

Section 9 The Laplace Expansion

In the last section, we defined the determinant of a (3×3) matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

to be $|\mathbf{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$.

In this section, we introduce a general formula for computing determinants. Rewriting

$$\begin{aligned} |\mathbf{A}| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

note that the terms outside the brackets are the terms along the first row of the matrix

$$\begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The term in brackets associated with a_{11} is the determinant of the (2×2) matrix after deleting the 1st row and 1st column of \mathbf{A} :

$$(a_{22}a_{33} - a_{23}a_{32}) = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

The term in brackets associated with a_{12} is the determinant of the (2×2) matrix after deleting the 1st row and 2nd column of \mathbf{A} :

$$(a_{21}a_{33} - a_{23}a_{31}) = \begin{vmatrix} \cancel{a_{11}} & \boxed{\cancel{a_{12}}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix}$$

The term in brackets associated with a_{13} is the determinant of the (2×2) matrix after deleting the 1st row and 3rd column of \mathbf{A} :

$$(a_{21}a_{32} - a_{22}a_{31}) = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \boxed{\cancel{a_{13}}} \\ a_{21} & a_{22} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{vmatrix}.$$

There is the matter of the minus sign in front of a_{12} . This can be achieved by multiplying into each term $(-1)^{i+j}$. Therefore, we can write

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{11} (-1)^{1+1} \underbrace{\begin{vmatrix} \cancel{a_{11}} & \cancel{a_{13}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix}}_{(1,1)\text{th Minor of } \mathbf{A}} + a_{12} (-1)^{1+2} \underbrace{\begin{vmatrix} \cancel{a_{11}} & \boxed{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix}}_{(1,2)\text{th Minor of } \mathbf{A}} + a_{13} (-1)^{1+3} \underbrace{\begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \boxed{a_{13}} \\ a_{21} & a_{22} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{vmatrix}}_{(1,3)\text{th Minor of } \mathbf{A}}$$

(1,1)th Cofactor of \mathbf{A} (1,2)th Cofactor of \mathbf{A} (1,3)th Cofactor of \mathbf{A}

Some names have been introduced in the formula above: the determinant of the matrix after removing the i th row and j th column is called the (i, j) th “minor” of \mathbf{A} ; including the sign $(-1)^{i+j}$ gives us the (i, j) th “cofactor” of \mathbf{A} .

Example The determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 3 \\ -3 & 3 & 2 \end{bmatrix}$$

is

$$|\mathbf{A}| = (0)(-1)^{1+1} \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} + (2)(-1)^{1+2} \begin{vmatrix} 1 & 3 \\ -3 & 2 \end{vmatrix} + (-1)(-1)^{1+3} \begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} = -28;$$

The cofactor expansion for computing determinants is not unique. For instance, we could have written the original formula as

$$\begin{aligned} |\mathbf{A}| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &= -a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\ &= -a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{23}(a_{11}a_{32} - a_{12}a_{31}) \end{aligned}$$

which can be written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{21} (-1)^{2+1} \underbrace{\begin{vmatrix} \cancel{a_{11}} & a_{12} & a_{13} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix}}_{\substack{(2,1)\text{th Minor of } \mathbf{A} \\ (2,1)\text{th Cofactor of } \mathbf{A}}} + a_{22} (-1)^{2+2} \underbrace{\begin{vmatrix} a_{11} & \cancel{a_{12}} & a_{13} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix}}_{\substack{(2,2)\text{th Minor of } \mathbf{A} \\ (2,2)\text{th Cofactor of } \mathbf{A}}} + a_{23} (-1)^{2+3} \underbrace{\begin{vmatrix} a_{11} & a_{12} & \cancel{a_{13}} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{vmatrix}}_{\substack{(2,3)\text{th Minor of } \mathbf{A} \\ (2,3)\text{th Cofactor of } \mathbf{A}}}$$

Alternatively, we could have expanded along a column:

$$\begin{aligned}
|\mathbf{A}| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\
&= a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} \\
&= -a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{32}(a_{11}a_{23} - a_{13}a_{21})
\end{aligned}$$

in which case, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{12} (-1)^{1+2} \underbrace{\begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix}}_{\substack{(1,2)\text{th Minor of } \mathbf{A} \\ (1,2)\text{th Cofactor of } \mathbf{A}}} + a_{22} (-1)^{2+2} \underbrace{\begin{vmatrix} a_{11} & \cancel{a_{12}} & a_{13} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix}}_{\substack{(2,2)\text{th Minor of } \mathbf{A} \\ (2,2)\text{th Cofactor of } \mathbf{A}}} + a_{32} (-1)^{3+2} \underbrace{\begin{vmatrix} a_{11} & \cancel{a_{12}} & a_{13} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} \end{vmatrix}}_{\substack{(3,2)\text{th Minor of } \mathbf{A} \\ (3,2)\text{th Cofactor of } \mathbf{A}}}$$

Note that we achieved the correct signs by multiplying the (i, j) th minor with $(-1)^{i+j}$.

Expanding along any row or column would in fact give us the same expression; we can write

$$|\mathbf{A}| = \sum_{j=1}^3 a_{ij} (-1)^{i+j} M_{ij} \text{ for any row } i$$

$$\text{or } |\mathbf{A}| = \sum_{i=1}^3 a_{ij} (-1)^{i+j} M_{ij} \text{ for any column } j.$$

where M_{ij} is the (i, j) th minor of \mathbf{A} . This formula is known as the Laplace Expansion.

The fact that you can expand along any row or column can simplify computations substantially if there is a row or column with many zeros.

Example Compute the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 11 \\ 2 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

using the Laplace Expansion (i) expanding along the first row, (ii) expanding down the second column.

$$(i) \quad \begin{vmatrix} 1 & 0 & 11 \\ 2 & 0 & 3 \\ 5 & 4 & 6 \end{vmatrix} = (1)(-1)^{1+1} \begin{vmatrix} 0 & 3 \\ 4 & 6 \end{vmatrix} + (0)(-1)^{1+2} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (11)(-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 5 & 4 \end{vmatrix} = -12 + 0 + 88 = 76$$

$$(ii) \quad \begin{vmatrix} 1 & 0 & 11 \\ 2 & 0 & 3 \\ 5 & 4 & 6 \end{vmatrix} = (0)(-1)^{1+2} \begin{vmatrix} 1 & 11 \\ 5 & 6 \end{vmatrix} + (0)(-1)^{2+2} \begin{vmatrix} 1 & 11 \\ 5 & 6 \end{vmatrix} + (4)(-1)^{3+2} \begin{vmatrix} 1 & 11 \\ 2 & 3 \end{vmatrix} = 76$$

(n × n) Determinants

Cramer's Rule and the Laplace Expansion extend to larger systems. The determinant for a general $(n \times n)$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is $|\mathbf{A}| = \sum_{j=1}^n a_{ij}(-1)^{i+j} M_{ij}$ for any row i

or $|\mathbf{A}| = \sum_{i=1}^n a_{ij}(-1)^{i+j} M_{ij}$ for any column j

where the (i, j) th minor M_{ij} of an $n \times n$ matrix \mathbf{A} is the determinant of the $(n-1) \times (n-1)$ matrix that remains after removing the i th row and j th column of \mathbf{A} .

Example

The determinant of $\mathbf{D} = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 1 & -1 & 0 & 3 \\ -2 & 0 & -1 & 3 \end{bmatrix}$ is

$$\begin{aligned} |\mathbf{D}| &= \underbrace{3(-1)^{1+3} \begin{vmatrix} 0 & -1 & 11 \\ 1 & -1 & 3 \\ -2 & 0 & 3 \end{vmatrix}}_{=-39} + \underbrace{0(-1)^{2+3} \begin{vmatrix} 2 & 2 & 4 \\ 1 & -1 & 3 \\ -2 & 0 & 3 \end{vmatrix}}_{=0} \dots \\ &\quad + \underbrace{0(-1)^{3+3} \begin{vmatrix} 2 & 2 & 4 \\ 0 & -1 & 11 \\ -2 & 0 & 3 \end{vmatrix}}_{=0} + \underbrace{(-1)(-1)^{4+3} \begin{vmatrix} 2 & 2 & 4 \\ 0 & -1 & 11 \\ 1 & -1 & 3 \end{vmatrix}}_{=42} = 3 \end{aligned}$$

where we have expanded down the third column.

The Laplace expansion includes the (2×2) case. Define the determinant of a single number as $|a| = a$. Note that in this context the symbol $|\cdot|$ does not refer to absolute values, e.g. $|-2| = -2$, not 2. Then taking, say, a first row expansion, we have

$$|\mathbf{A}| = \sum_{j=1}^2 a_{1j}(-1)^{1+j} M_{1j} = a_{11}(-1)^{1+1} M_{11} + a_{12}(-1)^{1+2} M_{12} = a_{11}a_{22} - a_{12}a_{21}.$$

Cramer's Rule also extends to general $(n \times n)$ systems of equations. The solution to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

is
$$x_i = \frac{|\mathbf{A}_i(\mathbf{b})|}{|\mathbf{A}|}, \quad i=1, \dots, n,$$

where
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

and $\mathbf{A}_i(\mathbf{b})$ is the matrix \mathbf{A} with its i th column replaced by \mathbf{b} .

Exercises

1. Find the determinants of the following matrices using the Laplace expansion.

$$(i) \begin{bmatrix} 4 & 0 & 1 \\ 19 & 1 & -3 \\ 7 & 1 & 0 \end{bmatrix} \quad (ii) \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix} \quad (iii) \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

$$(vi) \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (v) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad (vii) \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$(viii) \begin{bmatrix} 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \\ a_{41} & 0 & 0 \end{bmatrix} \quad (ix) \begin{bmatrix} 4 & 3 & 2 \\ 6 & 4.5 & 3 \\ 7 & 1 & 0 \end{bmatrix} \quad (x) \begin{bmatrix} 4 & 3 & 0 \\ 9 & 1 & 3 \\ 7 & 1 & 0 \end{bmatrix}$$

2. Use Cramer's Rule to solve

$$\begin{aligned} 4x &+ z = 4 \\ 19x + y - 3z &= 3 \\ 7x + y &= 1 \end{aligned}$$

3. Solve the following system of equations

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 + 4x_4 &= 2 \\ -x_2 + 11x_4 &= 3 \\ x_1 - x_2 + 3x_4 &= 1 \\ -2x_1 - x_3 + 3x_4 &= 2 \end{aligned}$$

4. Find the determinants using the Laplace expansion

$$(i) \begin{bmatrix} a_{11} & 0 & 0 & a_{41} \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad (ii) \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad (iii) \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ 0 & a_{22} & a_{32} & a_{42} \\ 0 & 0 & a_{33} & a_{43} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

$$(iv) \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \quad (v) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (vi) \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

5. Let \mathbf{A} be an arbitrary $(n \times n)$ matrix. Show that if we multiply every element of a single row or column by c , then the determinant of the new matrix is $c|\mathbf{A}|$. What is the determinant of $c\mathbf{A}$?

6. Let \mathbf{E} be the matrix obtained by switching the 1st and last rows of an $(n \times n)$ identity matrix, i.e.

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

- (a) Show using the Laplace expansion that $\det \mathbf{E} = -1$;
- (b) Does your answer depend on whether n is even or odd?
- (c) Use your result to prove that the determinant of the matrix formed by switching *any* two rows of the identity matrix is -1 .

7. (a) Let

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix}$$

and suppose \mathbf{A} is some $(3 \times n)$ matrix. Describe the rows of the product \mathbf{EA} in terms of the rows of \mathbf{A} ; What is the determinant of the matrix \mathbf{E} ?

(b) Let \mathbf{E} be a matrix that carries out the elementary row operation of subtracting a multiple of one row from another row. What is the determinant of \mathbf{E} ?