}\phi^{s-1}(y_{t-s}-m_1)\right] & \text{if } j=1 \\ 0 & \text{otherwise.}\end{cases}$$
$$= \begin{cases} a_1 \frac{1-\phi^{r-1}}{1-\phi} E(y_{t-s}-m_1) = 0 & \text{if } j=1 \\ 0 & \text{otherwise.}\end{cases} \tag{51}$$

since under H_0, $E(y_{t-s}-m_1) = 0$, for all s. Similarly,

$$E\left[\int_0^1 [\delta_1(y_t, y_{t-1}, ..., y_1)]^2\,dy_t\right] = E\left[a_1^2\int_0^1 \pi_1^2(y_t)\left(\sum_{s=1}^{r}\phi^{s-1}(y_{t-s}-m_1)\right)^2 dy_t\right]$$
$$= a_1^2 E\left[\sum_{s=1}^{r}\phi^{s-1}(y_{t-s}-m_1)\right]^2$$
$$= a_1^2\frac{1-\phi^{2(r)}}{1-\phi^2}E(y_{t-s}-m_1)^2 = a_1^2\frac{1-\phi^{2(r)}}{1-\phi^2}m_{11}, \tag{52}$$

since under H_0 all $y_t's$ are independent, and $E(y_{t-s}-m_1)^2 = \sigma_\varepsilon^2 = m_{11}$. Hence, the asymptotic information matrix is given by

$$\mathcal{I} = \begin{bmatrix} 1 & \mathbf{0}' & 0 \\ \mathbf{0} & I_{k-1} & \mathbf{0} \\ 0 & \mathbf{0}' & a_1^2\frac{1-\phi^{2(r)}}{1-\phi^2}m_{11} \end{bmatrix} = \begin{bmatrix} I_k & B \\ B' & D \end{bmatrix}. \tag{53}$$

Using the same notations as before we obtain the following results:

$(i)\, D - B'B = a_1^2\frac{(1-\phi^{2r})m_{11}}{1-\phi^2} - \left(\frac{0}{1-\phi}\right)^2 \Rightarrow E = (D-B'B)^{-1} = \left[a_1^2\frac{(1-\phi^{2r})m_{11}}{1-\phi^2}\right]^{-1}$.

$(ii)\, U'BEB'U = 0.$

$(iii)\, U'BEV = 0.$

$(iv)\, V'EV = v_1^2 E = v_1^2\left[a_1^2\frac{(1-\phi^{2r})m_{11}}{1-\phi^2}\right]^{-1}$.

Hence, if ϕ is a known constant,

$$\begin{aligned}
&\sum_{j=1}^{k} u_j^2 + v_1^2 E \\
&= \sum_{j=1}^{k} u_j^2 + v_1^2 \left[a_1^2 \frac{\left(1-\phi^{2r}\right) m_{11}}{1-\phi^2} \right]^{-1} \\
&\sim \chi_{k+1}^2.
\end{aligned} \tag{54}$$

5.0.3 Case 3: Weighted ARCH Model

The log-likelihood function is

$$\begin{aligned}
L &= Const - \frac{1}{2}\sum_{t=1}^{n} \ln h_t - \frac{1}{2}\sum_{t=1}^{n} \frac{u_t^2}{h_t} \\
&= Const - \frac{1}{2}\sum_{t=1}^{n} \ln \left[\alpha_0 + \alpha_1 \sum_{s=1}^{r} w_s u_{t-s}^2 \right] - \frac{1}{2}\sum_{t=1}^{n} \frac{u_t^2}{\alpha_0 + \alpha_1 \sum_{s=1}^{r} w_s u_{t-s}^2}.
\end{aligned} \tag{55}$$

Differentiating (55) with respect to α_1,

$$\begin{aligned}
\frac{\partial L}{\partial \alpha_1} &= -\frac{1}{2}\sum_{t=1}^{n} \frac{\left[\sum_{s=1}^{r} w_s u_{t-s}^2\right]}{\left[\alpha_0 + \alpha_1 \sum_{s=1}^{r} w_s u_{t-s}^2\right]} + \frac{1}{2}\sum_{t=1}^{n} \frac{u_t^2 \left[\sum_{s=1}^{r} w_s u_{t-s}^2\right]}{\left[\alpha_0 + \alpha_1 \sum_{s=1}^{r} w_s u_{t-s}^2\right]^2} \\
\Rightarrow \left.\frac{\partial L}{\partial \alpha_1}\right|_{H_0} &= \frac{1}{2\alpha_0^2}\sum_{t=1}^{n} \tilde{u}_t^2 \sum_{s=1}^{r} w_s u_{t-s}^2, \text{ where } \tilde{u}_t^2 = u_t^2 - \alpha_0.
\end{aligned} \tag{56}$$

From (56) differentiating again

$$\begin{aligned}
\frac{\partial^2 L}{\partial \alpha_1^2} &= \frac{1}{2}\sum_{t=1}^{n} \frac{\left[\sum_{s=1}^{r} w_s u_{t-s}^2\right]^2}{\left[\alpha_0 + \alpha_1 \sum_{s=1}^{r} w_s u_{t-s}^2\right]^2} - \frac{2}{2}\sum_{t=1}^{n} \frac{u_t^2 \left[\sum_{s=1}^{r} w_s u_{t-s}^2\right]^2}{\left[\alpha_0 + \alpha_1 \sum_{s=1}^{r} w_s u_{t-s}^2\right]^3} \\
\Rightarrow - \left.\frac{\partial^2 L}{\partial \alpha_1^2}\right|_{H_0} &= \frac{1}{2\alpha_0^3}\sum_{t=1}^{n} \left[2u_t^2 - \alpha_0\right] \left[\sum_{s=1}^{r} w_s u_{t-s}^2\right]^2.
\end{aligned} \tag{57}$$

Now taking expectation of (57) as $n \to \infty$ the asymptotic information matrix is

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} E\left[-\left.\frac{\partial^2 L}{\partial \alpha_1^2}\right|_{H_0} \right] &= \lim_{n\to\infty} \frac{1}{n} \frac{\alpha_0}{2\alpha_0^3} \sum_{t=1}^{n} E\left[\sum_{s=1}^{r} w_s u_{t-s}^2 \right]^2 \\
&= \frac{1}{2\alpha_0^2} \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{r} w_s^2 E\left[u_{t-s}^4\right], \, IID \text{ under } H_0 \\
&= \frac{1}{2\alpha_0^2} E\left[u_t^4\right] \sum_{s=1}^{r} \frac{\left[(r+1)-s\right]^2}{\left[\frac{1}{2} r(r+1)\right]^2} \\
&= \frac{1}{2\alpha_0^2} E\left[u_t^4\right] \frac{\frac{1}{6} r(r+1)(2r+1)}{\left[\frac{1}{2} r(r+1)\right]^2} \\
&= \frac{1}{2\alpha_0^2} E\left[u_t^4\right] \frac{2}{3} \frac{(2r+1)}{r(r+1)} \\
&= \frac{(2r+1)}{r(r+1)} \text{ putting } E\left[u_t^4\right] = 3\alpha_0^2. \qquad (58)
\end{aligned}$$

Hence, the Rao Score Statistic is

$$RS = n^{-1} \left[\frac{1}{2\hat{\alpha}_0^2} \sum_{t=1}^{n} \tilde{u}_t^2 \sum_{s=1}^{r} w_s u_{t-s}^2 \right]^2 \frac{r(r+1)}{(2r+1)} = n \frac{r(r+1)}{(2r+1)} \frac{\left[\sum_{t=1}^{n} \tilde{u}_t^2 \sum_{s=1}^{r} w_s u_{t-s}^2\right]^2}{4\left[\sum_{t=1}^{n} u_t^4\right]^2} \qquad (59)$$

which for testing for ARCH(1) becomes

$$RS = \frac{n}{12} \frac{\left[\sum_{t=1}^{n} \tilde{u}_t^2 u_{t-1}^2\right]^2}{\left[\sum_{t=1}^{n} u_t^4\right]^2}. \qquad (60)$$

Now let us setup the augmented Neyman Smooth test for incorporating several ARCH effects using a linear weighting scheme suggested by Engle (1982, 1983). We choose

$$\begin{aligned}
\delta_2(y_t, y_{t-1}, ..., y_1) &= \left(y_t^2 - \frac{1}{3}\right) \sum_{s=1}^{r} w_s \left(y_{t-s}^2 - m_2\right), \text{ where } w_s = \frac{(r+1)-s}{\frac{1}{2} r(r+1)} \\
\Rightarrow v_2 &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left(a_1 \pi_1(y_t) + a_2 \pi_2(y_t)\right) \sum_{s=1}^{r} w_s \left(y_{t-s}^2 - m_2\right) \text{ is the score function.}
\end{aligned} \qquad (61)$$

Furthermore,

$$E\left[\int_0^1 \pi_j(y_t)\,\delta_2(y_t, y_{t-1}, ..., y_1)\,dy_t\right] = E\left[\begin{array}{c}\int_0^1 \pi_j(y_t)\left(a_1\pi_1(y_t) + a_2\pi_2(y_t)\right) \\ \times \sum_{s=1}^r w_s\left(y_{t-s}^2 - m_2\right) dy_t\end{array}\right]$$

$$= \begin{cases} a_1 E\left[\sum_{s=1}^r w_s\left(y_{t-s}^2 - m_2\right)\right] & \text{if } j = 1 \\ a_2 E\left[\sum_{s=1}^r w_s\left(y_{t-s}^2 - m_2\right)\right] & \text{if } j = 2 \\ 0 & \text{otherwise.}\end{cases}$$

$$= \begin{cases} a_1 \sum_{s=1}^r w_s E\left(y_{t-s}^2 - m_2\right) & \text{if } j = 1 \\ a_2 \sum_{s=1}^r w_s E\left(y_{t-s}^2 - m_2\right) & \text{if } j = 2 \\ 0 & \text{otherwise.}\end{cases}$$

$$= \begin{cases} a_1 \sum_{s=1}^r w_s E\left(y_{t-s}^2 - m_2\right) = 0 & \text{if } j = 1 \\ a_2 \sum_{s=1}^r w_s E\left(y_{t-s}^2 - m_2\right) = 0 & \text{if } j = 2 \\ 0 & \text{otherwise.}\end{cases} \tag{62}$$

since under H_0, $E\left(y_{t-s}^2 - m_2\right) = 0$, for all s. Further,

$$E\left[\int_0^1 \left[\delta_2(y_t, y_{t-1}, ..., y_1)\right]^2 dy_t\right] = E\left[\begin{array}{c}\int_0^1 \left(a_1\pi_1(y_t) + a_2\pi_2(y_t)\right)^2 \\ \times \left(\sum_{s=1}^r w_s\left(y_{t-s}^2 - m_2\right)\right)^2 dy_t\end{array}\right]$$

$$= \left(a_1^2 + a_2^2\right) E\left[\sum_{s=1}^r w_s\left(y_{t-s}^2 - m_2\right)\right]^2$$

$$= a_3^2 \sum_{s=1}^r w_s^2 E\left(y_{t-s}^2 - m_2\right)^2$$

$$= a_3^2 m_{22} \sum_{s=1}^r w_s^2 \tag{63}$$

since under H_0 all $y_t s$ are independent and $E\left(y_{t-s}^2 - m_2\right)^2 = m_{22}$, for all s. Hence, the asymptotic information matrix is given by

$$\mathcal{I} = \begin{bmatrix} 1 & 0 & \mathbf{0}' & 0 \\ 0 & 1 & 0 & 0 \\ \mathbf{0} & 0 & I_{k-1} & \mathbf{0} \\ 0 & 0 & \mathbf{0}' & \frac{4(3+r)}{3r(r+1)} a_3^2 m_{22} \end{bmatrix} = \begin{bmatrix} I_k & B \\ B' & D \end{bmatrix}. \tag{64}$$

Using the same notations as before we obtain the following results:

(i) $D - B'B = \frac{4(3+r)}{3r(r+1)} a_3^2 m_{22} \Rightarrow E = \left(D - B'B\right)^{-1}$.

$(ii)\, U'BEB'U = 0.$
$(iii)\, U'BEV = 0.$
$(iv)\, V'EV = v_1^2 \left[\frac{4(3+r)}{3r(r+1)} a_3^2 m_{22}\right]^{-1}.$
Hence, if r is a known constant,

$$\begin{aligned} &\sum_{j=1}^{k} u_j^2 + v_2^2 E \\ &= \sum_{j=1}^{k} u_j^2 + v_2^2 \left[\frac{4\,(3+r)}{3r\,(r+1)} a_3^2 m_{22}\right]^{-1} \\ &\sim \chi_{k+1}^2. \end{aligned} \tag{65}$$

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