## Section 11 Rank and Linear Dependence

In an early section, we learnt that the number of solutions in a system of linear equations depends on the number of "independent" equations in the system relative to the number of unknowns. From Cramer's Rule, we learnt that a system of $n$-linear equations in $n$-unknowns will have a unique solution only when the determinant of the coefficient matrix is not zero. When will the determinant of a matrix be zero? This has a lot to do with the linear dependence in the coefficient matrix.

The presence of any zero row or column will produce a zero determinant (this is easy to verify: just use the Laplace expansion along that zero row or column).

In the $(2\times 2)$ case we obtained a zero determinant when one row was a multiple of the other (or is the same as the other), and this remains true of the $(3\times 3)$ case. A more subtle case is when one row is a linear combination of the other two (i.e., we can write one row as the sum of some multiple of a second row plus some multiple of the third). For instance, take the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 4 \\ 3 & 3 & 5 \end{bmatrix}$$

which you can verify has a zero determinant. Here the third row is equal to the first row plus half of the second row:

$$\begin{bmatrix} 3 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 4 & 2 & 4 \end{bmatrix}.$$

(*Alternatively, we can say that the first row is the third row minus half of the second*

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 5 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 4 & 2 & 4 \end{bmatrix}.$$

*or that the second row is two times the third row minus two times the first.)*

We say that the three row vectors that make up the matrix are "linearly dependent". This "linear dependence" between the rows leads to a zero determinant because it means that row operations can be applied to ultimately create a zero row.

A $(3\times 3)$ matrix will have a non-zero determinant only if it has three linearly <u>independent</u> rows (or columns). More generally, think of a $(3\times 3)$ square matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

as a stack of three row vectors

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} [a_{11} & a_{12} & a_{13}] \\ [a_{21} & a_{22} & a_{23}] \\ [a_{31} & a_{32} & a_{33}] \end{bmatrix}$$

We say that three (column or row) vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly dependent if there are constants $c_1$, $c_2$, and $c_3$, not all zero, such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = \mathbf{0}.$$

If the only case where this is true is $c_1 = c_2 = c_3 = 0$, then the vectors are said to be linearly independent. For now, define the **rank** of the matrix is the number of linearly independent row vectors that it contains.

*Examples*

1. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 1 & 3 \end{bmatrix}$. Let $\mathbf{a}_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$.

   Then these vectors are linearly dependent, since $2\mathbf{a}_1 - 1.\mathbf{a}_2 + 0.\mathbf{a}_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. The second row is just twice that of the first row (or the first is half of the second). Removing one of the dependent vectors, say $\mathbf{a}_2 = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$, the remaining vectors $\mathbf{a}_1$ and $\mathbf{a}_3$ are linearly independent: the only constants $c_1$ and $c_2$ such that $c_1\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + c_2\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ are $c_1 = c_2 = 0$. The matrix has two linear independent vectors, and is said to be of rank 2.

If we imagine these three vectors in the usual representation as 'arrows' in three-dimensional co-ordinate space, we will see that the vector $\mathbf{a}_2$ is just an extension of $\mathbf{a}_1$. Combinations of the three vectors are effectively combinations of only two vectors.

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = c_1\mathbf{a}_1 + c_2 2\mathbf{a}_1 + c_3\mathbf{a}_3 = (c_1 + 2c_2)\mathbf{a}_1 + c_3\mathbf{a}_3$$

Different combinations of the three vectors will therefore result in new vectors that all lie on a plane (a two-dimensional space), and cannot 'span' or cover the entire 3-d space.

2. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \\ 2 & 1 & 3 \end{bmatrix}$. Let $\mathbf{a}_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 & 3 & 6 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$.

   Then these vectors are linearly dependent, since $1\mathbf{a}_1 - 1.\mathbf{a}_2 + 1.\mathbf{a}_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. The second row "depends" on the other two in that it is the sum of $\mathbf{a}_1$ and $\mathbf{a}_3$.

(Equivalently, $\mathbf{a}_3$ "depends on" $\mathbf{a}_1$ and $\mathbf{a}_2$ because it is $\mathbf{a}_2 - \mathbf{a}_1$). Removing one of these 'dependent' vectors (any one of them) leaves two independent vectors. For instance, removing $\mathbf{a}_3$, then the only constants $c_1$ and $c_2$ such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 \;=\; c_1\begin{bmatrix}1 & 2 & 3\end{bmatrix} + c_2\begin{bmatrix}3 & 3 & 6\end{bmatrix} \;=\; \begin{bmatrix}0 & 0 & 0\end{bmatrix}$$

are $c_1 = c_2 = 0$. The matrix has two linear independent vectors, and is said to be of rank 2.

If we imagine these three vectors geometrically in three-dimensional co-ordinate space, we will see that although no one vector is an extension of another, all three vectors nonetheless lie in a single plane, so again combinations of the three vectors will 'create' new vectors that also lie on that plane. The three vectors cannot 'span', or cover, the entire 3-d space.

3. Let $\mathbf{A} = \begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{bmatrix}$. Let $\mathbf{a}_1 = \begin{bmatrix}1 & 2 & 3\end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix}2 & 4 & 6\end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix}3 & 6 & 9\end{bmatrix}$.

   Then these vectors are linearly dependent, since $1\mathbf{a}_1 + 1.\mathbf{a}_2 - 1.\mathbf{a}_3 \;=\; \begin{bmatrix}0 & 0 & 0\end{bmatrix}$. In this case, there is only one linearly independent vector. If we remove $\mathbf{a}_3$, we find that the remaining two are still linearly dependent: $\mathbf{a}_2$ is twice that of $\mathbf{a}_1$. The rank of this matrix is one: $rank(\mathbf{A}) = 1$.

Geometrically, all three vectors lie on a single line. The vector $\mathbf{a}_2$ is twice that of $\mathbf{a}_1$, and $\mathbf{a}_3$ is three times that of $\mathbf{a}_1$. Combinations of these vectors therefore will only create new vectors that are *also* extensions of $\mathbf{a}_1$.

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 \;=\; c_1\mathbf{a}_1 + c_2 2\mathbf{a}_1 + c_3 3\mathbf{a}_1 \;=\; (c_1 + 2c_2 + 3c_3)\mathbf{a}_1 \,.$$

4. Let $\mathbf{A} = \begin{bmatrix}0 & 0 & 0\\ 2 & 4 & 6\\ 2 & 1 & 3\end{bmatrix}$. Let $\mathbf{a}_1 = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix}2 & 4 & 6\end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix}2 & 1 & 3\end{bmatrix}$.

   Then these vectors are linearly dependent, since $c_1.\mathbf{a}_1 + 0.\mathbf{a}_2 + 0.\mathbf{a}_3 \;=\; \begin{bmatrix}0 & 0 & 0\end{bmatrix}$ for any $c_1$. Removing vector $\mathbf{a}_1$ leaves two linearly independent vectors. The rank of this matrix is two.

5. Let $\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 2 & 1 \end{bmatrix}$. Let $\mathbf{a}_1=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\mathbf{a}_2=\begin{bmatrix} 0 & 2 & 4 \end{bmatrix}$, and $\mathbf{a}_3=\begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$.

Then all three row vectors are linearly independent; it is impossible to find $c_1$, $c_2$ and $c_3$, not all zero, such that $c_1\mathbf{a}_1+c_2\mathbf{a}_2+c_3\mathbf{a}_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. This matrix is of "**full rank**".

A square matrix will have a non-zero determinant if and only if ('iff') it has full rank.

There are many ways to determine the rank of a matrix. One way is to do 'Gaussian Elimation" (the row operations) and see how many non-zero pivots you get (see Section 2 for 'pivots'). Other ways will be discussed in later sections, or in more advanced classes. For the time being we will proceed by observation.

We conclude this section with several remarks regarding rank. Earlier, we viewed a matrix as a stack of row vectors. The rank concept that we discussed could be called 'row rank'. But we can also view a matrix as the concatenation (joining together) of three column vectors

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} & \begin{bmatrix} a_{12} \\ a_{22} \\ a_{23} \end{bmatrix} & \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \end{bmatrix}$$

1. Everything that we said here follows for column vectors as for row vectors, and in fact a matrix will have the same number of linearly independent column vectors as there are linearly independent row vectors: if $\mathbf{A}$ has two linearly independent row vectors, it will also have two linearly independent column vectors; if it has only one linearly independent row vector, then it will one have one linearly independent column vector, and so on. A matrix's row rank is the same as its "column rank".

2. We can also speak of the rank of non-square matrices. Take for example

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 1 & 3 \end{bmatrix}$$

This is in fact the last two rows of an earlier example we were looking at. Looking at the rows, we see that these two rows are linearly independent (verify!) The 'row-rank' is two. What if we view this matrix as a concatenation of three two-dimensional vectors? Note that three two-dimensional vectors can only span a two-dimensional space. Draw three two dimensional vectors on a "$x$ - $y$" co-ordinate space on a piece of paper. Can you generate a three-dimensional space from this? (The answer is no.)

If we observe the three vectors

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

we observe that these *do* span the entire two-dimensional plane. So the column rank of this matrix is also 2. The principle that a matrix's row and column ranks are the same continues to hold, and holds generally.

As another illustration, consider

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

It is easy to see that the row rank of this matrix is one, and the column rank of this matrix is also one.

Because the row and column ranks are always the same, we can always speak unambiguously of "the rank of a matrix". Furthermore,

$$rank(\mathbf{A}) \le \min(\#\text{rows}, \#\text{columns})$$

3. Finally, we often need to find the rank of a product. Here are two very useful results:

3a If $\mathbf{A}$ is $(m \times n)$ and $\mathbf{B}$ is $(n \times p)$, then

$$rank(\mathbf{AB}) \le \min(rank(\mathbf{A}), rank(\mathbf{B}))$$

*Proof* Let $\mathbf{C} = \mathbf{AB}$. Every column of $\mathbf{C}$ is some linear combination of the columns of $\mathbf{A}$, therefore the columns in $\mathbf{C}$ cannot span a space of dimension greater than the column rank of $\mathbf{A}$. Similarly, every row of $\mathbf{C}$ is some linear combination of the rows of $\mathbf{B}$, therefore the rows in $\mathbf{C}$ cannot span a space of dimension greater than the row rank of $\mathbf{B}$. Since row rank and column rank are the same, we have our result.

3b If $\mathbf{A}$ is $(m \times n)$ and $\mathbf{B}$ is $(n \times n)$ and full rank, then

$$rank(\mathbf{AB}) = rank(\mathbf{A})$$

*Proof* If $x \le \min(a,b)$, then obviously $x \le a$ and $x \le b$ are both true. Since

$$rank(\mathbf{AB}) \;\le\; \min(rank(\mathbf{A}), rank(\mathbf{B})) .$$

we have

$$\begin{aligned} rank(\mathbf{A}) &= rank(\mathbf{ABB}^{-1}) && [\mathbf{B}^{-1}\text{exists since } \mathbf{B} \text{ is full rank}] \\ &\le \min(rank(\mathbf{AB}), \mathbf{B}^{-1}) \\ &\le rank(\mathbf{AB}) \\ &\le \min(rank(\mathbf{A}), rank(\mathbf{B})) \\ &\le rank(\mathbf{A}) \end{aligned}$$

This extends to the product $\mathbf{CAB}$ where both $\mathbf{C}$ and $\mathbf{B}$ are full rank:

$$rank(\mathbf{CAB}) = rank(\mathbf{A}) .$$

**Exercises**

1. Find the determinants of the following matrices using the Laplace expansion. If you find the determinant of a matrix to be zero, find out how many linear independent rows the matrix has.

$$\text{(i)} \quad \begin{bmatrix} 4 & 0 & 1 \\ 19 & 1 & -3 \\ 7 & 1 & 0 \end{bmatrix} \qquad \text{(ii)} \quad \begin{bmatrix} 4 & 3 & 0 \\ 19 & 1 & 0 \\ 7 & 1 & 0 \end{bmatrix} \text{(iii)} \begin{bmatrix} 4 & 3 & 0 \\ 1 & 1 & 2 \\ 7 & 6 & 6 \end{bmatrix} \text{(vi)} \begin{bmatrix} 1 & 3 & 1 \\ -2 & -6 & -2 \\ 4 & 12 & 4 \end{bmatrix}$$

2. Show using the formula

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

that the determinant of a $(3 \times 3)$ matrix will be zero if

(i) one row is a multiple of another;

(ii) one row is a linear combination of the other two.

3. Show that $rank(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = rank(\mathbf{A})$.